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NORMALITY CONCERNING TWO FAMILIES OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study normality concerning two families of meromorphic functions. In particular, we investigate the sharing conditions under which normality of one family implies the normality of other. Results obtained in this paper extend some earlier works of several authors.

1. Introduction and main results

A family $\mathcal{F}$ of meromorphic functions defined on a domain $D \subseteq \overline{\mathbb{C}}$ is said to be normal in D if every sequence of elements of $\mathcal{F}$ contains a subsequence which converges locally uniformly in D with respect to the spherical metric, to a meromorphic function or ∞. One can refer to [6] for all necessary background information.

Let f and g be meromorphic functions defined on a domain D, and a be a complex number. If $g(z) = a$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = a$, and say that f *share* a *partially with* g. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = a$, and say that f *and* g *share the value* a *in* D.

Schwick [7] was the first to draw a connection between shared values and normality of the family of meromorphic functions. Precisely, he proved that if there exist three distinct numbers a_1, a_2 and a_3 in $\mathbb{C}$ such that $f(z) = a_i \Leftrightarrow f'(z) = a_i$ $(i = 1, 2, 3)$ in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

In 2013, X. J. Liu et al. [3] posed the problem: *For two families of functions which share four values, if one is normal, is the other normal* ? Interestingly, X. J. Liu et al. [3] gave the answer to this problem in a positive way. They proved:

Theorem A. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain $D \subset \mathbb{C}$, a_1, a_2, a_3, a_4 be four distinct complex numbers. If $\mathcal{G}$ is normal,*

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and for each $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that

$$f(z) = a_i \Leftrightarrow g(z) = a_i \ (i = 1, 2, 3, 4),$$

then $\mathcal{F}$ is normal on D.

Following Theorem A, many authors (see [9, 10] etc.) attempt to find the normality criteria for family of meromorphic functions $\mathcal{F}$ when members of $\mathcal{F}$ share some value(s) with some members of a normal family. Recently, Yuan et al. [10] obtained the following results:

Theorem B. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions in a domain $D \subset \mathbb{C}$, and a_i $(i = 1, 2)$ be two distinct non zero complex numbers. If all of zeros of $f \in \mathcal{F}$ have multiplicities at least 3, $\mathcal{G}$ is normal and for each $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that $f' = a_i \Rightarrow g' = a_i$ for $i = 1, 2$, then $\mathcal{F}$ is normal in D.*

Theorem C. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions in a domain $D \subset \mathbb{C}$, k be a positive integer, and a_i $(i = 1, 2)$ be two distinct non zero complex numbers. Suppose for each $f \in \mathcal{F}$, all its zeros are of multiplicity at least $k + 1$ and all its poles are multiple. If $\mathcal{G}$ is normal and for each $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that $f^{(k)} = a_i \Rightarrow g^{(k)} = a_i$ for $i = 1, 2$, then $\mathcal{F}$ is normal in D.*

Let $n_0, n_1, n_2, \ldots, n_k$ be non negative integers with at least one of them non zero, and set

$$M[f] := f^{n_o}(f')^{n_1}(f'')^{(n_2)} \cdots (f^{(k)})^{n_k},$$

$\gamma_M := \sum_{j=0}^{k} n_j$ and $\Gamma_M := \sum_{j=0}^{k}(j + 1)n_j$. Then $M[f]$ is called *differential monomial* of f, γ_M is called degree of $M[f]$, and Γ_M is called the weight of $M[f]$.

Let $M_1[f]$, $M_2[f]$, …, $M_m[f]$ be differential monomials of f, and let a_1, a_2, …, a_m be holomorphic functions in D. Set

$$P[f] := \sum_{i=1}^{m} a_i M_i[f],$$

$\gamma_P := \max\{\gamma_i : 1 \leq i \leq m\}$ and $\Gamma_P := \max\{\Gamma_i : 1 \leq i \leq m\}$. Then $P[f]$ is called *differential polynomial* of f, γ_P is called degree of $P[f]$, and Γ_P is called the weight of $P[f]$. Also, we set

$$\frac{\Gamma}{\gamma}\Big|_P := \max\left\{\frac{\Gamma_{M_i}}{\gamma_{M_i}} : 1 \leq i \leq m\right\}.$$

Recently, Zeng [12] proved the following normality criterion for a family of meromorphic functions when differential polynomials of any pair of functions in the family shared a value.

Theorem D. *Let k and $q(\geq 2)$ be two positive integers, $a \neq 0$ be a complex number, and let $P[f]$ be a differential polynomial with $\frac{\Gamma}{\gamma}\big|_P < k + 1$. Let $\mathcal{F}$ be* *a family of meromorphic functions in a domain $D \subset \mathbb{C}$, all of whose zeros are of multiplicity at least $k+1$. If for each pair f and g in $\mathcal{F}$,*

$$(f^{(k)})^q + P[f] = a \Leftrightarrow (g^{(k)})^q + P[g] = a,$$

then $\mathcal{F}$ is normal in D.

In this paper we extend Theorem D as:

Theorem 1.1. *Let $M[f]$ be a differential monomial with $n_o \geq 2$, $n_k \geq 1$, $k \geq 1$, and $P[f]$ be a differential polynomial with $\frac{\Gamma}{\gamma}\big|_P < k+1$. Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions in a domain $D \subset \mathbb{C}$, and a_i $(i = 1, 2)$ be two distinct non zero complex numbers. Suppose that for each $f \in \mathcal{F}$, all its zeros are of multiplicity at least $k+1$. If $\mathcal{G}$ is normal, and for each $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that*

$$M[f] + P[f] = a_i \Rightarrow M[g] + P[g] = a_i$$

for $i = 1, 2$, then $\mathcal{F}$ is normal in D.

Another interesting parameter characterizing the normal families of meromorphic functions is the spherical derivative which is defined as

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

with an obvious modification if $f(z) = \infty$. By Marty's criterion, normality of any family of meromorphic functions on some domain D is equivalent to the local boundedness of the corresponding family of spherical derivatives. On the other hand, Grahl and Nevo [2] gave the reverse counterpart to Marty's theorem: A family of meromorphic functions in the unit disk $\mathbb{D}$ where spherical derivatives are uniformly bounded away from zero is normal.

Keeping in view the above facts, we prove the following normality criterion related to two families of meromorphic functions in terms of spherical derivative.

Theorem 1.2. *Let $M[f]$ be a differential monomial with $n_o \geq 2$, $n_k \geq 1$, $k \geq 1$, and $P[f]$ be a differential polynomial with $\frac{\Gamma}{\gamma}\big|_P < k+1$. Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions in a domain $D \subset \mathbb{C}$, and a_i $(i = 1, 2)$ be two distinct non zero complex numbers. Suppose that $0 < \epsilon_1 < \epsilon_2$ and for each $f \in \mathcal{F}$, all its zeros are of multiplicity at least $k+1$. If $\mathcal{G}$ is normal, and for each $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that*

$$M[f] + P[f] = a_1 \Rightarrow g^{\#} \leq \epsilon_1$$

and

$$M[f] + P[f] = a_2 \Rightarrow g^{\#} \geq \epsilon_2,$$

then $\mathcal{F}$ is normal in D.

2. Proof of the main results

For the proof of main results we shall use the famous rescaling lemma which was originally proved by L. Zalcman [11] and later extended by X. C. Pang ([4, 5]), and by H. Chen and Y. Gu [1]. Here we require the following general version of this rescaling lemma.

Lemma 2.1 (Zalcman-Pang Lemma). *Let $\mathcal{F}$ be a family of meromorphic functions in $\mathbb{D}$ all of whose zeros have multiplicity at least m and all of whose poles have multiplicity at least p. Then $\mathcal{F}$ is not normal at a point $z_0 \in \mathbb{D}$ if and only if there exist, for each $\alpha : -p < \alpha < m$,*
(i) *a real number $r : r < 1$,*
(ii) *points $z_n : |z_n| < r$,*
(iii) *positive numbers $\rho_n : \rho_n \to 0$,*
(iv) *functions $f_n \in \mathcal{F}$*
such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges locally uniformly with respect to the spherical metric to $g(\zeta)$, where $g(\zeta)$ is a non-constant meromorphic function on $\mathbb{C}$ and $g^{\#}(\zeta) \le g^{\#}(0) = 1$.

Lemma 2.2. *Let $M[f]$ be a differential monomial with $n_o \ge 2$, $n_k \ge 1$, $k \ge 1$, and a be a non zero complex number. Let f be a non constant rational function having zeros of multiplicity at least $k+1$. Then $M[f] - a$ has at least one zero.*

Proof. If f is a polynomial, then $M[f] - a$ is also a polynomial and so by Fundamental Theorem of Algebra $M[f] - a$ has at least one zero.

If f is rational but not polynomial, then we set

$$f(z) = A\frac{\prod_{i=1}^{s}(z-\alpha_i)^{m_i}}{\prod_{j=1}^{t}(z-\beta_j)^{l_j}}, \tag{2.1}$$

where A is non zero constant, $m_i \ge k+1$ $(i = 1, 2, \ldots, s)$ and $l_j \ge 1$ $(j = 1, 2, \ldots, t)$.

Put

$$M = \sum_{i=1}^{s} m_i \ge (k+1)s \text{ and } N = \sum_{j=1}^{t} l_j \ge t.$$

Now

$$M[f](z) = A^{\gamma_M}\frac{\prod_{i=1}^{s}(z-\alpha_i)^{(m_i+1)\gamma_M - \Gamma_M}}{\prod_{j=1}^{t}(z-\beta_j)^{(l_j-1)\gamma_M + \Gamma_M}}\, g_o(z), \tag{2.2}$$

where $g_o(z)$ is a polynomial such that $\deg(g_o(z)) \le (s+t-1)(\Gamma_M - \gamma_M)$.

On differentiating (2.2), we get

$$M'[f](z) = A^{\gamma_M} \frac{\prod_{i=1}^{s}(z-\alpha_i)^{(m_i+1)\gamma_M-\Gamma_M-1}}{\prod_{j=1}^{t}(z-\beta_j)^{(l_j-1)\gamma_M+\Gamma_M+1}} g_1(z), \tag{2.3}$$

where $g_1(z)$ is a polynomial such that $\deg(g_o(z)) \leq (s+t-1)(\Gamma_M - \gamma_M + 1)$.

Assume on the contrary that $M[f]-a$ has no zero, then

$$M[f](z) = a + \frac{C}{\prod_{j=1}^{t}(z-\beta_j)^{(l_j-1)\gamma_M+\Gamma_M}}, \tag{2.4}$$

where C is non zero constant.

Differentiating (2.4), we get

$$M'[f](z) = \frac{g_2(z)}{\prod_{j=1}^{t}(z-\beta_j)^{(l_j-1)\gamma_M+\Gamma_M+1}}, \tag{2.5}$$

where $g_2(z)$ is a polynomial such that $\deg(g_2(z)) \leq t-1$.

Comparing (2.3) and (2.5), we obtain

$$t-1 \geq \deg(g_2(z)) \geq \gamma_M M - (\Gamma_M - \gamma_M + 1)s$$

and since $N \geq t$, we find that

$$M < \frac{\Gamma_M - \gamma_M + 1}{\gamma_M} s + \frac{N}{\gamma_M}. \tag{2.6}$$

Now, comparing (2.2) and (2.4), we get

$$\begin{aligned}(N-t)\gamma_M + t\Gamma_M &\leq (M+s)\gamma_M - s\Gamma_M + \deg(g_o(z))\\ &\leq (M+s)\gamma_M - s\Gamma_M + (s+t-1)(\Gamma_M - \gamma_M),\end{aligned}$$

so that $N < M$. Therefore, from (2.6) and the fact that $M \geq (k+1)s$, we conclude that

$$M < \left[\frac{\Gamma_M - \gamma_M + 1}{(k+1)\gamma_M} + \frac{1}{\gamma_M}\right] M < M,$$

which is absurd. Hence $M[f]-a$ has at least one zero. □

Lemma 2.3. *Let $M[f]$ be a differential monomial with $n_o \geq 2$, $n_k \geq 1$, $k \geq 1$, and a be a non zero complex number. Let f be a transcendental meromorphic function such that f has only zeros of multiplicity at least $k+1$. Then $M[f]-a$ has infinitely many zeros.*

Proof. Since the zeros of f has multiplicity at least $k+1$, we have

$$\overline{N}\left(r, \frac{1}{f}\right) = \overline{N}_{(k+1}\left(r, \frac{1}{f}\right)$$

$$\leq \frac{1}{kn_0+\gamma_M-1}\left[N\left(r,\frac{1}{M[f]}\right)-\overline{N}\left(r,\frac{1}{M[f]}\right)\right]. \tag{2.7}$$

Also, one can see that

$$\overline{N}(r,f) \leq \frac{1}{\Gamma_M}N(r,M[f]). \tag{2.8}$$

Now,

$$\begin{aligned}\overline{N}\left(r,\frac{1}{M[f]}\right) &\leq \overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{k}\overline{N}_o\left(r,\frac{1}{f^{(j)}}\right)\\ &\leq \overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{k}j\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)\right]+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{f}\right)+\frac{k(k+1)}{2}\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)\right]+S(r,f),\end{aligned}$$

where $\overline{N}_o(r,1/f^{(j)})$ is the numbers of those zeros of $f^{(j)}$ which are not the zeros of f.

That is,

$$\overline{N}\left(r,\frac{1}{M[f]}\right) \leq \left(1+\sum_{j=1}^{k}j\right)\overline{N}\left(r,\frac{1}{f}\right)+\sum_{j=1}^{k}j\overline{N}(r,f)+S(r,f). \tag{2.9}$$

On substituting (2.7) and (2.8) in (2.9), we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{M[f]}\right) \leq \frac{1+\sum_{j=1}^{k}j}{kn_0+\gamma_M-1}&\left[N\left(r,\frac{1}{M[f]}\right)-\overline{N}\left(r,\frac{1}{M[f]}\right)\right]\\ &+\frac{1}{\Gamma_M}\sum_{j=1}^{k}jN(r,M[f])+S(r,f)\end{aligned}$$

$$\begin{aligned}\Rightarrow \left(1+\frac{1+\sum_{j=1}^{k}j}{kn_0+\gamma_M-1}\right)\overline{N}\left(r,\frac{1}{M[f]}\right) &\leq \frac{1+\sum_{j=1}^{k}j}{kn_0+\gamma_M-1}N\left(r,\frac{1}{M[f]}\right)\\ &\quad+\frac{1}{\Gamma_M}\sum_{j=1}^{k}jN(r,M[f])+S(r,f).\end{aligned}$$

Therefore

$$\begin{aligned}\overline{N}\left(r,\frac{1}{M[f]}\right) \leq \frac{1+\sum_{j=1}^{k}j}{kn_0+\gamma_M+\sum_{j=1}^{k}j}N\left(r,\frac{1}{M[f]}\right)+\frac{1}{\Gamma_M}\frac{\sum_{j=1}^{k}j(kn_0+\gamma_M-1)}{kn_0+\gamma_M+\sum_{j=1}^{k}j}N(r,M[f])\\ +S(r,f).\end{aligned} \tag{2.10}$$

Now, suppose on contrary that $M[f]-a$ has only finitely many zeros. Then, by Second fundamental theorem of Nevanlinna, we have

$$T(r,M[f]) \le \overline{N}(r,M[f]) + \overline{N}\left(r,\frac{1}{M[f]}\right) + \overline{N}\left(r,\frac{1}{M[f]-a)}\right) + S(r,M[f])$$

$$\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{M[f]}\right) + S(r,M[f]). \tag{2.11}$$

By using (2.7) and (2.10) in (2.11), we get

$$T(r,M[f]) \le \frac{1}{\Gamma_M}N(r,M[f]) + \frac{1+\sum_{j=1}^{k} j}{kn_0+\gamma_M+\sum_{j=1}^{k} j}N\left(r,\frac{1}{M[f]}\right)$$
$$+\frac{1}{\Gamma_M}\frac{\sum_{j=1}^{k} j(kn_0+\gamma_M-1)}{kn_0+\gamma_M+\sum_{j=1}^{k} j}N(r,M[f]) + S(r,M[f])$$

$$\Rightarrow \left[1-\frac{1+\sum_{j=1}^{k} j}{kn_0+\gamma_M+\sum_{j=1}^{k} j} - \frac{1}{\Gamma_M}\left(1+\frac{\sum_{j=1}^{k} j(kn_0+\gamma_M-1)}{kn_0+\gamma_M+\sum_{j=1}^{k} j}\right)\right]T(r,M[f])$$
$$\le S(r,M[f])$$
$$\Rightarrow T(r,M[f]) \le S(r,M[f]).$$

Therefore, by using the inequality (see [8])

$$T(r,f)+S(r,f) \le CT(r,M[f]) + S(r,M[f]),$$

where C is a constant and $S(r,f)=S(r,M[f])$, we get a contradiction. Hence the lemma follows. $\square$

Proof of Theorem 1.1. Suppose on the contrary that $\mathcal{F}$ is not normal at some point $z_o \in D$. Without loss of generality, we assume D to be the open unit disk $\mathbb{D}$, and $z_o = 0$. By Lemma 2.1, we can find a sequence $\{f_j\}$ in $\mathcal{F}$, a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ and a sequence $\{\rho_j\}$ of positive real numbers with $\rho_j \to 0$ such that

$$h_j(\zeta) = \rho_j^{-\beta} f_j(z_j+\rho_j\zeta)$$

converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $h(\zeta)$ on $\mathbb{C}$, all of whose zeros are of multiplicity at least $k+1$, such that $h^{\#}(\zeta) \le h^{\#}(0) = 1$ for all $\zeta \in \mathbb{C}$. Also, we take $\beta = \frac{\Gamma_M}{\gamma_M} - 1$.

It is easily seen that

$$\begin{aligned} P[h_j](\zeta) &= P[f_j](z_j+\rho_j\zeta) \\ &= \sum_{i=1}^{m} a_i(z_j+\rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} M_i[h_j](\zeta). \end{aligned}$$

Since all $a_i(z)$ $(i=1,2,\ldots,m)$ are holomorphic in D, by using $\frac{\Gamma}{\gamma}\Big|_P < k+1$, we deduce that

$$\sum_{i=1}^{m} a_i(z_j+\rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} M_i[h_j](\zeta)$$

converges uniformly to 0 on $\mathbb{C}$.

Hence on every compact subset of $\mathbb{C}$ that contain no poles of h, we have

$$M[h_j](\zeta)+P[h_j](\zeta) = M[f_j](z_j+\rho_j\zeta)+P[f_j](z_j+\rho_j\zeta) \to M[h](\zeta)$$

locally uniformly with respect to the spherical metric. By Lemma 2.2 and Lemma 2.3, $M[h]-a_i$ $(i=1,2)$ has at least one zero. Let ζ_o and ζ_o^* be the zeros of $M[h]-a_1$ and $M[h]-a_2$ respectively. Obviously, $\zeta_o \neq \zeta_o^*$. We choose r (>0) small enough such that $D_r(\zeta_o)\cap D_r(\zeta_o^*)=\phi$, where $D_r(\zeta_o)=\{\zeta\in D : |\zeta-\zeta_o|<r\}$ and $D_r(\zeta_o^*)=\{\zeta\in D : |\zeta-\zeta_o^*|<r\}$.

By Hurwitz's theorem, there exist points $\zeta_j\in D_r(\zeta_o)$ and $\zeta_j^*\in D_r(\zeta_o^*)$ such that for sufficiently large j,

$$M[f_j](z_j+\rho_j\zeta_j)+P[f_j](z_j+\rho_j\zeta_j)-a_1=0$$

and

$$M[f_j](z_j+\rho_j\zeta_j^*)+P[f_j](z_j+\rho_j\zeta_j^*)-a_2=0.$$

By hypothesis there exists a sequence $\{g_j\}$ of functions in $\mathcal{G}$ such that

$$M[g_j](z_j+\rho_j\zeta_j)+P[g_j](z_j+\rho_j\zeta_j)=a_1$$

and

$$M[g_j](z_j+\rho_j\zeta_j^*)+P[g_j](z_j+\rho_j\zeta_j^*)=a_2.$$

Since $\mathcal{G}$ is normal, without loss of generality, we assume that $g_j(z)\to g(z)$ locally uniformly with respect to the spherical metric. Thus taking $j\to\infty$, we have

$$\begin{aligned} 0<|a_1-a_2| &= |(M[g_j](z_j+\rho_j\zeta_j)+P[g_j](z_j+\rho_j\zeta_j)) \\ &\quad -(M[g_j](z_j+\rho_j\zeta_j^*)+P[g_j](z_j+\rho_j\zeta_j^*))| \\ &\to |M[g](0)+P[g](0)-M[g](0)-P[g](0)|=0. \end{aligned}$$

This is a contradiction. Hence $\mathcal{F}$ is normal at z_o. □

Proof of Theorem 1.2. Proceeding in the same way as in Theorem 1.1, on every compact subset of $\mathbb{C}$ that contain no poles of h, we have

$$M[h_j](\zeta)+P[h_j](\zeta) = M[f_j](z_j+\rho_j\zeta)+P[f_j](z_j+\rho_j\zeta) \to M[h](\zeta)$$

locally uniformly with respect to the spherical metric. By Lemma 2.2 and Lemma 2.3, $M[h] - a_i \, (i = 1, 2)$ has at least one zero. Let ζ_o and ζ_o^* be the zeros of $M[h] - a_1$ and $M[h] - a_2$ respectively. Obviously, $\zeta_o \neq \zeta_o^*$. We choose $r \; (> 0)$ small enough such that $D_r(\zeta_o) \cap D_r(\zeta_o^*) = \phi$, where $D(\zeta_o) = \{\zeta \in D : |\zeta - \zeta_o| < r\}$ and $D_r(\zeta_o^*) = \{\zeta \in D : |\zeta - \zeta_o^*| < r\}$.

By Hurwitz's theorem, there exist points $\zeta_j \in D_r(\zeta_o)$ and $\zeta_j^* \in D_r(\zeta_o^*)$ such that for sufficiently large j,

$$M[f_j](z_j + \rho_j \zeta_j) + P[f_j](z_j + \rho_j \zeta_j) - a_1 = 0$$

and

$$M[f_j](z_j + \rho_j \zeta_j^*) + P[f_j](z_j + \rho_j \zeta_j^*) - a_2 = 0.$$

By hypothesis there exists sequence $\{g_j\}$ in $\mathcal{G}$ such that

$$g_j^{\#}(z_j + \rho_j \zeta_j) \leq \epsilon_1 \text{ and } g_j^{\#}(z_j + \rho_j \zeta_j^*) \geq \epsilon_2.$$

Since $\mathcal{G}$ is normal, without loss of generality, we assume that $g_j(z) \to g(z)$ locally uniformly with respect to the spherical metric. Thus taking $j \to \infty$, we have

$$\epsilon_1 \geq \lim_{j \to \infty} g_j^{\#}(z_j + \rho_j \zeta_j) = g^{\#}(0) = \lim_{j \to \infty} g_j^{\#}(z_j + \rho_j \zeta_j^*) \geq \epsilon_2.$$

This is a contradiction to the fact that $0 < \epsilon_1 < \epsilon_2$. Hence $\mathcal{F}$ is normal at z_o. □

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