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ON A NEW CLASS OF FUNCTIONS RELATED WITH MITTAG-LEFFLER AND WRIGHT FUNCTIONS AND THEIR PROPERTIES

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Abstract. In the present paper, we define new class of functions $T_{\alpha,\beta}(\lambda; z)$ which is an extension of the classical Wright function and the Mittag-Leffler function. We show some mean value inequalities for the this function, such as Turán-type inequalities, Lazarević-type inequalities and Wilker-type inequalities. Moreover, integrals formula and integral inequality for the function $T_{\alpha,\beta}(\lambda; z)$ are presented.

1. Introduction

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, z \in \mathbb{C}). \tag{1.1}$$

This function was first introduced by G. Mittag-Leffler in 1903 for $\alpha = 1$ and by A. Winman in 1905 for the general case (1.1). The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. These functions interpolate between a purely exponential law and power-law like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts [4–6,12]. The most essential properties of these entire functions, investigated by many mathematicians, can be found in [1,2].

The Wright function, denoted by $W_{\alpha,\beta}(z)$, was studied in [13] in connection with the asymptotic of the number of some special partitions of the natural

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numbers. It was defined through the convergent series

$$W_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\alpha\,n+\beta)}\quad(\alpha>-1,\ \beta\in\mathbb{C},\ z\in\mathbb{C}). \tag{1.2}$$

If $\alpha>-1$, the series (1.2) is absolutely convergent for all $z\in\mathbb{C}$, while for $\alpha=-1$ this series is absolutely convergent in the unit disk $\mathbb{D}:=\{z:|z|<1\}$. Moreover, for $\alpha>-1$, $W_{\alpha,\beta}$ is entire function of z. The Wright functions have been used widely in the asymptotic theory of partitions, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Also, the Wright function appeared as the Green function while solving some initial- and boundary-value problems for the fractional diffusion-wave equation, *i.e.*, for the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first or second order time derivative by a fractional derivative of order α with $0<\alpha\leq 2$ (see, [3,7]).

The purpose of present work is to define a new function which enable us to study both Mittag-Leffler and Wright function simultaneously. For this we define a new function as follows:

$$T_{\alpha,\beta}(\lambda;z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)(\lambda\,n!+(1-\lambda))}. \tag{1.3}$$

It is easy to see that

$$T_{\alpha,\beta}(0;z)=E_{\alpha,\beta}(z)$$

and

$$T_{\alpha,\beta}(1;z)=W_{\alpha,\beta}(z).$$

Therefore this function enable us to study both Mittag-Leffler and Wright function.

Our aim in this paper is to investigate certain types inequalities for this new function, such as Turán-type inequalities, Lazarević-type inequalities and Wilker-type inequalities.

In the proof of the main results we will need the following lemma, see [11], for more details.

Lemma 1.1. *Consider the power series*

$$f(x)=\sum_{n\geq 0}a_nx^n \text{ and } g(x)=\sum_{n\geq 0}b_nx^n,$$

where for all $n\geq 0$ *we have* $a_n,b_n\in\mathbb{R}$ *and* $b_n>0$, *and suppose that both series converge on* $(-r,r)$, $r>0$. *If the sequence* $\{a_n/b_n\}_{n\geq 0}$ *is increasing* (*decreasing*), *then the function* $x\mapsto f(x)/g(x)$ *is increasing* (*decreasing*) *too on* $(0,r)$.

2. Turán type inequalities for the function $T_{\alpha,\beta}(\lambda;x)$

Our first main results are asserted in the following theorem.

Theorem 2.1. *Let $\alpha,\beta>0$ and $\lambda\in[0,1]$. Then the following assertions are true*:

(a) *The function $\beta\mapsto\Gamma(\beta)T_{\alpha,\beta}(\lambda;x)=\mathbb{T}_{\alpha,\beta}(\lambda;x)$ is log-convex on $(0,\infty)$.*

(b) *The following Turán type inequality*

$$[T_{\alpha,\beta+1}(\lambda;x)]^2\le[T_{\alpha,\beta}(\lambda;x)]\;[T_{\alpha,\beta+2}(\lambda;x)]$$

holds for all $x\in(0,\infty)$.

(c) *For $n\in\mathbb{N}$, we define the function $T^n_{\alpha,\beta}(\lambda;x)$ by*

$$\begin{aligned}T^n_{\alpha,\beta}(\lambda;x)&=T_{\alpha,\beta}(\lambda;x)-\sum_{k=0}^{n}\frac{x^k}{\Gamma(\alpha k+\beta)(\lambda\,k!+(1-\lambda))}\\&=\sum_{k=n+1}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)(\lambda\,k!+(1-\lambda))}.\end{aligned}$$

Then the following Turán-type inequality

$$T^n_{\alpha,\beta}(\lambda;x)T^{n+2}_{\alpha,\beta}(\lambda;x)\le[T^{n+1}_{\alpha,\beta}(\lambda;x)]^2$$

is valid for all $n\in\mathbb{N}$ and $\alpha,\beta>0$ and $x>0$.

Proof. (a) To prove log-convexity of $\beta\mapsto\Gamma(\beta)T_{\alpha,\beta}(\lambda;x)$, it is enough to show the log-convexity of each individual term and to use the fact that the sum of log-convex function is log-convex too. Thus, we need to show that for each $n\ge0$ we have

$$\frac{\partial^2}{\partial\beta^2}\log\left[\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\right]=[\psi'(\beta)-\psi'(\alpha n+\beta)]\ge0,$$

where

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function. But ψ is known to be concave, and consequently the function $\beta\mapsto\Gamma(\beta)T_{\alpha,\beta}(\lambda;x)$ is log-convex on $(0,\infty)$.

(b) Since the function $\beta\mapsto\mathbb{T}_{\alpha,\beta}(\lambda;x)$ is log-convex therefore for all $\beta_1,\beta_2>0$, $x>0$ and $\mu\in[0,1]$, we have

$$\mathbb{T}_{\alpha,(\mu\beta_1+(1-\mu)\beta_2)}(\lambda;x)\le[\mathbb{T}_{\alpha,\beta_1}(\lambda;x)]^{\mu}\;[\mathbb{T}_{\alpha,\beta_2}(\lambda;x)]^{1-\mu}.$$

Now choosing $\beta_1=\beta,\beta_2=\beta+2,\mu=1/2$, we have

$$[\mathbb{T}_{\alpha,\beta+1}(\lambda;x)]^2\le[\mathbb{T}_{\alpha,\beta}(\lambda;x)]\;[\mathbb{T}_{\alpha,\beta+2}(\lambda;x)].$$

This completes the proof of part (b).

(c) Let us take $L(k) = \lambda\ k! + (1-\lambda)$ and $\phi(k) = L(k)/L(k-1)$. Doing calculation, we get

$$\begin{aligned}\phi(k+1)-\phi(k) &= \frac{\lambda^2 k!(k-1)! + \lambda(1-\lambda)(k-1)!(k^2-k+1)}{(\lambda\ k! + (1-\lambda))(\lambda\ (k-1)! + (1-\lambda))} \\ &\geq 0 \text{ (for all } k \geq 1).\end{aligned} \tag{2.1}$$

This implies $\phi(k)$ is a non-decreasing function of k. Now, from the definition of the function $T^n_{\alpha,\beta}(\lambda; x)$, for each $n \in \mathbb{N}$, we have

$$T^n_{\alpha,\beta}(\lambda; x) = T^{n+1}_{\alpha,\beta}(\lambda; x) + \frac{x^{n+1}}{\Gamma(\alpha(n+1)+\beta)L(n+1)}$$

and

$$T^{n+2}_{\alpha,\beta}(\lambda; x) = T^{n+1}_{\alpha,\beta}(\lambda; x) - \frac{x^{n+2}}{\Gamma(\alpha(n+2)+\beta)L(n+2)}.$$

Now

$$\begin{aligned}&T^n_{\alpha,\beta}(\lambda; x)T^{n+2}_{\alpha,\beta}(\lambda; x) - \left[T^{n+1}_{\alpha,\beta}(\lambda; x)\right]^2 \\ &= T^{n+1}_{\alpha,\beta}(\lambda; x)\left(\frac{x^{n+1}}{\Gamma(\alpha(n+1)+\beta)L(n+1)} - \frac{x^{n+2}}{\Gamma(\alpha(n+2)+\beta)L(n+2)}\right) \\ &\quad - \frac{x^{2n+3}}{\Gamma(\alpha(n+1)+\beta)L(n+1)\Gamma(\alpha(n+2)+\beta)L(n+2)} \\ &= \sum_{k=n+3}^{\infty}\left(\frac{x^{k+n+1}}{\Gamma(\alpha k+\beta)L(k)\Gamma(\alpha(n+1)+\beta)L(n+1)} - \frac{x^{k+n+1}}{\Gamma(\alpha(k-1)+\beta)L(k-1)\Gamma(\alpha(n+2)+\beta)L(n+2)}\right) \\ &= \sum_{k=n+3}^{\infty}\frac{1}{L(k)L(n+2)\Gamma(\alpha k+\beta)\Gamma(\alpha(n+2)+\beta)} \\ &\qquad \times\left(\frac{\Gamma(\alpha(n+2)+\beta)L(n+2)}{\Gamma(\alpha(n+1)+\beta)L(n+1)} - \frac{\Gamma(\alpha k+\beta)L(k)}{\Gamma(\alpha(k-1)+\beta)L(k-1)}\right)x^{k+n+1}.\end{aligned}$$

In view of non-decreasingness of $\phi(k)$ and the fact that $\frac{\Gamma(x+a)}{\Gamma(x)}$ is a increasing function of x for all $a \in (0,\infty)$, it is easy to see that the second term in the bracket is a increasing function of k, while the first term is a constant. If we show that the first term of the summation is negative then all other term will also become negative. Now

$$\frac{\Gamma(\alpha(n+3)+\beta)L(n+3)}{\Gamma(\alpha(n+2)+\beta)L(n+2)} \geq \frac{\Gamma(\alpha(n+2)+\beta)L(n+2)}{\Gamma(\alpha(n+1)+\beta)L(n+1)}$$

by again using the same non-decreasingness of $\phi(k)$ and increasingness of $\frac{\Gamma(x+a)}{\Gamma(x)}$. Therefore the first term of summation is negative for $k = n+3$ and hence all the term of summation is also negative. Thus we can conclude that

$$T^n_{\alpha,\beta}(\lambda; x)T^{n+2}_{\alpha,\beta}(\lambda; x) - \left[T^{n+1}_{\alpha,\beta}(\lambda; x)\right]^2 \leq 0.$$

This completes the proof of part (c). □

Remark 2.2. For $\lambda = 0$ in Theorem 2.1, we get the results proved by Mehrez and Sitnik in [9, 10] and the results corresponding to $\lambda = 1$ are proved by Mehrez in [8].

Theorem 2.3. *Let $\alpha, \beta_1, \beta_2 > 0$ and $\lambda \in [0, 1]$*

a. *If $\beta_2 > \beta_1$ ($\beta_2 < \beta_1$), then the function $x \mapsto T_{\alpha,\beta_1}(\lambda; x)/T_{\alpha,\beta_2}(\lambda; x)$ is increasing (decreasing) on $(0, \infty)$.*

b. *If $\beta_2 > \beta_1 > 1$, then following Turán type inequality holds:*

$$T_{\alpha,\beta_2}(\lambda; x)T_{\alpha,\beta_1-1}(\lambda; x) - T_{\alpha,\beta_1}(\lambda; x)T_{\alpha,\beta_2-1}(\lambda; x) + (\beta_2 - \beta_1)T_{\alpha,\beta_1}(\lambda; x)T_{\alpha,\beta_2}(\lambda; x) \geq 0. \tag{2.2}$$

In particular, the Turán type inequality

$$T_{\alpha,\beta}(\lambda; x)T_{\alpha,\beta+2}(\lambda; x) - T^2_{\alpha,\beta+1}(\lambda; x) + T_{\alpha,\beta+1}(\lambda; x)T_{\alpha,\beta+2}(\lambda; x) \geq 0. \tag{2.3}$$

Proof. Using (1.3), we have

$$T_{\alpha,\beta_1}(\lambda; x)/T_{\alpha,\beta_2}(\lambda; x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta_1)(\lambda\, n! + (1-\lambda))} \Big/ \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta_2)(\lambda\, n! + (1-\lambda))}.$$

In view of Lemma 1.1, we need to study the monotonicity of sequence $\{u_k\}_{k\geq 0}$ defined by:

$$u_k = \frac{\Gamma(\alpha k + \beta_2)}{\Gamma(\alpha k + \beta_1)} = \frac{\Gamma(\alpha k + \beta_1 + (\beta_2 - \beta_1))}{\Gamma(\alpha k + \beta_1)} \; (k \geq 0); \tag{2.4}$$

$$u_k = \frac{\Gamma(\alpha k + \beta_2)}{\Gamma(\alpha k + \beta_1)} = \frac{\Gamma(\alpha k + \beta_2)}{\Gamma(\alpha k + \beta_2 + (\beta_1 - \beta_2))} \; (k \geq 0). \tag{2.5}$$

If $\beta_2 > \beta_1$, then (2.4) together with the fact that $\frac{\Gamma(x+a)}{\Gamma(x)}$ is a increasing function of x in $(0, \infty)$ implies that the $\{u_k\}_{k\geq 0}$ is a monotonically increasing sequence on $(0, \infty)$ and hence the function $x \mapsto T_{\alpha,\beta_1}(\lambda; x)/T_{\alpha,\beta_2}(\lambda; x)$ is increasing in view of Lemma 1.1.

If $\beta_1 > \beta_2$, then (2.5) together with the fact that $\frac{\Gamma(x)}{\Gamma(x+a)}$ is a decreasing function of x in $(0, \infty)$ implies that the $\{u_k\}_{k\geq 0}$ is a monotonically decreasing sequence on $(0, \infty)$ and hence the function $x \mapsto T_{\alpha,\beta_1}(\lambda; x)/T_{\alpha,\beta_2}(\lambda; x)$ is increasing in view of Lemma 1.1.

It is easy to verify that

$$\frac{d}{dx} T_{\alpha,\beta}(\lambda; x) = \frac{T_{\alpha,\beta-1}(\lambda; x) - (\beta - 1)T_{\alpha,\beta}(\lambda; x)}{\alpha x}. \tag{2.6}$$

Now using (2.6) and part a of Theorem 2.3 for $\beta_2 > \beta_1 > 1$, we have

$$\left[\frac{T_{\alpha,\beta_1}(\lambda; x)}{T_{\alpha,\beta_2}(\lambda; x)}\right]' = \frac{T_{\alpha,\beta_1-1}(x)T_{\alpha,\beta_2}(x) - T_{\alpha,\beta_2-1}(x)T_{\alpha,\beta_1}(x) + (\beta_2 - \beta_1)T_{\alpha,\beta_1}(x)T_{\alpha,\beta_2}(x)}{\alpha x T^2_{\alpha,\beta_2}(x)} \geq 0.$$

This completes the proof of (2.2). Finally, choosing $\beta_1 = \beta + 1$ and $\beta_2 = \beta + 2$ in the inequality (2.2), we obtain (2.3). □

3. Lazarević and Wilker-type inequalities for the function $T_{\alpha,\beta}(\lambda; x)$

Theorem 3.1. *Let $\alpha, \beta_1, \beta_2 > 0$ be such that $\beta_1 \geq \beta_2 > 1$. Then the following inequality holds for all $x \in \mathbb{R}$*

$$[\mathbb{T}_{\alpha,\beta_1}(\lambda; x)]^{\frac{1}{\beta_1 - 1}} \leq [\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{1}{\beta_2 - 1}}. \tag{3.1}$$

Proof. From Theorem 2.1 the function $\beta \mapsto \log[\mathbb{T}_{\alpha,\beta}(\lambda; x)]$ is convex and hence it follows that $\beta \mapsto \log[\mathbb{T}_{\alpha,\beta+a}(\lambda; x)] - \log[\mathbb{T}_{\alpha,\beta}(\lambda; x)]$ is increasing for each $a > 0$. Thus, choosing $a = 1$ we obtain that indeed the function $\beta \mapsto \frac{\mathbb{T}_{\alpha,\beta+1}(\lambda;x)}{\mathbb{T}_{\alpha,\beta}(\lambda;x)}$ is increasing on $(0, \infty)$. Now providing that $\beta_1 \geq \beta_2 > 1$

$$\Phi(x) = \frac{1}{\beta_1 - 1} \log[\mathbb{T}_{\alpha,\beta_1}(\lambda; x)] - \frac{1}{\beta_2 - 1} \log[\mathbb{T}_{\alpha,\beta_2}(\lambda; x)],$$

differentiating with respect to x

$$\Phi'(x) = \frac{1}{\alpha z} \left[\frac{\mathbb{T}_{\alpha,\beta_1-1}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} - \frac{\mathbb{T}_{\alpha,\beta_2-1}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_2}(\lambda; x)} \right].$$

Since the function $\beta \mapsto \frac{\mathbb{T}_{\alpha,\beta+1}(\lambda;x)}{\mathbb{T}_{\alpha,\beta}(\lambda;x)}$ is increasing on $(0, \infty)$ and $\beta_1 \geq \beta_2 > 1$, this implies

$$\frac{\mathbb{T}_{\alpha,\beta_1-1}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} \leq \frac{\mathbb{T}_{\alpha,\beta_2-1}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_2}(\lambda; x)}. \tag{3.2}$$

This gives $x \mapsto \Phi(x)$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. Consequently $\Phi(x) \leq \Phi(0) = 0$ for all $x \in \mathbb{R}$. This completes the proof. □

Corollary 3.2. *Let $\alpha, \beta_1, \beta_2 > 0$ such that $\beta_1 \geq \beta_2 > 1$. Then the following inequality*

$$\frac{\mathbb{T}_{\alpha,\beta_2}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} + [\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{\beta_1 - \beta_2}{\beta_2 - 1}} \geq 2, \tag{3.3}$$

holds true for all $x >\in \mathbb{R}$ and $\lambda \in [0, 1]$.

Proof. Keeping (3.1) in mind, we get

$$\frac{[\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{\beta_1 - 1}{\beta_2 - 1}}}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} = \frac{\mathbb{T}_{\alpha,\beta_2}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} [\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{\beta_1 - \beta_2}{\beta_2 - 1}} \geq 1.$$

In view of this expression and the arithmetic-geometric mean inequality, we get

$$\frac{1}{2} \left[\frac{\mathbb{T}_{\alpha,\beta_2}(\lambda; x)}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)} + [\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{\beta_1 - \beta_2}{\beta_2 - 1}} \right] \geq \sqrt{\frac{[\mathbb{T}_{\alpha,\beta_2}(\lambda; x)]^{\frac{\beta_1 - 1}{\beta_2 - 1}}}{\mathbb{T}_{\alpha,\beta_1}(\lambda; x)}} \geq 1.$$

□

4. Integral formula and integral inequality for the function $T_{\alpha,\beta}(\lambda; z)$

Using (1.3) it can be easily seen that

$$\frac{d}{dz}\left[z^{\beta}T_{\alpha,\beta+1}(\lambda;z^{\alpha})\right] = z^{\beta-1}T_{\alpha,\beta}(\lambda;z^{\alpha}). \tag{4.1}$$

Integrating (4.1) between the limits 0 to x, we have

$$\int_0^x z^{\beta-1}T_{\alpha,\beta}(\lambda;z^{\alpha})dz = x^{\beta}T_{\alpha,\beta+1}(\lambda;x^{\alpha}). \tag{4.2}$$

Theorem 4.1. *Let $\alpha > 0$ and $\beta > 0$. Then the following integral formula holds for all $x > 0$:*

$$\begin{aligned}&\int_0^x \left[\frac{T_{\alpha,\beta-1}(\lambda;z)T_{\alpha,\beta+1}(\lambda;z)}{T^2_{\alpha,\beta-1}(\lambda;z)} + (1-\alpha)\frac{T_{\alpha,\beta+1}(\lambda;z)}{T_{\alpha,\beta}(\lambda;z)}\right]dz\\ &= x\left(1-\alpha\frac{T_{\alpha,\beta+1}(\lambda;x)}{T_{\alpha,\beta}(\lambda;x)}\right).\end{aligned} \tag{4.3}$$

Proof. Let us consider a function

$$G^{\lambda}_{\alpha,\beta}(z) = \frac{T_{\alpha,\beta+1}(\lambda;z)}{T_{\alpha,\beta}(\lambda;z)},\ z > 0.$$

Differentiating it with respect to z and using (2.6), we obtain

$$\frac{d}{dz}\left(G^{\lambda}_{\alpha,\beta}(z)\right) = \frac{1}{\alpha z}\left[1 - G^{\lambda}_{\alpha,\beta}(z) - \frac{G^{\lambda}_{\alpha,\beta}(z)}{G^{\lambda}_{\alpha,\beta-1}(z)}\right] \tag{4.4}$$

and consequently

$$\frac{d}{dz}\left(z\, G^{\lambda}_{\alpha,\beta}(z)\right) = \frac{1}{\alpha}\left[(\alpha-1)G^{\lambda}_{\alpha,\beta}(z) + 1 - \frac{G^{\lambda}_{\alpha,\beta}(z)}{G^{\lambda}_{\alpha,\beta-1}(z)}\right]. \tag{4.5}$$

Integrating both sides of the above equation over $(0, x)$ and rearranging gives the required result. □

Theorem 4.2. *Let $\alpha > 0$ and $\beta > 1$ then the following inequality*

$$\frac{T_{\alpha,\beta+1}(\lambda;x^{\alpha})}{T_{\alpha,\beta}(\lambda;x^{\alpha})} \le \frac{1}{(\beta+1)} + \frac{T_{\alpha,\beta+2}(\lambda;x^{\alpha})}{T_{\alpha,\beta}(\lambda;x^{\alpha})}, \tag{4.6}$$

holds true for all $x > 0$. However, the inequality (4.6) is reversed if $0 < \beta < 1$.

Proof. Since the function

$$z \to \frac{T_{\alpha,\beta}(\lambda;z)}{T_{\alpha,\beta+1}(\lambda;z)}$$

is increasing on $(0, \infty)$, therefore we have

$$T_{\alpha,\beta+1}(\lambda;z) \le \frac{T_{\alpha,\beta}(\lambda;z)}{\beta}. \tag{4.7}$$

This implies that

$$\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz \leq \frac{1}{\beta-1}\int_0^x z^\beta T_{\alpha,\beta-1}(\lambda; z^\alpha)dz. \tag{4.8}$$

Integrating by parts, we get

$$\begin{aligned}\int_0^x z^\beta T_{\alpha,\beta-1}(\lambda; z^\alpha)dz &= \int_0^x z^2\left[z^{\beta-2}T_{\alpha,\beta-1}(\lambda; z^\alpha)\right]dz\\ &= \left[z^{\beta+1}T_{\alpha,\beta}(\lambda; z^\alpha)\right]_0^x - 2\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz\\ &= x^{\beta+1}T_{\alpha,\beta}(\lambda; x^\alpha) - 2\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz.\end{aligned} \tag{4.9}$$

In view of (4.8) and (4.9) we obtain

$$\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz \leq \frac{1}{\beta-1}\left(x^{\beta+1}T_{\alpha,\beta}(\lambda; x^\alpha) - 2\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz\right),$$

which implies that

$$\left(1+\frac{2}{\beta-1}\right)\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz \leq \frac{x^{\beta+1}}{\beta-1}T_{\alpha,\beta}(\lambda; x^\alpha),$$

and consequently the following inequality

$$\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz \leq \frac{x^{\beta+1}}{\beta+1}T_{\alpha,\beta}(\lambda; x^\alpha), \tag{4.10}$$

holds true for all $\beta > 1$ and reversed if $0 < \beta < 1$. Moreover, on using integration by parts and (4.2), we have

$$\begin{aligned}\int_0^x z^\beta T_{\alpha,\beta}(\lambda; z^\alpha)dz &= \int_0^x z\left(z^{\beta-1}T_{\alpha,\beta}(\lambda; z^\alpha)\right)dz\\ &= x^{\beta+1}T_{\alpha,\beta+1}(\lambda; x^\alpha) - \int_0^x z^\beta T_{\alpha,\beta+1}(\lambda; z^\alpha)dz\\ &= x^{\beta+1}\left[T_{\alpha,\beta+1}(\lambda; x^\alpha) - T_{\alpha,\beta+2}(\lambda; x^\alpha)\right].\end{aligned}$$

In view of the above formula and (4.10), we get required result. □

Theorem 4.3. *Let $\alpha > 0$ and $\beta > 0$, we define the functions*

$$I_{\alpha,\beta,0}(\lambda; x) = x^\beta T_{\alpha,\beta}(\lambda; x^\alpha) \tag{4.11}$$

and

$$I_{\alpha,\beta,n}(\lambda; x) = \int_0^x I_{\alpha,\beta,n-1}(\lambda; t)dt \ (n \in \mathbb{N}) \tag{4.12}$$

if $\beta > 1$ and $0 < x < 1$, then

$$I_{\alpha,\beta,n+1}(\lambda; x) \leq \frac{1}{\beta+1}I_{\alpha,\beta,n}(\lambda; x). \tag{4.13}$$

In addition, the inequality (4.13) is reversed if $0 < \beta < 1$.

Proof. Let $\beta > 1$ and $0 < x < 1$. Applying (4.10) we thus get

$$\begin{aligned} I_{\alpha,\beta,1}(\lambda;x) &= \int_0^x I_{\alpha,\beta,0}(\lambda;t)dt = \int_0^x t^{\beta} T_{\alpha,\beta}(\lambda;t^{\alpha}) \\ &\leq \frac{x^{\beta+1}}{\beta+1} T_{\alpha,\beta}(\lambda;x^{\alpha}) \\ &\leq \frac{x^{\beta} T_{\alpha,\beta}(\lambda;x^{\alpha})}{\beta+1} \\ &= \frac{I_{\alpha,\beta,0}(\lambda;x^{\alpha})}{\beta+1}. \end{aligned}$$

Integrating both sides of the above inequality n times with respect to x yields the desired result. □

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