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COMMON FIXED POINT FOR GENERALIZED MULTIVALUED MAPPINGS VIA SIMULATION FUNCTION IN METRIC SPACES

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Abstract. The purpose of this paper is to introduce the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings and establish common fixed point theorems for such mappings in complete metric spaces. Results obtained in this paper extend and generalize some well known fixed point results of the literature. We deduce some corollaries from our main result and provide examples in support of our results.

1. Introduction

One of the fundamental and most useful results in fixed point theory is Banach Contraction Principle [8]. This result has been extended in many directions for single and multivalued cases on a metric space. In 1969, Nadler [19] introduced the notion of multivalued contraction mapping and show that such mapping has a fixed point on complete metric space. Then many fixed point theorems have been proved by various authors as a generalization of the Nadler's theorem (see [4, 6, 9–11, 15, 17, 18]).

Recently, F. Khojasteha et al. [14] introduced the notion of a simulation function with a view to consider a new class of contraction called $\mathcal{Z}$-contraction. They studied the existence and uniqueness of fixed point for $\mathcal{Z}$-contraction type operators. Using the idea of a simulation function, different contractive conditions can be expressed in a simple and unified way. This class of $\mathcal{Z}$-contraction includes a large type of non-linear contraction existing in the literature (see [1–3, 7, 13, 16, 20, 22, 23]).

In this paper we introduce the notion of the generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of

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mappings and establish some common fixed point theorems for such mappings in complete metric spaces.

2. Preliminaries

Let (X, d) be a metric space and $CB(X)$ denote the collection of all non-empty closed and bounded subset of X. For $x \in X$ and $A, B \in CB(X)$, we have

$$D(x, A) = \inf\{d(x, y) : y \in A\}$$

and

$$H(A, B) = \max\big\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\big\}.$$

The function H is a metric on $CB(X)$ and is called a Hausdorff metric induced by the metric d.

Theorem 2.1 ([19])**.** *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a contraction mapping such that*

$$H(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$ and for some $r \in [0, 1)$. Then, there exists $x^* \in X$ such that $x^* \in Tx^*$.

Recently in 2015, Khojasteh et al. [14] introduced the notion of $\mathcal{Z}$-contraction with respect to ζ, which generalizes the Banach contraction principle and unifies several known types of contraction.

Definition 2.1 ([14])**.** Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $s, t > 0$;
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Argoubi et al. [5] slightly modified the definition of simulation function by withdrawing the condition (ζ_1).

Definition 2.2 ([5])**.** A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_2) $\zeta(t, s) < s - t$, for all $s, t > 0$;
(ζ_3') if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions by $\mathcal{Z}$. For examples of simulation function we may refer to [12, 14, 23].

Definition 2.3 ([20])**.** Let (X, d) be a metric space, $F : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then F is called a generalized $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(d(Fu, Fv), M(u, v)) \geq 0 \text{ for all } u, v \in X,$$

where

$$M(u, v) = \max\left\{d(u, v), d(u, Fu), d(v, Fv), \tfrac{d(u,Fv)+d(v,Fu)}{2}\right\}.$$

On the other hand, A. Padcharoen et al. [21] defined the notion of generalized Suzuki type $\mathcal{Z}$-contraction on a metric spaces as follows.

Definition 2.4 ([21])**.** Let (X, d) be a metric space, $F : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then F is called a generalized Suzuki type $\mathcal{Z}$-contraction with respect to ζ if

$$\frac{1}{2}d(u, Fu) < d(u, v) \Rightarrow \zeta(d(Fu, Fv), M(u, v)) \geq 0,$$

for all distinct $u, v \in X$, where

$$M(u, v) = \max\left\{d(u, v), d(u, Fu), d(v, Fv), \tfrac{d(u,Fv)+d(v,Fu)}{2}\right\}.$$

Motivated and inspired by Definition 2.3 and Definition 2.4, we introduce the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings in metric space.

Definition 2.5. Let (X, d) be a metric space and $F, G : X \to CB(X)$. Then the pair (F, G) is said to be a generalized multivalued $\mathcal{Z}$-contraction for pair of mappings with respect to ζ if

$$\zeta(H(Fu, Gv), M(u, v)) \geq 0, \tag{1}$$

for all $u, v \in X$, where

$$M(u, v) = \max\left\{d(u, v), D(u, Fu), D(v, Gv), \tfrac{D(u,Gv)+D(v,Fu)}{2}\right\}.$$

Definition 2.6. Let (X, d) be a metric space and $F, G : X \to CB(X)$. Then the pair (F, G) is said to be a generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings with respect to ζ if

$$\frac{1}{2}\min\{D(u, Fu), D(v, Gv)\} < d(u, v) \Rightarrow \zeta(H(Fu, Gv), M(u, v)) \geq 0, \tag{2}$$

for all $u, v \in X$ with $Fu \neq Gv$, where

$$M(u, v) = \max\left\{d(u, v), D(u, Fu), D(v, Gv), \tfrac{D(u,Gv)+D(v,Fu)}{2}\right\}.$$

3. Main results

Now we state our main results.

Theorem 3.1. *Let (X, d) be a complete metric space and $F, G : X \to CB(X)$ be a pair of generalized multivalued Suzuki type $\mathcal{Z}$-contractions with respect to ζ. Then F and G have a common fixed point.*

Proof. Let u_0 be an arbitrary point in X. Choose $u_1 \in Fu_0$. Then by the definition of Hausdorff metric there exists $u_2 \in Gu_1$ such that

$$0 < d(u_1, u_2) = D(u_1, Gu_1) \leq H(Fu_0, Gu_1). \tag{3}$$

Assume that $D(u_0, Fu_0) > 0$ and $D(u_1, Gu_1) > 0$ then

$$\frac{1}{2}\min\{D(u_0, Fu_0), D(u_1, Gu_1)\} < d(u_0, u_1).$$

Therefore from (2) we have

$$\begin{aligned} 0 &\leq \zeta(H(Fu_0, Gu_1), M(u_0, u_1)) \\ &< M(u_0, u_1) - H(Fu_0, Gu_1). \end{aligned}$$

Consequently, we get

$$d(u_1, u_2) \leq H(Fu_0, Gu_1) < M(u_0, u_1), \tag{4}$$

where

$$\begin{aligned} M(u_0, u_1) &= \max\left\{d(u_0, u_1), D(u_0, Fu_0), D(u_1, Gu_1), \tfrac{D(u_0,Gu_1)+D(u_1,Fu_0)}{2}\right\} \\ &\leq \max\left\{d(u_0, u_1), d(u_0, u_1), d(u_1, u_2), \tfrac{d(u_0,u_2)+d(u_1,u_1)}{2}\right\} \\ &= \max\left\{d(u_0, u_1), d(u_1, u_2), \tfrac{d(u_0,u_2)}{2}\right\}. \end{aligned}$$

Since

$$\frac{d(u_0, u_2)}{2} \leq \frac{d(u_0, u_1) + d(u_1, u_2)}{2} \leq \max\left\{d(u_0, u_1), d(u_1, u_2)\right\},$$

$$M(u_0, u_1) \leq \max\left\{d(u_0, u_1), d(u_1, u_2)\right\}.$$

Suppose that $\max\left\{d(u_0, u_1), d(u_1, u_2)\right\} = d(u_1, u_2)$, then (4) becomes

$$d(u_1, u_2) \leq H(Fu_0, Gu_1) < d(u_1, u_2),$$

which is a contradiction. Thus we conclude that

$$\max\left\{d(u_0, u_1), d(u_1, u_2)\right\} = d(u_0, u_1).$$

By (4) we get

$$d(u_1, u_2) < d(u_0, u_1).$$

Similarly, for $u_2 \in Gu_1$ and $u_3 \in Fu_2$ we have

$$d(u_2, u_3) \leq H(Gu_1, Fu_2) < d(u_1, u_2).$$

This implies

$$d(u_2, u_3) < d(u_1, u_2).$$

By continuing in this manner, we construct a sequence $\{u_n\}$ in X such that $u_{2n+1} \in Fu_{2n}$ and $u_{2n+2} \in Gu_{2n+1}$, $n = 0, 1, 2, \ldots$ such that

$$0 < d(u_{2n+1}, u_{2n+2}) = D(u_{2n+1}, Gu_{2n+1}) \leq H(Fu_{2n}, Gu_{2n+1})$$

and

$$\frac{1}{2}\min\{D(u_{2n}, Fu_{2n}), D(u_{2n+1}, Gu_{2n+1})\} < d(u_{2n}, u_{2n+1}).$$

Hence from (2) we have

$$\begin{aligned} 0 &\leq \zeta(H(Fu_{2n}, Gu_{2n+1}), M(u_{2n}, u_{2n+1})) \\ &< M(u_{2n}, u_{2n+1}) - H(Fu_{2n}, Gu_{2n+1}). \end{aligned}$$

Consequently, we get

$$d(u_{2n+1}, u_{2n+2}) \leq H(Fu_{2n}, Gu_{2n+1}) < M(u_{2n}, u_{2n+1}), \tag{5}$$

where

$$\begin{aligned} M(u_{2n}, u_{2n+1}) =&\ \max\big\{d(u_{2n}, u_{2n+1}), D(u_{2n}, Fu_{2n}), D(u_{2n+1}, Gu_{2n+1}), \\ &\tfrac{D(u_{2n}, Gu_{2n+1}) + D(u_{2n+1}, Fu_{2n})}{2}\big\} \\ \leq&\ \max\big\{d(u_{2n}, u_{2n+1}), d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2}), \\ &\tfrac{d(u_{2n}, u_{2n+2})}{2}\big\} \\ =&\ \max\big\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2}), \tfrac{d(u_{2n}, u_{2n+2})}{2}\big\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{d(u_{2n}, u_{2n+2})}{2} &\leq \frac{[d(u_{2n}, u_{2n+1}) + d(u_{2n+1}, u_{2n+2})]}{2} \\ &\leq \max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\}, \end{aligned}$$

$$M(u_{2n}, u_{2n+1}) \leq \max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\}.$$

Suppose that $\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} = d(u_{2n+1}, u_{2n+2})$, then from (5) we have

$$d(u_{2n+1}, u_{2n+2}) \leq H(Fu_{2n}, Gu_{2n+1}) < d(u_{2n+1}, u_{2n+2})$$

which is a contradiction. So

$$\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\} = d(u_{2n}, u_{2n+1}).$$

Then from (5) we have

$$d(u_{2n+1}, u_{2n+2}) \leq H(Fu_{2n}, Gu_{2n+1}) < d(u_{2n}, u_{2n+1}).$$

This implies that

$$d(u_{2n+1}, u_{2n+2}) < d(u_{2n}, u_{2n+1}). \tag{6}$$

Hence $d(u_{n+1}, u_{n+2}) < d(u_n, u_{n+1})$ for all n.

Therefore $\{d(u_n, u_{n+1})\}$ is a strictly decreasing sequence of non-negative real numbers. Thus there exists $L \geq 0$ such that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = L.$$

We shall prove that $L = 0$. Suppose on the contrary that $L > 0$. Now by using condition (ζ_3) with $t_n = H(Fu_{2n}, Gu_{2n+1})$ and $s_n = d(u_{2n}, u_{2n+1})$, we have

$$0 \leq \limsup_{n\to\infty} \zeta(H(Fu_{2n}, Gu_{2n+1}), d(u_{2n}, u_{2n+1})) < 0$$

which is a contradiction. Thus we conclude that $L = 0$, i.e.,

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{7}$$

Now we prove that $\{u_n\}$ is a Cauchy sequence. Suppose to contrary, that it is not a Cauchy sequence. We assume that there exist $\epsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that

$$n(k) > m(k) > k, \;\; d(u_{n(k)}, u_{m(k)}) \geq \epsilon, \;\; d(u_{n(k)-1}, u_{m(k)}) < \epsilon. \tag{8}$$

Using the triangular inequality, we get

$$\begin{aligned} \epsilon \leq d(u_{n(k)}, u_{m(k)}) &\leq d(u_{n(k)}, u_{n(k)-1}) + d(u_{n(k)-1}, u_{m(k)}) \\ &< d(u_{n(k)}, u_{n(k)-1}) + \epsilon. \end{aligned}$$

Now, by taking the limit as $k \to \infty$ and using (7) we obtain that

$$\lim_{k\to\infty} d(u_{n(k)}, u_{m(k)}) = \epsilon. \tag{9}$$

Using the triangle inequality, we have

$$\epsilon \leq d(u_{n(k)}, u_{m(k)}) \leq d(u_{n(k)}, u_{m(k)+1}) + d(u_{m(k)+1}, u_{m(k)})$$

and

$$d(u_{n(k)}, u_{m(k)+1}) \leq d(u_{n(k)}, u_{m(k)}) + d(u_{m(k)}, u_{m(k)+1}).$$

Again, by taking the limit as $k \to \infty$ and using (7), (8) and (9) we get

$$\lim_{n\to\infty} d(u_{n(k)}, u_{m(k)+1}) = \epsilon. \tag{10}$$

Similarly, we obtain

$$\lim_{n\to\infty} d(u_{n(k)+1}, u_{m(k)}) = \epsilon. \tag{11}$$

Also, we observe that

$$d(u_{n(k)+1}, u_{m(k)+1}) \leq d(u_{n(k)+1}, u_{m(k)}) + d(u_{m(k)}, u_{m(k)+1})$$

and

$$d(u_{n(k)+1}, u_{m(k)}) \leq d(u_{n(k)+1}, u_{m(k)+1}) + d(u_{m(k)+1}, u_{m(k)}).$$

By taking the limit $k \to \infty$ and using (7), (9), (10) and (11) we obtain

$$\lim_{n\to\infty} d(u_{n(k)+1}, u_{m(k)+1}) = \epsilon. \tag{12}$$

From (7) and (8) we can choose a positive integer $n_0 \geq 1$ such that

$$\frac{1}{2} \min\{D(u_{n(k)}, Fu_{n(k)}), D(u_{m(k)}, Gu_{m(k)})\} < \frac{\epsilon}{2} < d(u_{n(k)}, u_{m(k)}).$$

Hence, from (2) we get

$$\begin{aligned} 0 &\le \zeta(H(Fu_{n(k)}, Gu_{m(k)}), M(u_{n(k)}, u_{m(k)})) \\ &< M(u_{n(k)}, u_{m(k)}) - H(Fu_{n(k)}, Gu_{m(k)}). \end{aligned}$$

Consequently, we get

$$d(u_{n(k)+1}, u_{m(k)+1}) \le H(Fu_{n(k)}, Gu_{m(k)}) < M(u_{n(k)}, u_{m(k)}), \tag{13}$$

where

$$\begin{aligned} M(u_{n(k)}, u_{m(k)}) = &\max\big\{d(u_{n(k)}, u_{m(k)}), D(u_{n(k)}, Fu_{n(k)}), D(u_{m(k)}, Gu_{m(k)}), \\ &\tfrac{D(u_{n(k)}, Gu_{m(k)}) + D(u_{m(k)}, Fu_{n(k)})}{2}\big\} \\ \le &\max\big\{d(u_{n(k)}, u_{m(k)}), d(u_{n(k)}, u_{n(k)+1}), d(u_{m(k)}, u_{m(k)+1}), \\ &\frac{d(u_{n(k)}, u_{m(k)+1}) + d(u_{m(k)}, u_{n(k)+1})}{2}\big\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (7), (9), (10) and (11), we obtain

$$\lim_{n\to\infty} M(u_{n(k)}, u_{m(k)}) = \epsilon.$$

By using condition (ζ_3) and (13) with $t_n = H(Fu_{n(k)}, Gu_{m(k)})$ and $s_n = M(u_{n(k)}, u_{m(k)})$ we have

$$0 \le \limsup_{n\to\infty} \zeta\big(H(Fu_{n(k)}, Gu_{m(k)}), M(u_{n(k)}, u_{m(k)})\big) < 0$$

which is a contradiction. Hence $\{u_n\}$ is a Cauchy sequence. Since X is complete we can ensure that $\{u_n\}$ converges to some $u^* \in X$, i.e.,

$$\lim_{n\to\infty} d(u_n, u^*) = 0$$

and so

$$\lim_{n\to\infty} d(u_n, u^*) = \lim_{n\to\infty} d(u_{2n}, u^*) = \lim_{n\to\infty} d(u_{2n+1}, u^*) = 0. \tag{14}$$

Now we claim that

$$\frac{1}{2}\min\{D(u_n, Fu_n), D(u^*, Gu^*)\} < d(u_n, u^*)$$

or

$$\frac{1}{2}\min\{D(u^*, Fu^*), D(u_{n+1}, Gu_{n+1})\} < d(u^*, u_{n+1}) \tag{15}$$

for all $n \in \mathbb{N}$. Suppose that it is not the case. Then there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}\min\{D(u_m, Fu_m), D(u^*, Gu^*)\} \ge d(u_m, u^*) \tag{16}$$

and

$$\frac{1}{2}\min\{D(u^*, Fu^*), D(u_{m+1}, Gu_{m+1})\} \ge d(u^*, u_{m+1}). \tag{17}$$

Therefore

$$\begin{aligned}2d(u_m,u^*) &\le \min\big\{D(u_m,Fu_m),D(u^*,Gu^*)\big\}\\ &\le \min\big\{d(u_m,u^*)+D(u^*,Fu_m),D(u^*,Gu^*)\big\}\\ &\le d(u_m,u^*)+D(u^*,Fu_m)\\ &\le d(u_m,u^*)+d(u^*,u_{m+1}),\end{aligned}$$

which implies that

$$d(u_m,u^*)\le d(u^*,u_{m+1}). \tag{18}$$

From (17) and (18)

$$d(u_m,u^*)\le d(u_{m+1},u^*)\le \frac{1}{2}\min\big\{D(u^*,Fu^*),D(u_{m+1},Gu_{m+1})\big\}. \tag{19}$$

Since $\frac{1}{2}\min\big\{D(u_m,Fu_m),D(u^*,Gu^*)\big\}<d(u_m,u_{m+1})$, from (2) we have

$$\begin{aligned}0&\le \zeta(H(Fu_m,Gu_{m+1}),M(u_m,u_{m+1}))\\ &< M(u_m,u_{m+1})-H(Fu_m,Gu_{m+1}).\end{aligned}$$

Consequently, we get

$$d(u_{m+1},u_{m+2})\le H(Fu_m,Gu_{m+1})<M(u_m,u_{m+1}), \tag{20}$$

where

$$\begin{aligned}M(u_m,u_{m+1}) =&\ \max\big\{d(u_m,u_{m+1}),D(u_m,Fu_m),D(u_{m+1},Gu_{m+1}),\\ &\tfrac{D(u_m,Gu_{m+1})+D(u_{m+1},Fu_m)}{2}\big\}\\ \le&\ \max\big\{d(u_m,u_{m+1}),d(u_m,u_{m+1}),d(u_{m+1},u_{m+2}),\\ &\tfrac{d(u_m,u_{m+2})+d(u_{m+1},u_{m+1})}{2}\big\}\\ =&\ \max\big\{d(u_m,u_{m+1}),d(u_{m+1},u_{m+2}),\tfrac{d(u_m,u_{m+2})}{2}\big\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{d(u_m,u_{m+2})}{2}&\le \frac{d(u_m,u_{m+1})+d(u_{m+1},u_{m+2})}{2}\\ &\le \max\big\{d(u_m,u_{m+1}),d(u_{m+1},u_{m+2})\big\},\end{aligned}$$

$$M(u_m,u_{m+1})\le \max\big\{d(u_m,u_{m+1}),d(u_{m+1},u_{m+2})\big\}.$$

Suppose that $\max\{d(u_m,u_{m+1}),d(u_{m+1},u_{m+2})\}=d(u_{m+1},u_{m+2})$, then from (20) we have

$$d(u_{m+1},u_{m+2})\le H(Fu_m,Gu_{m+1})<d(u_{m+1},u_{m+2}),$$

which is a contradiction. Thus we conclude that

$$\max\{d(u_m,u_{m+1}),d(u_{m+1},u_{m+2})\}=d(u_m,u_{m+1}).$$

By (20) we get that

$$d(u_{m+1},u_{m+2})<d(u_m,u_{m+1}). \tag{21}$$

From (17), (19) and (21), we get

$$\begin{aligned} d(u_{m+1}, u_{m+2}) &< d(u_m, u_{m+1}) \\ &\leq d(u_m, u^*) + d(u^*, u_{m+1}) \\ &\leq \frac{1}{2} \min\left\{D(u^*, Fu^*), D(u_{m+1}, Gu_{m+1})\right\} \\ &\quad + \frac{1}{2} \min\left\{D(u^*, Fu^*), D(u_{m+1}, Gu_{m+1})\right\} \\ &= \min\left\{D(u^*, Fu^*), D(u_{m+1}, Gu_{m+1})\right\} \\ &\leq d(u_{m+1}, u_{m+2}), \end{aligned}$$

which is a contradiction. Hence (15) holds, i.e., for every $n \geq 2$

$$\frac{1}{2} \min\left\{D(u_n, Fu_n), D(u^*, Gu^*)\right\} < d(u_n, u^*)$$

holds. Hence from (2), it follows that for every $n \geq 2$

$$\begin{aligned} 0 &\leq \zeta(H(Fu_n, Gu^*), M(u_n, u^*)) \\ &< M(u_n, u^*) - H(Fu_n, Gu^*). \end{aligned} \tag{22}$$

Consequently, we get

$$D(u_{n+1}, Gu^*) \leq H(Fu_n, Gu^*) < M(u_n, u^*), \tag{23}$$

where

$$\begin{aligned} M(u_n, u^*) &= \max\left\{d(u_n, u^*), D(u_n, Fu_n), D(u^*, Gu^*), \tfrac{D(u_n, Gu^*) + D(u^*, Fu_n)}{2}\right\} \\ &\leq \max\left\{d(u_n, u^*), d(u_n, u_{n+1}), D(u^*, Gu^*), \tfrac{D(u_n, Gu^*) + d(u^*, u_{n+1})}{2}\right\}. \end{aligned}$$

Letting $n \to \infty$ and by using (7) and (14), we obtain

$$\lim_{n\to\infty} M(u_n, u^*) = D(u^*, Gu^*). \tag{24}$$

Now we prove that $u^* \in Gu^*$. Suppose on the contrary that $D(u^*, Gu^*) > 0$. Letting $n \to \infty$ in (23) we obtain

$$\begin{aligned} D(u^*, Gu^*) &= \lim_{n\to\infty} D(u_{n+1}, Gu^*) \\ &\leq \lim_{n\to\infty} H(Fu_n, Gu^*) \\ &< \lim_{n\to\infty} M(u_n, u^*) = D(u^*, Gu^*) \end{aligned}$$

which is a contradiction. Therefore $u^* \in Gu^*$.

Similarly, we can show that $u^* \in Fu^*$. Thus F and G have a common fixed point. $\square$

Example 3.2. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(0,1) = 1, d(0,2) = 2, d(1,2) = 3 \text{ and } d(0,0) = d(1,1) = d(2,2) = 0.$$

Then (X, d) is a complete metric space.

Define the mappings $F, G : X \times X \to CB(X)$ by

$$Fu = \begin{cases} \{0\} & u \in \{0,1\}, \\ \{0,1\} & u = 2 \end{cases} \quad \text{and } Gu = \begin{cases} \{0\} & u \in \{0,1\}, \\ \{1\} & u = 2. \end{cases}$$

Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\zeta(t,s) = \frac{9}{10}s - t$ for all $s,t \in [0,\infty)$. Now we verify inequality (2) for all $u,v \in X$ with $Fu \neq Gu$. Note that for all $u,v \in \{0,1,2\}$ with $Fu \neq Gu$ the inequality $\frac{1}{2}\min\{D(u,Fu), D(v,Gv)\} < d(u,v)$ gives

$$(u,v) \in \big\{(0,2),(2,0),(1,2),(2,1)\big\}.$$

Then from (2), we have

$$\zeta\big(H(Fu,Gv), M(u,v)\big) = \frac{9}{10}M(u,v) - H(Fu,Gv) \geq 0.$$

This implies that

$$H(Fu,Gv) \leq \frac{9}{10}M(u,v).$$

Case (i) for $u = 0,\ v = 2$;

$$H(F0,G2) = H(\{0\},\{1\}) = 1 \leq \frac{9}{10}M(0,2).$$

Case (ii) for $u = 2,\ v = 0$;

$$H(F2,G0) = H(\{0,1\},\{0\}) = 1 \leq \frac{9}{10}M(2,0).$$

Case (iii) for $u = 1,\ v = 2$;

$$H(F1,G2) = H(\{0\},\{1\}) = 1 \leq \frac{9}{10}M(1,2).$$

Case (iv) for $u = 2,\ v = 1$;

$$H(F2,G1) = H(\{0,1\},\{0\}) = 1 \leq \frac{9}{10}M(2,1).$$

Thus all the hypothesis of Theorem 3.1 are satisfied. Hence 0 is a common fixed point of F and G.

Example 3.3. Let $X = \{0,2,4\}$ be endowed with the usual metric. Let $F, G : X \to CB(X)$ be defined by

$$Fu = \begin{cases} \{\frac{u}{6}\} & \text{if } u \in \{0,4\}, \\ \{0, \frac{1}{6}\} & \text{if } u = 2, \end{cases} \quad \text{and} \quad Gu = \{\tfrac{u}{4}\} \text{ for all } u \in X.$$

We now define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(t,s) = \frac{5}{6}s - t$ for all $s,t \in [0,\infty)$. Now we verify inequality (2) for all $u,v \in X$ with $Fu \neq Gv$. Note that for all $u,v \in X$ with $Fu \neq Gv$ the inequality $\frac{1}{2}\min\{D(u,Fu), D(v,Gv)\} < d(u,v)$ gives

$$(u,v) \in \big\{(0,2),(2,0),(0,4),(4,0),(2,4),(4,2)\big\}.$$

Then from (2), we have

$$\zeta\big(H(Fu,Gv),M(u,v)\big)=\frac{5}{6}M(u,v)-H(Fu,Gv)\geq 0.$$

This implies that

$$H(Fu,Gv)\leq\frac{5}{6}M(u,v).$$

Case (i) for $u=0,\ v=2$;

$$H(F0,G2)=H(\{0\},\{\frac{1}{2}\})=\frac{1}{2}\leq\frac{5}{6}M(0,2).$$

Case (ii) for $u=2,\ v=0$;

$$H(F2,G0)=H(\{0,\frac{1}{6}\},\{0\})=\frac{1}{6}\leq\frac{5}{6}M(2,0).$$

Case (iii) for $u=0,\ v=4$;

$$H(F0,G4)=H(\{0\},\{1\})=1\leq\frac{5}{6}M(0,4).$$

Case (iv) for $u=4,\ v=0$;

$$H(F4,G0)=H(\{\frac{2}{3}\},\{0\})=\frac{2}{3}\leq\frac{5}{6}M(4,0).$$

Case (v) for $u=2,\ v=4$;

$$H(F2,G4)=H(\{0,\frac{1}{6}\},\{1\})=1\leq\frac{5}{6}M(2,4).$$

Case (vi) for $u=4,\ v=2$;

$$H(F4,G2)=H(\{\frac{2}{3}\},\{\frac{1}{2}\})=\frac{1}{6}\leq\frac{5}{6}M(4,2).$$

Thus all the hypothesis of Theorem 3.1 are satisfied. Hence 0 is a common fixed point of F and G.

Corollary 3.4. *Let (X,d) be a complete metric space and $F,G:X\to CB(X)$ be a pair of generalized multivalued $\mathcal{Z}$-contractions with respect to ζ. Then F and G have a common fixed point $u^*\in X$ and for $u\in X$ the sequence $\{F^nu\}$ converges to u^*.*

Corollary 3.5. *Let (X,d) be a complete metric space and $F:X\to CB(X)$ be a generalized multivalued Suzuki type $\mathcal{Z}$-contraction with respect to ζ, i.e.,*

$$\frac{1}{2}D(u,Fu)<d(u,v)\Rightarrow\zeta(H(Fu,Fv),M(u,v))\geq 0, \tag{25}$$

for all $u,v\in X$ with $u\neq v$, where

$$M(u,v)=\max\big\{d(u,v),D(u,Fu),D(v,Fv),\tfrac{D(u,Fv)+D(v,Fu)}{2}\big\}.$$

Then F has a fixed point $u^*\in X$ and for $u\in X$ the sequence $\{F^nu\}$ converges to u^*.

Proof. The proof follows from Theorem 3.1 by taking $F = G$. $\square$

Example 3.6. Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be defined by $d(0, 1) = 1$, $d(0, 2) = 2$, $d(1, 2) = 2$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then (X, d) is complete metric space. Define the mapping $F : X \to CB(X)$ by

$$F0 = F1 = \{0\}, \quad F2 = \{0, 1\}.$$

Note that for all distinct $u, v \in X$ the inequality $\frac{1}{2}D(u, Fu) < d(u, v)$. Hence from (25), we shall show that

$$\zeta(H(Fu, Fv), M(u, v)) \geq 0.$$

We now define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta(t, s) = \frac{7}{8}s - t$ for all $s, t \in [0, \infty)$. Thus we have

$$\zeta(H(Fu, Fv), M(u, v)) = \frac{7}{8}M(u, v) - H(Fu, Fv) \geq 0.$$

So, this implies that

$$H(Fu, Fv) \leq \frac{7}{8}M(u, v).$$

For this, we consider the following cases:

Case 1: $u, v \in \{0, 1\}$, we have

$$H(Fu, Fv) = d(0, 0) = 0 \leq \frac{7}{8}M(u, v).$$

Case 2: $u \in \{0, 1\}$, $v = 2$, we have

$$H(Fu, Fv) = H(\{0\}, \{0, 1\}) = \max\{0, 1\} = 1 \leq \frac{7}{8}d(u, v) = 1.75 \leq \frac{7}{8}M(u, v).$$

Thus all the hypothesis of Corollary 3.5 are satisfied. Here, $u = 0$ is the unique fixed point of F.

Corollary 3.7. *Let (X, d) be a complete metric space and $F, G : X \to X$ be a pair of generalized single valued Suzuki type $\mathcal{Z}$-contractions with respect to ζ, i.e.,*

$$\frac{1}{2}\min\big\{d(u, Fu), d(v, Gv)\big\} < d(u, v) \Rightarrow \zeta(d(Fu, Gv), M(u, v)) \geq 0, \tag{26}$$

for all $u, v \in X$ with $Fu \neq Gv$, where

$$M(u, v) = \max\big\{d(u, v), d(u, Fu), d(v, Gv), \tfrac{d(u, Gv) + d(v, Fu)}{2}\big\}.$$

Then F and G have a unique common fixed point $u^* \in X$ and for $u \in X$ the sequence $\{F^n u\}$ converges to u^*.

Proof. It can be proved easily by taking F and G as single valued mappings in Theorem 3.1. Uniqueness of the common fixed point is obvious. $\square$

Remark 3.8. Corollary 3.7 is a generalization of Theorem 2.4 [21].

Corollary 3.9. *Let (X, d) be a complete metric space and $F, G : X \to X$ be a pair of generalized single valued $\mathcal{Z}$-contractions with respect to ζ, i.e.,*

$$\zeta(d(Fu, Gv), M(u, v)) \geq 0, \tag{27}$$

for all $u, v \in X$, where

$$M(u, v) = \max\left\{d(u, v), d(u, Fu), d(v, Gv), \tfrac{d(u,Gv)+d(v,Fu)}{2}\right\}.$$

Then F and G have a unique common fixed point $u^* \in X$ and for $u \in X$ the sequence $\{F^n u\}$ converges to u^*.

Remark 3.10. Corollary 3.9 is a generalization of Theorem 2 [20].

The following example shows that Corollary 3.7 is a generalization of Corollary 3.9.

Example 3.11. Let $X = \big\{(0, 0), (0, 3), (3, 0), (0, 4), (4, 0), (3, 4), (4, 3)\big\}$ and define metric d on X by

$$d[(u_1, u_2), (v_1, v_2)] = |u_1 - v_1| + |u_2 - v_2|.$$

Let $F, G : X \to X$ be such that

$$Fu = \begin{cases} (u_1, 0), & u_1 \leq u_2, \\ (0, u_2), & u_1 > u_2 \end{cases} \quad \text{and } Gu = \begin{cases} (0, u_1), & u_1 \leq u_2, \\ (0, u_2), & u_1 > u_2. \end{cases}$$

We now define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta(t, s) = \frac{6}{7}s - t$ for all $s, t \in [0, \infty)$. Then F and G do not satisfy the condition (27) of Corollary 3.9 at $u = (3, 4)$ and $v = (4, 3)$. However, this is readily verified that all the hypotheses of Corollary 3.7 are satisfied for the maps F and G.

Conclusion. In this article, we introduced the notion of generalized multivalued $\mathcal{Z}$-contraction and generalized multivalued Suzuki type $\mathcal{Z}$-contraction for pair of mappings and establish common fixed point theorems for such mappings in complete metric spaces. Our theorems and corollaries are sharpened version of well known results. Our results extend and generalize some well known fixed point result exists in the literature.

References

[1] A. S. S. Alharbi, H. H. Alsulami, and E. Karapinar, *On the power of simulation and admissible functions in metric fixed point theory*, J. Funct. Spaces **2017** (2017), Art. ID 2068163, 7 pp. `https://doi.org/10.1155/2017/2068163`

[2] B. Alqahtani, A. Fulga, and E. Karapinar, *Fixed point results on Δ-symmetric quasi-metric space via simulation function with an application to Ulam stability*. Math. **6** (2018), Article No. 208. `https://doi.org/10.3390/math6100208`

[3] O. Alqahtani and E. Karapınar, *A bilateral contraction via simulation function*, Filomat **33** (2019), no. 15, 4837–4843. `https://doi.org/10.2298/FIL1915837A`

[4] E. Ameer, M. Arshad, D. Shin, and S. Yun, *Common fixed point theorems of generalized multivalued (ψ, ϕ)-contractions in complete metric spaces with application*. Math. **7** (2019), Article No. 194. `https://doi.org/10.3390/math7020194`

[5] H. Argoubi, B. Samet, and C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. **8** (2015), no. 6, 1082–1094. https://doi.org/10.22436/jnsa.008.06.18

[6] H. Aydi, M. Abbas, and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl. **159** (2012), no. 14, 3234–3242. https://doi.org/10.1016/j.topol.2012.06.012

[7] H. Aydi, E. Karapinar, and V. Rakočcević, *Nonunique fixed point theorems on b-metric spaces via simulation functions*, Jordan J. Math. Stat. **12** (2019), no. 3, 265–288.

[8] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181

[9] M. Berinde and V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl. **326** (2007), no. 2, 772–782. https://doi.org/10.1016/j.jmaa.2006.03.016

[10] L. Ćirić, *Multi-valued nonlinear contraction mappings*, Nonlinear Anal. **71** (2009), no. 7-8, 2716–2723. https://doi.org/10.1016/j.na.2009.01.116

[11] H. Covitz and S. B. Nadler, Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11. https://doi.org/10.1007/BF02771543

[12] E. Karapınar, *Fixed points results via simulation functions*, Filomat **30** (2016), no. 8, 2343–2350. https://doi.org/10.2298/FIL1608343K

[13] E. Karapınar and R. P. Agarwal, *Interpolative Rus-Reich-Ćirić type contractions via simulation functions*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **27** (2019), no. 3, 137–152. https://doi.org/10.2478/auom-2019-0038

[14] F. Khojasteh, S. Shukla, and S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat **29** (2015), no. 6, 1189–1194. https://doi.org/10.2298/FIL1506189K

[15] D. Klim and D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl. **334** (2007), no. 1, 132–139. https://doi.org/10.1016/j.jmaa.2006.12.012

[16] X.-L. Liu, A. H. Ansari, S. Chandok, and S. Radenović, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl. **24** (2018), no. 6, 1103–1114.

[17] J. T. Markin, *A fixed point theorem for set valued mappings*, Bull. Amer. Math. Soc. **74** (1968), 639–640. https://doi.org/10.1090/S0002-9904-1968-11971-8

[18] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141** (1989), no. 1, 177–188. https://doi.org/10.1016/0022-247X(89)90214-X

[19] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488. http://projecteuclid.org/euclid.pjm/1102978504

[20] M. Olgun, Ö. Biçer, and T. Alyıldız, *A new aspect to Picard operators with simulation functions*, Turkish J. Math. **40** (2016), no. 4, 832–837. https://doi.org/10.3906/mat-1505-26

[21] A. Padcharoen, P. Kumam, P. Saipara, and P. Chaipunya, *Generalized Suzuki type $\mathcal{Z}$-contraction in complete metric spaces*, Kragujevac J. Math. **42** (2018), no. 3, 419–430. https://doi.org/10.5937/kgjmath1803419p

[22] S. Radenović and S. Chandok, *Simulation type functions and coincidence points*, Filomat **32** (2018), no. 1, 141–147. https://doi.org/10.2298/fil1801141r

[23] A.-F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345–355. https://doi.org/10.1016/j.cam.2014.07.011

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