Commun. Korean Math. Soc. **35** (2020), No. 4, pp. 1095–1106
https://doi.org/10.4134/CKMS.c200116
pISSN: 1225-1763 / eISSN: 2234-3024

ON ϕ-PSEUDO-KRULL RINGS

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Abstract. The purpose of this paper is to introduce a new class of rings that is closely related to the class of pseudo-Krull domains. Let $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. Let $R \in \mathcal{H}$ be a ring with total quotient ring $T(R)$ and define $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ by $\phi(\frac{a}{b}) = \frac{a}{b}$ for any $a \in R$ and any regular element b of R. Then ϕ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. We say that R is a ϕ-pseudo-Krull ring if $\phi(R) = \bigcap R_i$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$ and for every non-nilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. We show that the theories of ϕ-pseudo Krull rings resemble those of pseudo-Krull domains.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. If R is a ring, we denote by $\mathrm{Nil}(R)$ and $\mathrm{J}(R)$ the ideal of all nilpotent elements of R and the Jacobson radical of R respectively. In [22], Hedstrom and Houston introduced the class of pseudo-valuation domains, which is closely related to the class of valuation domains. A domain R with quotient field K is called a *pseudo-valuation domain* (in short, *PVD*) if, whenever P is a prime ideal of R and $xy \in P$ with $x \in K$ and $y \in K$, then either $x \in P$ or $y \in P$. Any valuation domain is a PVD [22, Proposition 1.1]; any PVD is quasilocal, in the sense that it has a unique maximal ideal [22, Corollary 1.3]. Additional information about PVDs can be found in the interesting survey article [9]. In [11], D. F. Anderson, Badawi and Dobbs generalized the study of PVDs to the context of arbitrary rings (possibly with nontrivial zero-divisors). Recall from [5, 19] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R, equivalently if $P \subseteq (x)$ for any $x \in R \setminus P$. A ring R is called a *divided ring* if every prime ideal of R is divided. Recently A. Badawi, in [4, 6–8, 10], has studied the following class of rings: $\mathcal{H} = \{R \mid R$ is a

Received April 2, 2020; Accepted July 2, 2020.
2010 *Mathematics Subject Classification.* Primary 13F05; Secondary 13A15, 13G05, 13B21.
Key words and phrases. Amalgamated algebra, nonnil-Noetherian ring, pseudo-Krull domain, pseudo-valuation ring, ϕ-pseudo-Krull ring, trivial ring extension.

commutative ring and Nil(R) is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a *ϕ-ring*. It is easy to see that every integral domain is a ϕ-ring. An ideal I of R is said to be a *nonnil ideal* if $I \nsubseteq$ Nil(R). If I is a nonnil ideal of a ϕ-ring R, then Nil$(R) \subseteq I$. Let R be a ring with the total quotient ring $T(R)$ such that Nil(R) is a divided prime ideal of R. As in [6], we define $\phi : T(R) \longrightarrow K := R_{\text{Nil}(R)}$ by $\phi(\frac{a}{b}) = \frac{a}{b}$ for every $a \in R$ and every regular element b of R. Then ϕ is a ring homomorphism from $T(R)$ into K and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = \frac{x}{1}$ for every $x \in R$. Denote by $Z(R)$ the set of all zerodivisors of R. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subseteq Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. In [6], the author gave another generalization of pseudo valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). A prime ideal Q of $\phi(R)$ is said to be *K-strongly prime* if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be *ϕ-strongly prime* if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring* (in short *ϕ-PVR*). It is shown in [6, Corollary 7(2)] that a ring R is a ϕ-PVR if and only if Nil(R) is a divided prime ideal and for every $a, b \in R \setminus$ Nil(R), either $a \mid b$ or $b \mid ac$ in R for each nonunit element $c \in R$. A ring R is called a *ϕ-chained ring* if Nil(R) is divided prime and for each $x \in R_{\text{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$ [8]. Then it is easy to see that R is a ϕ-chained ring if and only if Nil(R) is divided prime and for every $a, b \in R \setminus$ Nil(R) either $a \mid b$ or $b \mid a$. Also, recall from [10] that a ring $R \in \mathcal{H}$ is called a *nonnil-Noetherian ring* if every nonnil ideal of R is finitely generated. It was shown in [10] that a ring $R \in \mathcal{H}$ is a nonnil-Noetherian ring if and only if $R/Nil(R)$ is a Noetherian domain. Recall that a ring R is called a *t-closed ring* if for every element $(a, r, c) \in R^3$ such that $a^3 + arc - c^2 = 0$, there exists an element $b \in R$ such that $a = b^2 - rb$ and $c = b^3 - rb^2$. Also, from [27] an integral domain R is called an *infra-Krull domain* if $R = \bigcap\{R_P \mid P$ is a height-one prime ideal of $R\}$, where this representation is of finite character and each R_P is a Noetherian ring for every height-one prime ideal P of R.

A ring S is called an *overring* of a ring R if $R \subseteq S \subseteq T(R)$. Recall from [28] that an integral domain R is called a *pseudo-Krull domain* if there exists a family $\mathcal{F} = \{R_i\}$ of overrings of R such that $R = \bigcap R_i$, where each R_i is a Noetherian pseudo-valuation overring of R and every nonzero element of R is not a unit in only a finite number of members of $\mathcal{F}$. Many characterizations and properties of pseudo-Krull domains are given in [25], [28], and [29]. In this article, we are interested in extending the pseudo-Krull property to ϕ-rings. Let $R \in \mathcal{H}$. We say that R is a *ϕ-pseudo-Krull ring* if $\phi(R) = \bigcap R_i$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$ and for every non-nilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. Among many results in this paper, we introduce the notions of ϕ-t-closed rings and

ϕ-infra-Krull rings and we show that a ϕ-ring R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-infra-Krull ring. We also use the trivial ring extension and the amalgamation of rings to construct examples of ϕ-pseudo-Krull rings which are not integral domains.

2. Results

We start this section by one of the main results on ϕ-pseudo Krull rings. Before we proceed, we recall the following lemma from [1].

Lemma 2.1 ([1, Lemma 2.5]). *Let $R \in \mathcal{H}$ and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$.*

Lemma 2.2. *Let $R \in \mathcal{H}$. Then R is a nonnil-Noetherian ϕ-pseudo valuation ring if and only if $R/Nil(R)$ is a Noetherian pseudo-valuation domain.*

Proof. This follows from [4, Proposition 2.9] and [10, Theorem 1.2]. □

Theorem 2.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo Krull ring if and only if $R/Nil(R)$ is a pseudo-Krull domain.*

Proof. Assume that R is a ϕ-pseudo Krull ring. Then there exists a family $\mathcal{F} = \{R_i\}$ of overring of $\phi(R)$ such that $\phi(R) = \bigcap R_i$ where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$ and for every non-nilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many R_i. Since $T(\phi(R)/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R))$, it follows from Lemma 2.2 that each $R_i/Nil(\phi(R))$ is a Noetherian pseudo-valuation overring of $\phi(R)/Nil(\phi(R))$. Then $\phi(R)/Nil(\phi(R)) = \bigcap R_i/Nil(\phi(R))$ and every nonzero element of $\phi(R)/Nil(\phi(R))$ is a unit in all but finitely many $R_i/Nil(\phi(R))$. Thus $\phi(R)/Nil(\phi(R))$ is a pseudo-Krull domain. As $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.1, we get that $R/Nil(R)$ is a pseudo-Krull domain.

Conversely, assume that $R/Nil(R)$ is a pseudo-Krull domain. As $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.1, we get $\phi(R)/Nil(\phi(R))$ is a pseudo-Krull domain. By Lemma 2.2 and the fact that

$$T(\phi(R)/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R)),$$

we conclude that $\phi(R)/Nil(\phi(R)) = \bigcap R_i/Nil(\phi(R))$, where each R_i is a nonnil-Noetherian ϕ-pseudo valuation overring of $\phi(R)$. Thus $\phi(R) = \bigcap R_i$. Since for every non-nilpotent element $x \in R$, $\phi(x) + Nil(\phi(R))$ is a unit in all but finitely many $R_i/Nil(\phi(R))$, we get that $\phi(x)$ is a unit in all but finitely many R_i. Therefore R is a ϕ-pseudo Krull ring. □

For a ring $R \in \mathcal{H}$, let ϕ_R denote the ring homomorphism $\phi : T(R) \longrightarrow R_{Nil(R)}$. We have the following result which is an analog of [28, Theorem 1.6]

Theorem 2.4. *Let $R \in \mathcal{H}$ be a ϕ-pseudo Krull ring such that $dim(R) \geq 1$. Then we have*:

(1) *R_P is a nonnil-Noetherian ϕ-pseudo valuation ring for every height-one prime ideal P of R.*
(2) *$\phi_R(R) = \bigcap \phi_{R_P}(R_P)$ where P runs all height-one prime ideals of R.*
(3) *Any nonzero proper nonnil principal ideal of R may be expressed as a finite intersection of (nonnil) primary ideals of height-one.*

Proof. Assume that R is a ϕ-pseudo Krull ring. Set $D := R/Nil(R)$ and let P be a height-one prime ideal of R.

(1) It follows from Theorem 2.3 that D is a pseudo-Krull domain, which implies that $D_{P/Nil(R)}$ is a Noetherian pseudo-valuation domain by [28, Theorem 1.6]. As $D_{P/Nil(R)}$ is ring-isomorphic to $R_P/Nil(R_P)$, we get that R_P is a nonnil-Noetherian ϕ-pseudo valuation ring by Lemma 2.2.

(2) First observe that $Nil(\phi_{R_P}(R_P)) = Nil(\phi_R(R))$. Since $R_P/Nil(R_P)$ is ring-isomorphic to $\phi_{R_P}(R_P)/Nil(\phi_{R_P}(R_P))$, Lemma 2.2 implies that $\phi_{R_P}(R_P)$ is a nonnil-Noetherian ϕ-pseudo valuation ring. Since D is a pseudo-Krull domain by Theorem 2.3 and D is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.1 and [2, Lemma 3.8], we conclude that $\phi(R)/Nil(\phi(R))$ is a pseudo-Krull domain. Thus

$$\begin{aligned}\phi(R)/Nil(\phi(R)) &= \bigcap \phi(R)_{\phi_R(P)}/Nil(\phi(R) \\ &= \bigcap \phi_{R_P}(R_P)/Nil(\phi_R(R))\end{aligned}$$

for each height-one prime ideal P of R. Hence one can easily see that $\phi(R) = \bigcap \phi_{R_P}(R_P)$ where P runs all height-one prime ideals of R.

(3) Since D is a pseudo Krull domain by Theorem 2.3, for any nonzero proper nonnil principal ideal I of R, $I/Nil(R)$ may be expressed as a finite intersection of primary ideals of height-one by [28, Theorem 1.6]. Therefore I may be expressed as a finite intersection of (nonnil) primary ideals of height-one. □

Let $R \in \mathcal{H}$. We say that R satisfies *property* $(*)$ if each non-nilpotent element of R is contained in only finitely many prime ideals of height-one. Recall from [28, Proposition 1.15] that an integral domain R is a pseudo Krull domain if and only if R_P is a pseudo-Kull domain for every prime ideal P of R and R satisfies property $(*)$, if and only if R_M is a pseudo-Krull domain for every maximal ideal M of R and R satisfies property $(*)$. The following is the analogous characterization of ϕ-pseudo-Krull rings.

Proposition 2.5. *Let $R \in \mathcal{H}$. Then the following statements are equivalent.*

(1) *R is a ϕ-pseudo-Krull ring.*
(2) *R_P is a ϕ-pseudo-Kull ring for every prime ideal P of R and R satisfies property $(*)$.*
(3) *R_M is a ϕ-pseudo-Krull ring for every maximal ideal M of R and R satisfies property $(*)$.*

Proof. Set $D := R/Nil(R)$.

(1) $\Rightarrow$ (2) Since D is a pseudo-Krull domain by Theorem 2.3, we conclude that $D_{P/Nil(R)}$ is a pseudo-Kull domain for every prime ideal P of R by [28, Proposition 1.15]. As $D_{P/Nil(R)}$ is ring-isomorphic to $R_P/Nil(R)R_P = R_P/Nil(R_P)$ and $R_P \in \mathcal{H}$, we get that R_P is a ϕ-pseudo-Kull ring by Theorem 2.3. Also, since D is a pseudo-Krull domain, [28, Proposition 1.15] implies that each nonzero element of R is contained in only finitely many prime ideal of height-one. Thus each non-nilpotent element of R is contained in only finitely many prime ideals of height-one.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) As $R_M \in \mathcal{H}$ for each maximal ideal M of R, we conclude that $R_M/Nil(R_M)$ is a pseudo-Krull domain for each maximal ideal M of R by Theorem 2.3 and D satisfies property $(*)$. Hence $D_{M/Nil(R)}$ is a pseudo-Krull domain for each maximal ideal M of R. Thus $R/Nil(R)$ is a pseudo-Krull domain by [28, Proposition 1.15]. Therefore R is a ϕ-pseudo-Krull ring by Theorem 2.3. ☐

We give the following definitions of ϕ-t-closed rings and ϕ-infra Krull rings.

Definition. Let $R \in \mathcal{H}$. Then:

(1) We say that R is a ϕ-infra-Krull ring if $\phi_R(R) = \bigcap\{\phi_{R_P}(R_P) \mid P$ is a height-one prime ideal of $R\}$, where this representation is of finite character and each R_P is a nonnil-Noetherian ring for every height-one prime ideal P of R.

(2) We say that R is a ϕ-t-closed ring if for every non-nilpotent elements a, c of R and each $r \in R$ such that $a^3 + arc - c^2 = 0$, there exist an element $b \in R$ and $w_1, w_2 \in Nil(R)$ such that $a = b^2 - (r - w_1)b$ and $c = b^3 - (r - w_2)b^2$.

We need the next lemma.

Lemma 2.6. *Let $R \in \mathcal{H}$. Then*

(1) *R is a ϕ-infra-Krull ring if and only if $R/Nil(R)$ is an infra-Krull domain.*

(2) *R is a ϕ-t-closed ring if and only if $R/Nil(R)$ is a t-closed domain.*

Proof. (1) Suppose that R is a ϕ-infra-Krull ring. Then $\phi_R(R) = \bigcap \phi_{R_P}(R_P)$, where this representation is of finite character and for each height-one prime ideal P of R, R_P is a nonnil-Noetherian ring. Since

$$Nil(\phi_{R_P}(R_P)) = Nil(\phi_R(R)),$$

each $\phi_{R_P}(R_P)/Nil(\phi_R(R))$ is a Noetherian domain by [10, Theorem 1.4]. Then $\phi_R(R)/Nil(\phi_R(R)) = \bigcap \phi_{R_P}(R_P)/Nil(\phi_R(R))$, where P runs through the height-one prime ideals of R and this representation is of finite character. It follows that $\phi_R(R)/Nil(\phi_R(R))$ is an infra-Krull domain. As $\phi_R(R)/Nil(\phi_R(R))$ is ring-isomorphic to $R/Nil(R)$ by Lemma 2.1, we get that $R/Nil(R)$ is an infra-Krull domain.

Conversely, assume that $R/Nil(R)$ is an infra-Krull domain. Since $R/Nil(R)$ is ring-isomorphic to $\phi_R(R)/Nil(\phi_R(R))$ by Lemma 2.1, $\phi_R(R)/Nil(\phi_R(R))$ is an infra-Krull domain. By [10, Theorem 1.4] and [2, Lemma 3.8], we conclude that $\phi_R(R)/Nil(\phi_R(R)) = \bigcap \phi_{R_P}(R_P)/Nil(\phi(R))$, where P runs through the height-one prime ideals of R and each R_P is a nonnil-Noetherian ring. Thus $\phi_R(R) = \bigcap \phi_{R_P}(R_P)$, where this representation is of finite character. Therefore R is a ϕ-infra Krull ring, as desired.

(2) Set $D := R/Nil(R)$. Assume that R is a ϕ-t-closed ring and let $(\bar{a}, \bar{r}, \bar{c}) \in D^3$ such that $\bar{a}^3 + \bar{a}\bar{r}\bar{c} - \bar{c}^2 = 0$. We may assume that $\bar{a} \neq 0$ and $\bar{c} \neq 0$, and so a and c are non-nilpotent elements of R. Since $a^3 + arc - c^2 \in Nil(R)$ and $Nil(R)$ is a divided prime ideal of R, $a^3 + (r - z)ac - c^2 = 0$ for some $z \in Nil(R)$. As R is a ϕ-t-closed ring, there exist an element $b \in R$ and $w_1, w_2 \in Nil(R)$ such that $a = b^2 - (r - z - w_1)b$ and $c = b^3 - (r - z - w_2)b^2$. That implies $\bar{a} = \bar{b}^2 - \bar{r}\bar{b}$ and $\bar{c} = \bar{b}^3 - \bar{r}\bar{b}^2$. Hence D is a t-closed domain.

Conversely, assume that D is a t-closed ring and let $(a, r, c) \in R^3$ such that a and c are non-nilpotents and $a^3 + arc - c^2 = 0$. Since D is a t-closed domain and $\bar{a}^3 + \bar{a}\bar{r}\bar{c} - \bar{c}^2 = 0$, there exists an element $(0 \neq)\bar{b} \in D$ such that $\bar{a} = \bar{b}^2 - \bar{r}\bar{b}$ and $\bar{c} = \bar{b}^3 - \bar{r}\bar{b}^2$. Hence $a - b^2 + rb \in Nil(R)$ and $c - b^3 + rb^2 \in Nil(R)$. That implies $a - b^2 + rb = w_1 b$ and $c - b^3 + rb^2 = w_2 b^2$ for some $w_1, w_2 \in Nil(R)$ since $Nil(R)$ is divided and $b \notin Nil(R)$. Thus $a = b^2 - (r - w_1)b$ and $c = b^3 - (r - w_2)b^2$. Therefore R is a ϕ-t-closed ring. □

It is well-known [29, Proposition 3.6] that an integral domain R is a pseudo-Krull domain if and only if R is a t-closed infra-Krull domain. We have a similar characterization for ϕ-pseudo-Krull rings.

Theorem 2.7. *Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-infra-Krull ring.*

Proof. Set $D := R/Nil(R)$. Assume that R is a ϕ-pseudo-Krull ring. Since D is a pseudo-Krull domain by Theorem 2.3, we conclude that D is a t-closed infra-Krull domain by [29, Proposition 3.6]. Hence Lemma 2.6 implies that R is a ϕ-t-closed ϕ-infra-Krull ring.

Conversely, since R is a ϕ-t-closed ϕ-infra-Krull ring, it follows by Lemma 2.6 that D is a t-closed infra-Krull domain. Thus D is a pseudo-Krull domain by [29, Proposition 3.6], and so R is a ϕ-pseudo Krull ring by Theorem 2.3. □

We have the following pullback characterization of ϕ-pseudo-Krull rings.

Theorem 2.8. *Let $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, $S := A/M$ is a pseudo-Krull subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose that $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathcal{H}$ and $Nil(A) = Z(A) = M$. Since A/M is a pseudo-Krull domain, A is a ϕ-pseudo-Krull ring by Theorem 2.3, and thus R is a ϕ-pseudo-Krull ring.

Conversely, suppose that R is a ϕ-pseudo-Krull ring. Setting $T := R_{Nil(R)}$, $M := Nil(R_{Nil(R)})$, and $A := \phi(R)$ yields the desired pullback diagram. □

Recall that an ideal J of a commutative ring R is called a Glaz-Vasconcelos ideal or a GV-ideal, denoted by $J \in GV(R)$, if J is finitely generated and the natural homomorphism $\alpha : R \to \mathrm{Hom}_R(J, R)$, defined by $\alpha(r)(a) = ra$, $\forall r \in R$, $\forall a \in J$, is an isomorphism. An R-module M is called a GV-torsion-free module if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, one has $x = 0$. A GV-torsion-free R-module M is said to be a w-module if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. Let $R \in \mathcal{H}$. Recall from [26] that a nonnil ideal J is a ϕ-GV-ideal of R if $\phi(J)$ is a GV-ideal of $\phi(R)$. Recall that a nonnil ideal I is a ϕ-w-ideal if $\phi(I)$ is a w-ideal of $\phi(R)$, and R is called a *ϕ-strong-Mori ring* if it satisfies the ascending chain condition (a.c.c.) on ϕ-w-ideals.

It is well-known [25, Proposition 3.3] that if R is a one-dimensional integral domain, then R is a pseudo-Krull domain if and only if R is a t-closed strong-Mori domain. We have the following analogous result for ϕ-pseudo-Krull rings.

Proposition 2.9. *Let $R \in \mathcal{H}$ be a one-dimensional ring. Then R is a ϕ-pseudo-Krull ring if and only if R is a ϕ-t-closed ϕ-strong-Mori ring.*

Proof. Set $D := R/Nil(R)$. Clearly D is a one-dimensional integral domain. Now assume that R is a ϕ-pseudo-Krull ring. Hence D is a pseudo-Krull domain by Theorem 2.3, and so D is a t-closed strong-Mori domain by [25, Proposition 3.3]. Thus R is a ϕ-t-closed ϕ-strong-Mori ring by Lemma 2.6(2) and [26, Theorem 2.4]. Conversely, assume R is a ϕ-t-closed ϕ-strong-Mori ring. Then D is a t-closed strong-Mori domain by Lemma 2.6(2) and [26, Theorem 2.4]. That implies D is a pseudo-Krull domain by [25, Proposition 3.3], and hence R is a ϕ-pseudo-Krull ring by Theorem 2.3. □

For a ring R, let R' denote the integral closure of R in $T(R)$ and let R^* denote the complete integral closure of R in $T(R)$. Recall that a ring $R \in \mathcal{H}$ is said to be *ϕ-integrally closed* (resp., *ϕ-completely integrally closed*) if $\phi(R)$ is integrally closed (resp., completely integrally closed) in $T(\phi(R)) = R_{Nil(R)}$.

Lemma 2.10 ([2, Lemma 2.8])**.** *Let $R \in \mathcal{H}$ and set $D = \phi(R)/Nil(\phi(R))$. Then one has that $D' = \phi(R)'/Nil(\phi(R))$ and $D^* = \phi(R)^*/Nil(\phi(R))$. In particular, R is ϕ-integrally closed* (*resp., ϕ-completely integrally closed*) *if and only if D is integrally closed* (*resp., completely integrally closed*)*, if and only if $R/Nil(R)$ is integrally closed* (*resp., completely integrally closed*)*.*

Recall from [2] that a ring $R \in \mathcal{H}$ is called a *ϕ-Krull ring* if $\phi(R) = \bigcap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$ and for every non-nilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i.

It is well-known that if R is a pseudo-Krull domain, then the complete integral closure of R is a Krull domain. Also, an integral domain R is a Krull domain if and only if R is an integrally closed pseudo-Krull domain. We have the following analogous result.

Theorem 2.11. *Let $R \in \mathcal{H}$. Then*

1. *If R is a ϕ-pseudo-Krull ring, then $\phi(R)^*$ is a ϕ-Krull ring.*
2. *R is a ϕ-Krull ring if and only if R is a ϕ-integrally closed ϕ-pseudo-Krull ring.*

Proof. (1) Set $D := \phi(R)/Nil(\phi(R))$. Since D is ring-isomorphic to $R/Nil(R)$ by Lemma 2.1, we conclude that D is a pseudo-Krull domain by Theorem 2.3. Since $D^* = \phi(R)^*/Nil(\phi(R))$ by Lemma 2.10 and D^* is a Krull domain, we get that $\phi(R)^*$ is a ϕ-Krull ring by [2, Theorem 3.1].

(2) Set $D := R/Nil(R)$. By using [2, Theorem 3.1], R is a ϕ-Krull ring if and only if D is a Krull domain, if and only if D is an integrally closed pseudo-Krull domain, if and only if R is a ϕ-integrally closed ϕ-pseudo-Krull ring by Lemma 2.10 and Theorem 2.3. □

It is well-known [28, Proposition 5.5] that a flat overring of a pseudo-Krull domain is a pseudo-Krull domain. We have the next result.

Proposition 2.12. *Let $R \in \mathcal{H}$ be a ϕ-pseudo-Krull ring. Then every flat overring of R is a ϕ-pseudo-Krull ring.*

Proof. Let T be a flat overring of R. Then $T \in \mathcal{H}$, $Nil(T) = Nil(R)$, and $T/Nil(R)$ is a flat overring of $R/Nil(R)$. Since $R/Nil(R)$ is a pseudo-Krull domain and $T/Nil(R)$ is a flat overring of $R/Nil(R)$, we conclude that $T/Nil(R)$ is a pseudo-Krull domain by [28, Proposition 5.5]. Thus T is a ϕ-pseudo-Krull ring by Theorem 2.3. □

3. Examples of ϕ-pseudo-Krull rings

In this section, we use the trivial ring extension and the amalgamation of rings to construct examples of ϕ-pseudo-Krull rings which are not integral domains.

Let A be a ring and E be an A-module. Then $A \ltimes E$, the *trivial* (*ring*) *extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [21,23]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf., [3, 12, 24]).

Proposition 3.1. *Let $A \in \mathcal{H}$ be a ring and $E = A_{Nil(A)}$ an A-module. Then $R := A \ltimes E$ is a ϕ-pseudo-Krull ring if and only if A is a ϕ-pseudo-Krull ring.*

Proof. First we show that R is a ϕ-ring. It is easy to see that $Nil(R) = Nil(A) \ltimes E$. Since $Nil(A)$ is prime, $Nil(R)$ is a prime ideal of R. Now we show that $Nil(R)$ is divided. Let $(a, e) \in R \setminus Nil(R)$ and let $(b, f) \in Nil(R)$. Hence $a \in A \setminus Nil(A)$ and so $b = ac$ for some $c \in Nil(A)$ since $A \in \mathcal{H}$. Then $(b, f) = (ac, ad + ce)$ for some $d \in E$. Thus $(b, f) = (a, e)(c, d) \in (a, e)R$, and hence $Nil(R)$ is a divided prime ideal of R. Now observe that $R/Nil(R) \cong A/Nil(A)$. Then R is a ϕ-pseudo-Krull ring if and only if $R/Nil(R)$ is a pseudo-Krull domain, if and only if so is $A/Nil(A)$, if and only if A is a ϕ-pseudo-Krull ring. □

We combine this proposition with [23, Theorem 25.1] to get the following corollary.

Corollary 3.2. *Let A be a pseudo-Krull domain with quotient field K and Krull dimension n. Then $A \ltimes K \in \mathcal{H}$ is a ϕ-pseudo-Krull ring with Krull dimension n.*

The following is an example of a ϕ-pseudo-Krull ring which is not a ϕ-Krull ring.

Example 3.3. Let A be a pseudo-Krull domain with quotient field K which is not a Krull-domain. Then $R := A \ltimes E$ is a ϕ-pseudo-Krull ring by Proposition 3.1 which is not a ϕ-Krull ring by [2, Theorem 3.1] since $R/Nil(R)$ is ring-isomorphic to A.

Example 3.4. Let $L \subseteq K$ be a proper finite algebraic field extension and let $T = K[[X_1, \ldots, X_n]]$ be a ring of power series in $n \geq 2$ variables over K. Then $T = K + M$, where $M = X_1T + \cdots + X_nT$ is the unique maximal ideal of T. Note that T is an n-dimensional Noetherian domain. Let $A = L + M$ with quotient field E and let $R = A \ltimes E$. Then:

(1) R is a ϕ-t-closed ϕ-strong-Mori ring with Krull dimension n.

(2) R is not a ϕ-pseudo-Krull ring.

Proof. (1) By [25, Remark 3.5] A is a t-closed strong-Mori domain. Since $Nil(R) = 0 \ltimes E$ and $R/Nil(R)$ is ring-isomorphic to A, it follows that R is a ϕ-t-closed ϕ-strong-Mori ring with Krull dimension n by Lemma 2.6(2), [26, Theorem 2.4], and [23, Theorem 25.1].

(2) Note that A is not a pseudo-Krull domain by [28, Example 3.6]. Then R is not a ϕ-pseudo-Krull ring by Proposition 3.1. □

Let A and B be two rings, J be an ideal of B, and $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \,|\, a \in A, j \in J\},$$

called the *amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [14–16]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [13, 17, 18] and denoted by $A \bowtie I$).

In [20, 30], the authors studied when the amalgamation of rings is a ϕ-ring. Now we study when $A \bowtie^f J$ is a ϕ-pseudo-Krull ring.

Proposition 3.5. *Let A and B be two rings, J an ideal of B such that $J \subseteq Nil(B)$, and $f : A \longrightarrow B$ be a ring homomorphism. Set $R := A \bowtie^f J$ and assume that $R \in \mathcal{H}$. Then R is a ϕ-pseudo-Krull ring if and only if so is A.*

Proof. First observe that $\mathrm{Nil}(R) = \{(a, f(a) + j) \mid a \in \mathrm{Nil}(A), j \in J \cap Nil(B)\}$. It is easy to see that if $J \subseteq Nil(B)$, then $Nil(R) = Nil(A) \bowtie^f J$. Now, assume that R is a ϕ-pseudo-Krull ring. Hence $R/Nil(R)$ is a pseudo-Krull domain by Theorem 2.3. Since $A/Nil(A)$ is ring-isomorphic to $R/Nil(R)$, it follows that $A/Nil(A)$ is a pseudo-Krull domain. Thus A is a ϕ-pseudo-Krull ring. The converse follows by similar reasoning. □

We combine [20, Corollary] with Proposition 3.5 to get the following examples.

Example 3.6. Let B be a pseudo-Krull domain with quotient field K. Set $A := B \ltimes K$ and $I := 0 \ltimes K$. Then the amalgamated duplication $R := A \bowtie I$ is a ϕ-pseudo-Krull ring.

Proof. By [20, Corollary 2.5], R is a ϕ-ring since $I = Nil(R)$. Thus Proposition 3.5 completes the proof. □

Example 3.7. Let C be a pseudo-Krull domain with quotient field K. Let $f : C \ltimes K := A \longrightarrow B := K \ltimes K$ be the natural injection and set $J := 0 \ltimes K$. Then $A \bowtie^f J$ is a ϕ-pseudo-Krull ring.

Acknowledgements. The first and the third author dedicated this work to Professor Abdelmoujib Benkirane.

References

[1] D. F. Anderson and A. Badawi, *On ϕ-Prüfer rings and ϕ-Bezout rings*, Houston J. Math. **30** (2004), no. 2, 331–343.

[2] ———, *On ϕ-Dedekind rings and ϕ-Krull rings*, Houston J. Math. **31** (2005), no. 4, 1007–1022.

[3] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009), no. 1, 3–56. https://doi.org/10.1216/JCA-2009-1-1-3

[4] A. Badawi, *Pseudo-valuation rings*, in Commutative ring theory (Fès, 1995), 57–67, Lecture Notes in Pure and Appl. Math., 185, Dekker, New York, 1997.

[5] ———, *On divided commutative rings*, Comm. Algebra **27** (1999), no. 3, 1465–1474. https://doi.org/10.1080/00927879908826507

[6] ———, *On ϕ-pseudo-valuation rings*, in Advances in commutative ring theory (Fez, 1997), 101–110, Lecture Notes in Pure and Appl. Math., 205, Dekker, New York, 1999.

[7] ______, *On ϕ-pseudo-valuation rings. II*, Houston J. Math. **26** (2000), no. 3, 473–480.
[8] ______, *On ϕ-chained rings and ϕ-pseudo-valuation rings*, Houston J. Math. **27** (2001), no. 4, 725–736.
[9] ______, *Pseudo-valuation domains: a survey*, in Mathematics & mathematics education (Bethlehem, 2000), 38–59, World Sci. Publ., River Edge, NJ, 2002.
[10] ______, *On nonnil-Noetherian rings*, Comm. Algebra **31** (2003), no. 4, 1669–1677. https://doi.org/10.1081/AGB-120018502
[11] A. Badawi, D. F. Anderson, and D. E. Dobbs, *Pseudo-valuation rings*, Lecture Notes Pure Appl. Math. 185, 57–67, Marcel Dekker, New York/Basel, 1997.
[12] C. Bakkari, S. Kabbaj, and N. Mahdou, *Trivial extensions defined by Prüfer conditions*, J. Pure Appl. Algebra **214** (2010), no. 1, 53–60. https://doi.org/10.1016/j.jpaa.2009.04.011
[13] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306** (2006), no. 2, 507–519. https://doi.org/10.1016/j.jalgebra.2005.12.023
[14] M. D'Anna, C. A. Finocchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, in Commutative algebra and its applications, 155–172, Walter de Gruyter, Berlin, 2009.
[15] ______, *Properties of chains of prime ideals in an amalgamated algebra along an ideal*, J. Pure Appl. Algebra **214** (2010), no. 9, 1633–1641. https://doi.org/10.1016/j.jpaa.2009.12.008
[16] ______, *New algebraic properties of an amalgamated algebra along an ideal*, Comm. Algebra **44** (2016), no. 5, 1836–1851. https://doi.org/10.1080/00927872.2015.1033628
[17] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45** (2007), no. 2, 241–252. https://doi.org/10.1007/s11512-006-0038-1
[18] ______, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), no. 3, 443–459. https://doi.org/10.1142/S0219498807002326
[19] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. **67** (1976), no. 2, 353–363. http://projecteuclid.org/euclid.pjm/1102817497
[20] A. El Khalfi, H. Kim, and N. Mahdou, *Amalgamated algebras issued from ϕ-chained rings and ϕ-pseudo valuation rings*, To appear in "Bull. Iranian Math. Soc."; https://doi.10.1007/s41980-020-00461-y
[21] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989. https://doi.org/10.1007/BFb0084570
[22] J. R. Hedstrom and E. G. Houston, *Pseudo-valuation domains*, Pacific J. Math. **75** (1978), no. 1, 137–147. http://projecteuclid.org/euclid.pjm/1102810151
[23] J. A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics, 117, Marcel Dekker, Inc., New York, 1988.
[24] S.-E. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra **32** (2004), no. 10, 3937–3953. https://doi.org/10.1081/AGB-200027791
[25] H. Kim and Y. S. Park, *Some remarks on pseudo-Krull domains*, Comm. Algebra **33** (2005), no. 6, 1745–1751. https://doi.org/10.1081/AGB-200063361
[26] H. Kim and F. Wang, *On ϕ-strong Mori rings*, Houston J. Math. **38** (2012), no. 2, 359–371.
[27] M. B. Martin and M. Zafrullah, *t-linked overrings of Noetherian weakly factorial domains*, Proc. Amer. Math. Soc. **115** (1992), no. 3, 601–604. https://doi.org/10.2307/2159205
[28] S. Oda, *On pseudo-Krull domains*, Math. Rep. Toyama Univ. **10** (1987), 85–106.
[29] G. Picavet and M. Picavet-L'Hermitte, *When is length a length function?*, J. Algebra **293** (2005), no. 2, 561–594. https://doi.org/10.1016/j.jalgebra.2005.08.014

[30] M. Tamekkante, K. Louartiti, and M. Chhiti, *Chain conditions in amalgamated algebras along an ideal*, Arab. J. Math. (Springer) **2** (2013), no. 4, 403–408. https://doi.org/10.1007/s40065-013-0075-0

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