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ON ENGEL QUASI-NORMAL SUBGROUP IN DIVISION RINGS WITH UNCOUNTABLE CENTER

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Abstract. The aim of this paper is to show that every Engel quasi-normal subgroups of the unit group of a division ring with uncountable center is central.

1. Introduction

Let G be a group. For $a, b \in G$ and $n > 1$, we denote $[a, {}_1b]$ by $a^{-1}b^{-1}ab$ and $[a, {}_nb]$ by $[[a, {}_{n-1}b], {}_1b]$. An element $a \in G$ is said to be *left Engel* in G if for any $g \in G$ there exists a positive integer n_g such that $[g, {}_{n_g}a] = 1$. If every element of G is left-Engel, then G is called an Engel group. The topic of Engel subgroups of the unit groups of algebras has been recently receiving a great attention (e.g., see [1,2,4,5,11,13,15,16,18]). Also, the subject of locally nilpotent groups and free subgroups of unit groups of division rings (e.g., see [8–10,14] and citations therein) is also drawing a remarkable attention.

In this article, our work involves mainly with division rings and Engel quasi-normal subgroups of their unit groups. The problem is inspired from a conjecture that was posed in [15].

Conjecture 1.1 ([15, Question]). *Let D be a division ring. Then Engel subnormal subgroups of D^* are all central.*

This conjecture is still an open problem even when we assume D^* to be an Engel group. However, several other special cases of the conjecture have been addressed. One example is where D is given to be a locally finite division ring [15]. Recall that a division ring D with center F is called *locally finite* if for every finite subset S of D, the division subring $F(S)$ of D generated by S over F is finite dimensional over F. Recently, the case in which F is uncountable has also been resolved [2].

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The answer to the conjecture would have great significance to the study of Engel subgroups in general rings. One of the reasons for this comes from the tactics that are used to solve such problem. For example, in [3], in order to prove their results, the authors had to make use of skew linear groups. This article mainly studies about a class of Engel quasi-normal subgroups of the unit groups of division rings. Recall that, a subgroup H of the group G is called a *quasi-normal subgroup* of G if $HK = KH$ for any subgroup K of G. Clearly, every normal subgroup is also a quasi-normal subgroup. Recently, a class of quasi-normal subgroups of $\mathrm{GL}_n(D)$ were also studied in [6]. In detail, if $n > 1$, then every quasi-normal subgroup of $\mathrm{GL}_n(D)$ is also normal in $\mathrm{GL}_n(D)$. For the case when $n = 1$, there is a counterexample for a division ring in which the unit group contains a quasi-normal but non-subnormal subgroup [6, Section 4]. There is also a result in [6] showing that if D is a locally finite division ring, then every non-central quasi-normal subgroup of D^* contains a non-abelian free subgroup. In particular, every Engel quasi-normal subgroup of the unit group of a locally finite division ring is central. The aim of this paper is to consider the conjecture for the case when N is a quasi-normal subgroup of D^* in case the center of D is uncountable.

The technique we use in this paper is a modification of [2]. Our notations are standard. For example, for a ring R, the notations $Z(R)$ and R^* are the center and unit group of R. A subset S of R is called *commuting* if $ab = ba$ for every $a, b \in S$.

2. Main results

Let R be a ring and x a commuting indeterminate. By $R[x]$, we denote the polynomial ring in x over R. In case $R = D$ is a division ring, then $D[x]$ has the (universal) division ring of fractions which is denoted by $D(x)$. Every element of $D(x)$ has the form $f(x)g(x)^{-1}$ where $f(x), g(x) \in D[x]$ with $g(x) \neq 0$. Inductively, we denote by $D(x, y)$ the division ring $D(x)(y)$. Similarly, every element of $D(x, y)$ has the form $f(x, y)g(x, y)^{-1}$ where $f(x, y), g(x, y) \in D[x, y]$. The formal power series ring in x over R is denoted by $R[[x]]$, that is, $R[[x]]$ is the ring of all formal power series $\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + \cdots$, where $a_i \in R$. Observe that if a_0 is invertible in R, then so is $\sum_{i=0}^{\infty} a_i x^i$ in $R[[x]]$ and $(\sum_{i=0}^{\infty} a_i x^i)^{-1} = (1 - \alpha + \alpha^2 - \cdots + (-1)^i \alpha^i + \cdots) a_0^{-1}$, where $\alpha = a_0^{-1} \sum_{i=1}^{\infty} a_i x^i$. In particular, $(1 - ax)^{-1} = 1 + ax + a^2 x^2 + \cdots$ for every $a \in R$.

Lemma 2.1 ([12, Lemma 1]). *Let D be a division ring with center F and $D(x, y)$ the division ring of fractions of the polynomial ring $D[x, y]$ in commuting indeterminates x, y over D. If $f(x, y) \in D(x, y)$ which satisfies $f(\lambda_1, \lambda_2) \in F$ for infinitely many pairs $(\lambda_1, \lambda_2) \in F \times F$, then $f(x, y) \in F(x, y)$.*

Lemma 2.2. *Let D be a division ring with center F and N an abelian subgroup of D^*. Then either $N \subseteq F$ or D^* is not radical over N.*

Proof. Assume on the contrary that $N \nsubseteq F$ and D^* is radical over N. We will now seek for a contradiction. Since N is not central, D is then a non-commutative ring. Denote $K = F(N)$ as a division subring of D generated by N over F. Since N is abelian, K is a subfield of D. Therefore $K \neq D$. Since D^* is radical over N, D^* is also radical over K. Since neither $D = F$ nor $D = K$, according to [7, Theorem B], this is a contradiction. Therefore, either $N \subseteq F$ or D^* is not radical over N. The proof is now complete. □

Lemma 2.3. *Let D be a division ring with uncountable center F. Assume that N is an Engel subgroup of D^* such that D^* is radical over N. Then, for every $a \in N \subseteq D(x,y)^*$ and $f(x,y) \in D(x,y)^*$, there exist positive integers n_f, m_f such that $[f(x,y)^{n_f}, {}_{m_f} a] = 1$.*

Proof. Let $a \in N$ and $f(x,y) \in D(x,y)^*$. We have $f(x,y) = f_1(x,y)f_2(x,y)^{-1}$ where $f_1(x,y), f_2(x,y) \in D[x,y]$ with $f_2(x,y) \neq 0$. By [17, the Vandermonde argument, Propositions 2.3.26 and 2.3.27], $f_1(\lambda_1, \lambda_2) = 0$ or $f_2(\lambda_1, \lambda_2) = 0$ only for finitely many $\lambda_1, \lambda_2 \in F$. There exist uncountably many central pairs $(\lambda_1, \lambda_2) \in F \times F$ such that $f(\lambda_1, \lambda_2) \in D^*$. For each such λ_1, λ_2, since D^* is radical over N, there exists a positive integer n_{λ_1,λ_2} such that $f(\lambda_1, \lambda_2)^{n_{\lambda_1,\lambda_2}} \in N$. Moreover, because N is Engel, there exists $m_{\lambda_1,\lambda_2} > 0$ such that $[f(\lambda_1, \lambda_2)^{n_{\lambda_1,\lambda_2}}, {}_{m_{\lambda_1,\lambda_2}} a] = 1$. Using the pigeon-hole principle, there exist positive integers n_f, m_f such that $[f(\lambda_1, \lambda_2)^{n_f}, {}_{m_f} a] = 1$ for infinite amount of pairs $(\lambda_1, \lambda_2) \in F \times F$. Hence by Lemma 2.1, $[f(x,y)^{n_f}, {}_{m_f} a] \in F(x,y)$. Therefore, $[f(x,y)^{n_f}, {}_{m_f+1} a] = 1$. The proof is now complete. □

In order to prevent the hassle of redundancy, we will here present some notion that may cause repetition in the rest of this paper.

For convenience, from now, D is a division ring with center F. Assume that $a, b \in D^*$ such that $ab \neq ba$ and $[b, {}_2 a] = 1$. Let $c = b^{-1}ab$, it should be trivial that $a \neq c$ and $bc = ab$. From $[b, {}_2 a] = 1$, it is also easy to see that $a^{-1}ca^{-1}c^{-1}a^2 = 1$. In other words, $ac = ca$.

Let $R = D[x][[y]]$ be the formal power series ring in y over in $D[x]$ and let $\circ$ denote the operation defined via the formal power series term action $dx^iy^j \circ \alpha = d\alpha a^i y^j$ for $d \in D, \alpha \in R, i, j \in \mathbb{N}$. R is then a left R-module with $\circ$. Let $K = F[a, a^{-1}, c, c^{-1}][[y]]$. Then, R is also a right K-module via the right multiplication. It is then easy to verify that R is an (R, K)-bimodule. Let $M = 1K + bK$ be the right K-submodule of R generated by $B = \{1, b\}$. Assume there exist $k_1, k_2 \in K$ such that $bk_2 = -k_1$, thus $abk_2 = bk_2a = bak_2$, as a commutes with k_1, k_2. Thus $(ab - ba)k_2 = 0$, implying $k_2 = 0$. As a consequence, $k_1 = 0$ and hence B is linearly independent. M is thus a free K-module whose basis is B.

Let

$$S = \{r \mid r \in R,\ r \circ M \subseteq M\}.$$

It can then be shown that S is a subring of R. Hence we define a map

$$\phi : S \to \mathrm{End}(M_K)$$
$$r \mapsto \phi_r,$$

where $\phi_r \in \mathrm{End}(M_K)$, $\phi_r(v) = r \circ v$ for any $v \in M$. It is easy to verify that ϕ is a ring homomorphism and each ϕ_r is indeed a right K-endomorphism of M.

Since M is a free module of rank 2, $\mathrm{End}(M_K) \cong M_2(K)$. For each module endomorphism of M there is a corresponding matrix of order 2 with respect to the basis of B. For $f \in \mathrm{End}(M_K)$, let $k_{11}, k_{12}, k_{21}, k_{22} \in K$ such that $f(1) = 1k_{11} + bk_{21}$ and $f(b) = 1k_{12} + bk_{22}$. We might as well say that $f = \left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$. For such arbitrary $f \in M_2(K)$, there exists $\gamma = (k_{11} + bk_{21})a^{-1}(b-1)^{-1}(ba - x) - (k_{12} + bk_{22})a^{-1}(b-1)^{-1}(a-x) \in S$ such that $\phi(\gamma) = f$. Hence, ϕ is also an epimorphism.

Let $r \in R$. If $r \circ 1 \in M$ and $r \circ b \in M$, then $r \circ M \subset M$, that is $r \in S$. Also, let $s \in S$ such that ϕ_s is an invertible matrix. Since $\phi_s \phi_{s^{-1}} = I$, it is clear that $\phi_{s^{-1}} = \phi_s^{-1}$.

Hence, from $\phi_a(1) = a \circ 1 = a$ and $\phi_a(b) = a \circ b = ab = bc$, we conclude that

$$\phi_a = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}.$$

Also, it is easy to see that $a^{-1} \in S$ and

$$\phi_{a^{-1}} = \phi_a^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & c^{-1} \end{bmatrix}.$$

To show the main result, we also need the following result.

Lemma 2.4. *Let* $d_{11} = -b(ab-ba)^{-1}(bab^{-1} - x)$; $d_{12} = (ab-ba)^{-1}(a - x)$; $d_{21} = -b^2(ab-ba)^{-1}(bab^{-1} - x)$; *and* $d_{22} = b(ab-ba)^{-1}(a-x)$. *Then,* $d_{11}, d_{12}, d_{21}, d_{22} \in S$ *and*

$$\phi_{d_{11}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \phi_{d_{12}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \phi_{d_{21}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \text{ and } \phi_{d_{22}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. The result comes from these calculation:

$$\begin{aligned} d_{11} \circ 1 &= -b(ab-ba)^{-1}(bab^{-1} - x) \circ 1 = -b(ab-ba)^{-1}(bab^{-1} - a) \\ &= -b(ab-ba)^{-1}(ba - ab)b^{-1} = 1. \end{aligned}$$

$$d_{11} \circ b = -b(ab-ba)^{-1}(bab^{-1} - x) \circ b = -b(ab-ba)^{-1}(bab^{-1}b - ba) = 0.$$

$$d_{12} \circ 1 = (ab-ba)^{-1}(a-x) \circ 1 = (ab-ba)^{-1}(a-a) = 0.$$

$$d_{12} \circ b = (ab-ba)^{-1}(a-x) \circ b = (ab-ba)^{-1}(ab-ba) = 1.$$

$$d_{21} \circ 1 = -b^2(ab-ba)^{-1}(bab^{-1} - x) \circ 1 = -b^2(ab-ba)^{-1}(bab^{-1} - a) = b.$$

$$d_{21} \circ b = -b^2(ab-ba)^{-1}(bab^{-1} - x) \circ b = -b^2(ab-ba)^{-1}(ba-ba) = 0.$$

$$d_{22} \circ 1 = b(ab-ba)^{-1}(a-x) \circ 1 = b(ab-ba)^{-1}(a-a) = 0.$$

$$d_{22} \circ b = b(ab-ba)^{-1}(a-x) \circ b = b(ab-ba)^{-1}(ab-ba) = b.$$

All of the calculated elements are in M, thus $d_{11}, d_{12}, d_{21}, d_{22} \in S$. By applying the convention that corresponds each module homomorphism with a matrix, our proof is now complete. $\square$

The verification of this result is easy, thus it is skipped on purpose.

Lemma 2.5. *Suppose* $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} b & 0 \\ 0 & c \end{smallmatrix}\right] \in \mathrm{GL}_2(D)$. *If* $\{a, b, c\}$ *is commuting, then for any positive integer* m, $[\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right], {}_m \left[\begin{smallmatrix} b & 0 \\ 0 & c \end{smallmatrix}\right]] = \left[\begin{smallmatrix} 1 & (b^{-1}c-1)^m a \\ 0 & 1 \end{smallmatrix}\right]$.

We are now ready to take on the main result.

Theorem 2.6. *Every Engel quasi-normal subgroup of the unit group of a division ring with uncountable center is central.*

Proof. Let D be a division ring with uncountable center F. Suppose N is a quasi-normal subgroup of D^* which is Engel. We must show that $N \subseteq F$. Assume that $N \not\subseteq F$. According to [2] and [6, Lemma 2.4], it suffices to consider the case when D^* is radical over N. As a consequence of Lemma 2.2, N is a non-abelian group. There exist $a, b \in N$ such that $ab \neq ba$. Since N is Engel, there exists a positive integer $n = n_b$ such that $[b, {}_n a] = 1$. It is clear that $n \geq 2$ and by replacing b by $[b, {}_{n-2} a]$, without loss of generality, we can assume that $n = 2$. Now we use the notations before Lemma 2.4 and consider two cases:

Case 1. $\mathrm{Char}(D) = 0$. Assign $f = 1 + d_{12} y = 1 + (ab - ba)^{-1}(a - x) y \in D[x, y]$. It is easy to see that $\phi_f(1) = 1$ and $\phi_f(b) = y + b$, hence we can then say that $\phi_f = \left[\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right]$. Moreover,

$$f^{-1} = 1 - d_{12} y + d_{12}^2 y^2 - d_{12}^3 y^3 + \cdots \in R.$$

Then since $\phi_f \in GL_2(K)$, $f \circ M = M$. So, we have

$$M = 1 \circ M = (f^{-1} f) \circ M = f^{-1} \circ (f \circ M) = f^{-1} \circ M,$$

which implies $f^{-1} \in S$.

Hence, $\phi_{f^{-1}} = \left[\begin{smallmatrix} 1 & -y \\ 0 & 1 \end{smallmatrix}\right]$. It can be seen that $f \in D[x, y] \subset D(x, y)$, hence $f \in D(x, y)^*$. By Lemma 2.3, there exist positive integers n_f, m_f satisfying $[f^{n_f}, {}_{m_f} a] = 1$. Thus, according to Lemma 2.5,

$$I_2 = \phi([f^{n_f}, {}_{m_f} a]) = \begin{bmatrix} 1 & n_f (a^{-1} c - 1)^{m_f} y \\ 0 & 1 \end{bmatrix}.$$

We conclude that $a = c$, a contradiction.

Case 2. $\mathrm{Char}(D) = p > 0$. Put $\alpha = (1 - ac^{-1})^n d_{12}$, $\beta = d_{21}$, $v = 1 + y\alpha, w = 1 + y\beta\alpha\beta$ and $u = 1 + y(1 - \beta)\alpha\beta\alpha(1 + \beta)$. Then, by Lemma 2.4, $\phi_\alpha = \left[\begin{smallmatrix} 0 & (1 - ac^{-1})^n \\ 0 & 0 \end{smallmatrix}\right]$ and $\phi_\beta = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Then, $\phi_\alpha^2 = \phi_\beta^2 = 0$, and $\phi_{\beta\alpha} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & (1 - ac^{-1})^n \end{smallmatrix}\right]$, a non-nilpotent matrix. For $s_1, s_2, s_3 \in S$, it is easily seen that $\phi_{s_1} + \phi_{s_2} \phi_{s_3} = \phi_{s_1 + s_2 s_3}$. Hence, by [10, Lemma 5.2], ϕ_v, ϕ_w, ϕ_u freely generate a subgroup of $\mathrm{GL}_2(K)$ which is isomorphic to the free product $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p. Using the same arguments in the first case, since u, v and w all have inverses in R and corresponding matrices which are invertible, so $u^{-1}, v^{-1}, w^{-1} \in S^*$. Hence, $\langle u, v, w\rangle \cong \langle \phi_u, \phi_v, \phi_w\rangle$. Therefore, u, v, w freely generate a subgroup in S^* which is isomorphic to the free product $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$. As a corollary, if $g = uvu$ and $h = vwv$, then g and h freely generate a free subgroup of S^*. Following [17, the Vandermonde argument, Propositions 2.3.26 and 2.3.27], there exist infinitely many λ_1 and λ_2 in F such that $g(\lambda_1, \lambda_2)h(\lambda_1, \lambda_2) \neq 0$. Since D^* is radical over N, there is a positive integer k_g such that $g(\lambda_1, \lambda_2)^{k_g} \in N$. Applying Lemma 2.3, there exist two positive integers $n_{h,g}, m_{h,g}$ such that $[h(x, y)^{n_{h,g}},_{m_{h,g}} g(\lambda_1, \lambda_2)^{k_g}] = 1$. In particular, $[h(\lambda_1, \lambda_2)^{n_{h,g}},_{m_{h,g}} g(\lambda_1, \lambda_2)^{k_g}] = 1$. By using Lemma 2.1, what follows is that $[h^{n_{h,g}},_{m_{h,g}} g^{k_g}] \in F(x, y)$. Hence, $[h^{n_{h,g}},_{m_{h,g}+1} g^{k_g}] = 1$, which contradicts the fact that g and h generate a free group.

In either case, our assumption leads to a contradiction. Hence, N is central. The proof is now complete. □

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