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EXACTNESS OF COCHAIN COMPLEXES VIA ADDITIVE FUNCTORS

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Abstract. We investigate the relation between the notion of e-exactness, recently introduced by Akray and Zebary, and some functors naturally related to it, such as the functor $P\colon \text{Mod-}R \to \text{Spec}(\text{Mod-}R)$, where $\text{Spec}(\text{Mod-}R)$ denotes the spectral category of Mod-R, and the localization functor with respect to the singular torsion theory.

1. Introduction

This paper has been inspired by the article [1], in which the authors considered the following type of "e-exact" cochain complexes of modules over a ring R.

Definition 1.1. Let R be a ring, not necessarily commutative. A cochain complex

$$\cdots \longrightarrow M^{i-1} \xrightarrow{f^{i-1}} M^i \xrightarrow{f^i} M^{i+1} \xrightarrow{f^{i+1}} \cdots$$

of right R-modules and right R-module morphisms is said to be *e-exact* if $f^{i-1}(M^{i-1})$ is an essential submodule of $\ker(f^i)$ for every i. In particular, a *short e-exact sequence* is a cochain complex of the type $0 \longrightarrow A_R \xrightarrow{f} B_R \xrightarrow{g} C_R \longrightarrow 0$ with f a monomorphism, $f(A_R)$ an essential submodule of $\ker(g)$, and $g(B_R)$ an essential submodule of C_R.

In [1], R denoted a commutative ring. It is immediately clear that the hypothesis of R commutative is not necessary in this definition, and that one can consider right modules over any ring R, possibly noncommutative. Here, we study the relation between this interesting notion and three functors naturally related to it. The first functor is the functor $P\colon \text{Mod-}R \to \text{Spec}(\text{Mod-}R)$ into the spectral category of the category Mod-R, as defined by Gabriel and Oberst

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in [5]. The second functor is the localization with respect to the singular torsion theory (Goldie topology) studied in particular by Goodearl ([7, Chapter 2] and [9, Examples VI.6.2 and IX.2.2]). This allows us to extend all the results in the first part of [1] from modules over commutative rings to modules over arbitrary, possibly non-commutative, rings, essentially replacing the usual torsion-theory with the singular torsion-theory, and replacing commutative integral domains with right non-singular rings. In particular, this applies to our third functor, the functor $-\otimes_R Q$, where R is a right Ore domain and Q is the classical right ring of fractions of R. It is interesting to notice, as we prove in Theorem 3.8, that for a ring R, except for the trivial case of R artinian semisimple, there do not exist additive functors $F\colon \text{Mod-}R \to \mathcal{A}$ into any abelian category $\mathcal{A}$ with the property that a cochain complex $A_R \xrightarrow{f} B_R \xrightarrow{g} C_R$ is e-exact in the sense of Definition 1.1 if and only if the cochain complex $F(A_R) \xrightarrow{F(f)} F(B_R) \xrightarrow{F(g)} F(C_R)$ is exact.

In the last section, we partially extend our setting to arbitrary Gabriel topologies.

In the following, all rings R are associative rings with identity $1 \neq 0$, and all modules are unitary right R-modules.

2. e-exactness and the spectral category

For any Grothendieck category $\mathcal{A}$, it is possible to construct the *spectral category* $\operatorname{Spec}(\mathcal{A})$, which is a Grothendieck category obtained from $\mathcal{A}$ by formally inverting all essential monomorphisms of $\mathcal{A}$ [5]. More precisely, for any fixed object A in $\mathcal{A}$, the set of all the essential subobjects of A is downward directed, because the intersection of two essential subobjects of A is an essential subobject of A. We will write $A' \leq_e A$ for "A' is an essential subobject of A". If we fix another object B of $\mathcal{A}$ and apply the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, B)$ to the essential subobjects A' of A and to the embeddings $A' \to A''$, where $A' \leq_e A$, $A'' \leq_e A$ and $A' \subseteq A''$, we get an upward directed family of abelian groups $\operatorname{Hom}_{\mathcal{A}}(A', B)$ and abelian group morphisms $\operatorname{Hom}_{\mathcal{A}}(A'', B) \to \operatorname{Hom}_{\mathcal{A}}(A', B)$. Take the direct limit $\varinjlim \operatorname{Hom}_{\mathcal{A}}(A', B)$, where A' ranges in the set of all essential subobjects of A. The *spectral category* $\operatorname{Spec}(\mathcal{A})$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and, for objects A and B of $\mathcal{A}$,

$$\operatorname{Hom}_{\operatorname{Spec}(\mathcal{A})}(A, B) := \varinjlim \operatorname{Hom}_{\mathcal{A}}(A', B),$$

where the direct limit is taken over all essential subobjects A' of A. The category $\operatorname{Spec}(\mathcal{A})$ can also be constructed as follows. Let $\mathcal{E}$ be the full subcategory of $\mathcal{A}$ consisting of all the injective objects of $\mathcal{A}$. Let $\mathcal{I}$ be the ideal of the category $\mathcal{E}$, where, for every $E_R, F_R \in \operatorname{Ob}(\mathcal{E})$, $\mathcal{I}(E_R, F_R)$ consists of all morphisms $E_R \to F_R$ whose kernel is essential in E_R. Then $\operatorname{Spec}(\mathcal{A})$ is the quotient category $\mathcal{E}/\mathcal{I}$. A third equivalent presentation of $\operatorname{Spec}(\mathcal{A})$ is as the category of fractions $\mathcal{A}[S^{-1}]$ (see [2, 4, 6]), where S is the class of all essential monomorphisms. There is a canonical functor $P\colon \mathcal{A} \to \mathrm{Spec}(\mathcal{A})$, which is the identity on objects. It is a left exact functor that sends essential monomorphisms to isomorphisms.

The category Spec(Mod-R) is a Grothendieck category in which every short exact sequence splits, i.e., in which every object is projective and injective. From the discussion made above, it is clear that every right R-module M_R becomes isomorphic to its injective envelope $E(M_R)$ in Spec(Mod-R). In particular, $P(M_R) = 0$ if and only if $M_R = 0$. Moreover, a right R-module morphism $f\colon M_R \to N_R$ becomes a zero morphism in Spec(Mod-R) whenever $\ker(f)$ is an essential submodule of M_R.

If we view the morphisms $f\colon A_R \to B_R$ in Spec(Mod-R) as morphisms $f'\colon A'_R \to B_R$ for some essential submodule A'_R of A_R, the description of the kernel, cokernel and image of f is the following. The kernel of f' in Mod-R has a complement C'_R in A'_R, that is, there exists $C'_R \leq A_R$ with $\ker(f') \oplus C'_R$ essential in A'_R (equivalently, C'_R is maximal in the set of all submodules X'_R of A'_R with $\ker(f') \cap X'_R = 0$). Then the submodule $f'(C'_R) \cong C'_R$ of B_R has a complement D'_R in B_R, i.e., $f'(C'_R) \oplus D'_R$ is essential in B_R. Then $\ker(f)$ is the kernel of f in Spec(Mod-R), D'_R is the cokernel of f in Spec(Mod-R) and $f'(C'_R)$ is its image.

In the description of Spec(Mod-R) as the quotient category $\mathcal{E}/\mathcal{I}$, the kernel and the cokernel of a morphism $f\colon E_R \to F_R$ are as follows. Assume that f is represented by a morphism $f'\colon E_R \to F_R$ in Mod-R. The kernel of f' has a complement in E_R, that is, there exists $C_R \leq E_R$ with $\ker(f') \oplus C_R$ essential in E_R (it suffices to take C_R maximal in the set of all submodules X_R of E_R with $\ker(f') \cap X_R = 0$). Then E_R decomposes as $E_R = E(\ker(f')) \oplus E(C_R)$. Moreover, $f'(E(C_R)) \cong E(C_R)$ is an injective submodule of F_R, so we can write $F_R = f'(E(C_R)) \oplus D_R$ for a suitable submodule D_R of F_R. Then $E(\ker(f'))$ is the kernel of f in Spec(Mod-R), D_R is the cokernel of f in Spec(Mod-R) and $f'(E(C_R))$ is the image of f.

We want to investigate the relation between the notion of e-exactness (Definition 1.1) and the functor $P\colon$ Mod-$R \to$ Spec(Mod-R).

Lemma 2.1. *Let $M^{i-1} \xrightarrow{f^{i-1}} M^i \xrightarrow{f^i} M^{i+1}$ be a cochain complex of right R-modules and right R-module morphisms. The following conditions are equivalent*:

(a) *The sequence*

$$P(M^{i-1}) \xrightarrow{P(f^{i-1})} P(M^i) \xrightarrow{P(f^i)} P(M^{i+1}) \tag{1}$$

is exact in Spec(Mod-R).

(b) *If C_R is a complement of* $\ker(f^{i-1})$ *in M^{i-1}, then $f^{i-1}(C_R)$ is essential in* $\ker(f^i)$.

Proof. The kernel of f^{i-1} in Mod-R and the kernel of $P(f^{i-1})$ in Spec(Mod-R) coincide because P is left exact. Similarly for the kernels of f^i and $P(f^i)$.

Let C_R be a complement of the kernel $\ker(f^{i-1})$ of f in Mod-R, so that $\ker(f^{i-1}) \oplus C_R$ is essential in M^{i-1}. Then $f^{i-1}(C_R)$ is the image of $P(f^{i-1})$.

Hence the sequence (1) is exact in $P(M^i)$ if and only if $f^{i-1}(C_R)$ is essential in $\ker(f^i)$. □

From the previous lemma we immediately get the following proposition.

Proposition 2.2. *Let* $0 \longrightarrow A_R \xrightarrow{f} B_R \xrightarrow{g} C_R \longrightarrow 0$ *be a cochain complex of right* R*-modules and right* R*-module morphisms. The following conditions are equivalent*:

(a) *The sequence* $0 \longrightarrow P(A_R) \xrightarrow{P(f)} P(B_R) \xrightarrow{P(g)} P(C_R) \longrightarrow 0$ *is a short exact sequence in* $\mathrm{Spec}(\text{Mod-}R)$.

(b) f *is a monomorphism in* Mod-R, $f(A_R)$ *an essential submodule of* $\ker(g)$, *and if* B'_R *is a complement of* $f(A_R)$ *in* B_R, *then* $g(B'_R)$ *is essential in* C_R.

Corollary 2.3. *Let*

$$0 \longrightarrow A_R \xrightarrow{f} B_R \xrightarrow{g} C_R \longrightarrow 0 \tag{2}$$

be a cochain complex of right R*-modules and right* R*-module morphisms such that the sequence* $0 \longrightarrow P(A_R) \xrightarrow{P(f)} P(B_R) \xrightarrow{P(g)} P(C_R) \longrightarrow 0$ *is a short exact sequence in* $\mathrm{Spec}(\text{Mod-}R)$*. Then the cochain complex* (2) *is a short e-exact sequence.*

3. Homological lemmas with e-exact sequences

The aim of this section is generalizing some results of [1] to the non-commutative case. We will replace commutative domains with right non-singular rings and torsion [resp. torsionfree] modules with singular [resp. non-singular] modules. Let R be a ring and let $\mathfrak{G}$ denote the family of all essential right ideals of R. Given a right R module M_R, the singular submodule of M_R is defined by $Z(M_R) := \{x \in M \mid xI = 0 \text{ for some } I \in \mathfrak{G}\} = \{x \in M \mid \mathrm{Ann}_r(x) \in \mathfrak{G}\}$ [7, Ch. 1, Sec. D]. A module M_R is called *singular* if $Z(M_R) = M_R$ and it is called *non-singular* if $Z(M_R) = 0$. If M is a module over a commutative domain R, then $Z(M)$ is equal to the torsion submodule $t(M)$ of M, so that M is singular (resp. non-singular) if and only if M is torsion (resp. torsionfree). Notice that $Z(-)$ defines a functor Mod-$R \to$ Mod-R. Indeed, given a right R-module morphism $f\colon M_R \to N_R$, we have $f(Z(M)) \leq Z(N)$, and so a morphism $Z(f)\colon Z(M) \to Z(N)$. The functor $Z(-)$ is an idempotent preradical (cf. [9, Chapter VI]), but it is not a radical in general. The smallest radical containing Z is denoted by Z_2 and it is defined, for every right R-module M_R, by $Z_2(M)/Z(M) = Z(M/Z(M))$ [9, Proposition 6.2]. A ring R is *right non-singular* if $Z_r(R) := Z(R_R) = 0$ (notice that the dual situation of "singular ring" never occurs, since $1 \notin Z(R_R)$). If R is a right non-singular ring, then $Z(-)$ is actually a radical, that is, $Z(M/Z(M)) = 0$ for every right R-module M_R (i.e., $Z_2(M) = Z(M)$ for every module M_R) [7, Proposition 1.23].

The following results extend to the non-commutative case results proved in [1]. A morphism $f\colon A \to B$ is said to be *e-epi* if $f(A)$ is essential in B.

Proposition 3.1 (4-lemma for e-exact sequences)**.** *Let R be a right non-singular ring and consider the following commutative diagram of right R-module morphisms with e-exact rows.*

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4. \end{array}$$

(1) *If t_1, t_3 are e-epis, t_4 is monic and B_2 is non-singular, then t_2 is e-epi.*
(2) *If t_1 is e-epi and t_2, t_4 are monic, then* $\ker(t_3)$ *is a singular module. In particular, if A_3 is a non-singular module, then t_3 is monic.*

Proposition 3.2 (5-lemma for e-exact sequences)**.** *Let R be a right non-singular ring and consider the following commutative diagram of right R-module morphisms with e-exact rows*:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} & & \downarrow{\scriptstyle t_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

where B_3 is a non-singular right R-modules.

(1) *If t_2 and t_4 are e-epis and t_5 is monic, then t_3 is e-epi.*
(2) *If t_1 is e-epi and t_2, t_4 are monic, then* $\ker(t_3)$ *is singular. In particular, if A_3 is non-singular, then t_3 is monic.*

Proposition 3.3 (3×3-lemma for e-exact sequences)**.** *Let R be a non-singular ring and consider the following commutative diagram of right R-modules*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle p_1} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle p_2} & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where A_2 and A_3 are non-singular modules. If the columns and the two bottom rows are e-exact, then the top row is also e-exact.

In order to prove Propositions 3.1, 3.2 and 3.3, recall a procedure to obtain a functor that, in some sense, can be viewed as a generalization of the "localization functor" for commutative rings to the non-commutative setting. We refer the reader to [7, Chapter 2] for more details about this construction. Anyway, we prefer to adopt the notation used in [9], in which these topics are treated from a more general point of view (we will discuss this more general framework in Section 4). Let R be a right non-singular ring, that is, $Z_r(R) = 0$. For any right R-module A, fix an injective envelope for $A/Z(A)$ and denote it by $A_{\mathfrak{G}}$. According to [7, Proposition 1.23(a)], $A_{\mathfrak{G}}$ is a non-singular module. Moreover, [7, Lemma 2.1] ensures that, for any given right R-module morphism $f\colon A_R \to B_R$, the induced morphism $\bar{f} : A/Z(A) \to B/Z(B)$ extends uniquely to a morphism $f_{\mathfrak{G}}\colon A_{\mathfrak{G}} \to B_{\mathfrak{G}}$. We have that $R_{\mathfrak{G}}$ is a ring containing R, and $(-)_{\mathfrak{G}}\colon$ Mod-$R \to$ Mod-$R_{\mathfrak{G}}$ turns out to be an exact functor.

Lemma 3.4. *Let R be a right non-singular ring and let $f\colon A \to B$ be an e-epi right R-module morphism. Then*

(1) $\bar{f}\colon A/Z(A) \to B/Z(B)$ *is e-epi*;
(2) $f_{\mathfrak{G}}\colon A_{\mathfrak{G}} \to B_{\mathfrak{G}}$ *is surjective.*

Proof. (1) Let $b \in B \setminus Z(B)$. By hypothesis, there exists $I \leq_e R_R$ such that $0 \neq bI \subseteq f(A)$, and therefore $(b + Z(B))I \subseteq \bar{f}(A/Z(A))$. Moreover, $bI \nsubseteq Z(B)$. Indeed, if $bI \subseteq Z(B)$, then $b + Z(B) \in Z(B/Z(B)) = 0$, which contradicts the fact that $b \notin Z(B)$. It follows that $\bar{f}$ is e-epi.

(2) By (1), $\bar{f}(A/Z(A))$ is essential in $B_{\mathfrak{G}} = E(B/Z(B))$, therefore $f_{\mathfrak{G}}(A_{\mathfrak{G}}) \leq_e B_{\mathfrak{G}}$. Since $B_{\mathfrak{G}}$ is a non-singular module, then so is $f_{\mathfrak{G}}(A_{\mathfrak{G}})$. In particular, $f_{\mathfrak{G}}(A_{\mathfrak{G}})$ is injective by [9, Proposition 7.4], hence $f_{\mathfrak{G}}(A_{\mathfrak{G}}) = B_{\mathfrak{G}}$. □

Corollary 3.5. *Let R be a right non-singular ring. If* $0 \longrightarrow A_R \longrightarrow B_R \longrightarrow C_R \longrightarrow 0$ *is a short e-exact sequence, then the sequence* $0 \longrightarrow A_{\mathfrak{G}} \longrightarrow B_{\mathfrak{G}} \longrightarrow C_{\mathfrak{G}} \longrightarrow 0$ *is exact.*

Proof. Since $(-)_{\mathfrak{G}}$ is an exact functor, it preserves kernels and images. By Lemma 3.4(2), if $f\colon M \to N$ is an e-epi right R-module morphism, then $f_{\mathfrak{G}}\colon M_{\mathfrak{G}} \to N_{\mathfrak{G}}$ is surjective. □

Lemma 3.6. *Let R be a right non-singular ring and let $f\colon A \to B$ be a right R-module morphism, where B is non-singular. If $f_{\mathfrak{G}}\colon A_{\mathfrak{G}} \to B_{\mathfrak{G}}$ is surjective, then f is e-epi.*

Proof. Assume, by contradiction, $f_{\mathfrak{G}}\colon A_{\mathfrak{G}} \to B_{\mathfrak{G}}$ surjective, but f not e-epi. Then there exists a nonzero submodule C of B such that $f(A) \cap C = 0$. Now f factorizes as $f = \varepsilon \circ f'$, where $f' = f|^{f(A)}\colon A \to f(A)$ is the corestriction of f and $\varepsilon\colon f(A) \hookrightarrow B$ is the inclusion. It follows that the image $f(A)$ of ε is not essential in B, so that $E(B) = E(f(A)) \oplus C'$ for some non-zero module C'. Applying the functor $(-)_{\mathfrak{G}}$ to the factorization $f = \varepsilon \circ f'$, we get that $f_{\mathfrak{G}} = \varepsilon_{\mathfrak{G}} \circ f'_{\mathfrak{G}}$. Now $f_{\mathfrak{G}}$ epi implies $\varepsilon_{\mathfrak{G}}$ epi. But since $f(A) \leq B$ are non-singular modules, $\varepsilon_{\mathfrak{G}}$ is the splitting embedding of $E(f(A))$ into $E(B)$, which is not epi because $C' \neq 0$.

By assumption, $B \leq_e B_{\mathfrak{G}} = E(B)$. Fix $b \in B \setminus \{0\}$. Then there exists $\alpha \in A_{\mathfrak{G}}$ such that $f_{\mathfrak{G}}(\alpha) = b$. Moreover, there exists $I \leq_e R_R$ such that $\alpha I \subseteq A/Z(A)$. Hence for every $i \in I$ there exists $a^{(i)} \in A$ such that $\alpha i = a^{(i)}$. Thus $\bar{f}(a^{(i)} + Z(A)) = bi + Z(B) = bi$, and so $f(a^{(i)}) = bi$. It follows that $bI \subseteq f(A)$ and $bI \neq 0$, because B is non-singular. Therefore f is e-epi. □

We are now in a position to prove Propositions 3.1, 3.2 and 3.3. Since all these results can be proved using the same procedure, we only write the proof of Proposition 3.1(1) and we leave the other proofs to the reader.

Applying the functor $(-)_{\mathfrak{G}}$ to the diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 \end{array}$$

we get a commutative diagram of $R_{\mathfrak{G}}$-modules with exact rows. Since t_1 and t_3 are e-epi, $(t_1)_{\mathfrak{G}}$ and $(t_3)_{\mathfrak{G}}$ are surjective by Lemma 3.4. The classical 4-lemma ensures that $(t_2)_{\mathfrak{G}}$ is surjective, hence t_2 is e-epi by Lemma 3.6. This proves Proposition 3.1(1). The other proofs are similar.

We now want to discuss the previous situation in the particular case in which R is a right Ore domain. A domain R is said to be *right Ore* if $aR \cap bR \neq 0$ for every pair of nonzero elements $a, b \in R$, that is, if every nonzero (principal) right ideal of R is essential. Given a right Ore domain R, it is possible to construct its classical right ring of fractions $Q := Q^r_{cl}(R)$ (see [8, Chapter 4]). We have that Q is a division ring which turns out to be a flat left R-module. Every element of Q is of the form rs^{-1} for suitable elements $r, s \in R$ and s nonzero. Moreover, R can be embedded into Q via the ring homomorphism defined by $r \mapsto r1^{-1}$. For any right R-module M_R, we can consider the "right localization" $M(R \setminus \{0\})^{-1} \cong M \otimes_R Q$, which is a right Q-module with elements of the form ms^{-1} for $m \in M$ and $s \in R \setminus \{0\}$. There is a natural map $M \to M(R \setminus \{0\})^{-1}$ whose kernel is the torsion submodule of M, $t(M) := \{m \in M \mid mr = 0 \text{ for some nonzero } r \in R\} = Z(M_R)$. Since in a right Ore domain all nonzero right ideals are essential, $Q \cong R_{\mathfrak{G}}$ as rings and the functor $- \otimes_R Q$ is naturally isomorphic to $(-)_{\mathfrak{G}}$. Hence we have the following result.

Lemma 3.7. *Let R be a right Ore domain with classical right ring of fractions Q and let $f\colon A_R \to B_R$ be a right R-module morphism.*

(1) *If f is e-epi, then $A \otimes_R Q \xrightarrow{f \otimes 1} B \otimes_R Q$ is surjective.*

(2) *If $A \otimes_R Q \xrightarrow{f \otimes 1} B \otimes_R Q$ is surjective and B is non-singular, then f is e-epi.*

In particular, if $0 \to A_R \to B_R \to C_R \to 0$ *is a short e-exact sequence, then the sequence* $0 \to A_R \otimes_R Q \to B_R \otimes_R Q \to C_R \otimes_R Q \to 0$ *is exact.*

Corollaries 2.3 and 3.5 show that the notion of short e-exact sequence is intermediate between that of becoming exact applying the functor P and that of becoming exact applying the functor $(-)_{\mathfrak{G}}$. The next result shows that there are no additive functors such that the notion of short e-exact sequence coincides with that of becoming exact applying F.

Theorem 3.8. *Let R be a ring that is not artinian semisimple. Then there do not exist additive functors $F\colon \text{Mod-}R \to \mathcal{A}$ into any abelian category $\mathcal{A}$ with the property that a cochain complex $A_R \xrightarrow{f} B_R \xrightarrow{g} C_R$ is e-exact if and only if the cochain complex $F(A_R) \xrightarrow{F(f)} F(B_R) \xrightarrow{F(g)} F(C_R)$ is exact.*

Proof. Suppose that such a category $\mathcal{A}$ and functor $F\colon \text{Mod-}R \to \mathcal{A}$ exist. First of all, notice that the functor F must be exact, because every exact sequence of R-modules is e-exact. Also, notice that if $f\colon A_R \to B_R$ is any essential monomorphism, then the cochain complex $0 \longrightarrow A_R \xrightarrow{f} B_R \longrightarrow 0$ is e-exact, so that the cochain complex $0 \longrightarrow F(A_R) \xrightarrow{F(f)} F(B_R) \longrightarrow 0$ is exact (F is additive, hence maps zero objects to zero objects and zero morphisms to zero morphisms). It follows that F maps essential monomorphisms to isomorphisms. Now for any singular module C_R, there exists an exact sequence $0 \longrightarrow A_R \xrightarrow{f} B_R \xrightarrow{g} C_R \longrightarrow 0$ with f an essential monomorphism [7, Proposition 1.20(b)]. Applying the exact functor F, we get an exact sequence $0 \longrightarrow F(A_R) \xrightarrow{F(f)} F(B_R) \xrightarrow{F(g)} F(C_R) \longrightarrow 0$ with $F(f)$ an isomorphism, so $F(C_R) = 0$. This proves that F necessarily annihilates all singular right R-modules. Now R is not artinian semisimple, so it has a maximal right ideal that is not a direct summand [3, Lemma 3.16], hence there is a simple right R-module S_R that is not projective. By [7, Proposition 1.24], S_R is singular. Now consider the embedding $f\colon S_R \to S_R \oplus S_R$ into the first direct summand. The cochain complex $S_R \xrightarrow{f} S_R \oplus S_R \longrightarrow 0$ is not e-exact, but, if we apply to it the functor F, we get the exact sequence $F(S_R) \xrightarrow{F(f)} F(S_R \oplus S_R) \longrightarrow 0$. This contradicts the hypothesis on F in the statement of the Theorem. □

4. Gabriel topologies

The results we have seen in Section 3 for the functor $(-)_{\mathfrak{G}}$ can be extended to a much more general setting. Let $\mathfrak{F}$ be any right Gabriel topology on a ring R [9, Chapter VI §5]. Thus $\mathfrak{F}$ corresponds to a hereditary torsion theory in Mod-R. Suppose this hereditary torsion theory *stable*, that is, the class of $\mathfrak{F}$-torsion modules is closed under injective envelopes [9, Chapter VI §7] and set for every right R-module M_R, $M_{\mathfrak{F}} := \varinjlim \operatorname{Hom}_R(I, M_R)$, where the direct limit is taken over the downwards directed family of ideals $I \in \mathfrak{F}$. The assignment

$M \mapsto M_{\mathfrak{F}}$ defines a left exact functor Mod-$R \to$ Mod-$R_{\mathfrak{F}}$ (see [9, Chapter IX §1]). We say that a cochain complex

$$\cdots \longrightarrow M^{i-1} \xrightarrow{f^{i-1}} M^i \xrightarrow{f^i} M^{i+1} \xrightarrow{f^{i+1}} \cdots$$

of right R-modules is $\mathfrak{F}$-*exact* if the cohomology modules $\ker(f^i)/f^{i-1}(M^{i-1})$ are $\mathfrak{F}$-torsion R-modules for every i. That is, if $(f^{i-1}(M^{i-1}) : x) \in \mathfrak{F}$ for every i and every $x \in \ker(f^i)$.

Lemma 4.1. *Let $\mathfrak{F}$ be a perfect stable Gabriel topology on R and let $f\colon A_R \to B_R$ be a right R-module morphism.*

(a) *If $B/f(A)$ is $\mathfrak{F}$-torsion, then the induced morphism $\widetilde{f}\colon A_{\mathfrak{F}} \to B_{\mathfrak{F}}$ is surjective.*

(b) *Conversely, if the morphism $\widetilde{f}\colon A_{\mathfrak{F}} \to B_{\mathfrak{F}}$ is surjective and B_R is $\mathfrak{F}$-torsionfree, then $B/f(A)$ is $\mathfrak{F}$-torsion.*

Proof. (a) Assume $B/f(A)$ is $\mathfrak{F}$-torsion. We have an exact sequence $A_R \xrightarrow{f} B_R \to B/f(A) \to 0$. Since $\mathfrak{F}$ is perfect stable, the localization functor $(-)_{\mathfrak{F}}$ and the tensor product functor $-\otimes_R R_{\mathfrak{F}}$ are naturally isomorphic [9, Proposition XI.3.4(e)] and ${}_RR_{\mathfrak{F}}$ is a flat left R-module [9, Proposition XI.3.11(b)]. It follows that the induced sequence $A_{\mathfrak{F}} \to B_{\mathfrak{F}} \to (B/f(A)_{\mathfrak{F}} \to 0$ is exact. But $(B/f(A)_{\mathfrak{F}}) = 0$ because $B/f(A)$ is $\mathfrak{F}$-torsion.

(b) Suppose $\widetilde{f}\colon A_{\mathfrak{F}} \to B_{\mathfrak{F}}$ surjective and B_R $\mathfrak{F}$-torsionfree. There is a commutative square

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ \downarrow{\scriptstyle\psi_A} & & \downarrow{\scriptstyle\psi_B} & & \\ A_{\mathfrak{F}} & \xrightarrow[\widetilde{f}]{} & B_{\mathfrak{F}} & \longrightarrow & 0. \end{array}$$

In order to prove that $B/f(A)$ is $\mathfrak{F}$-torsion, we must show that for every element $b \in B$ there exists $I \in \mathfrak{F}$ such that $bI \subseteq f(A)$. Now $\psi_B(b) \in B_{\mathfrak{F}}$ and $\widetilde{f}$ is onto, so that there exists an element $\widetilde{a} \in A_{\mathfrak{F}}$ with $\widetilde{f}(\widetilde{a}) = \psi_B(b)$. Since $A_{\mathfrak{F}} = \varinjlim \operatorname{Hom}(I, A_R)$ by [9, Proposition IX.1.7], the element $\widetilde{a} \in A_{\mathfrak{F}}$ is represented by a morphism $g\colon I' \to A_R$ in $\operatorname{Hom}(I', A_R)$ for some right ideal $I' \in \mathfrak{F}$. But $\widetilde{f}(\widetilde{a}) = \psi_B(b)$ in $B_{\mathfrak{F}}$, so there exists a right ideal $I \subseteq I'$, $I \in \mathfrak{F}$, such that the morphisms $\rho_b\colon R \to B$ (right multiplication by b) and $fg\colon I' \to B$ have the same restriction to I'. Then, for every $i \in I$, we get that $bi = fg(i)$. Therefore $bI \subseteq f(A_R)$, as desired. □

Lemma 4.2. *Let $\mathfrak{F}$ be a perfect stable Gabriel topology on a ring R. If the cochain complex $A_R \xrightarrow{f} B_R \xrightarrow{g} C_R$ is $\mathfrak{F}$-exact, then the sequence $A_{\mathfrak{F}} \xrightarrow{f} B_{\mathfrak{F}} \xrightarrow{g} C_{\mathfrak{F}}$ of right $R_{\mathfrak{F}}$-modules is exact.*

Proof. We have that $f = \varepsilon \circ f'$, where $\varepsilon\colon \ker g \to B$ is the inclusion and $f'\colon A_R \to \ker g$ is the corestriction of f to $\ker g$ (which contains $f(A_R)$).

Then $f'\colon A_R \to \ker g$ has an $\mathfrak{F}$-torsion cokernel, so that the induced mapping $\widetilde{f'}\colon A_{\mathfrak{F}} \to (\ker g)_{\mathfrak{F}}$ is onto by Lemma 4.1(a). Applying the exact functor $-\otimes R_{\mathfrak{F}}$ to the exact sequence $0 \to \ker g \xrightarrow{\varepsilon} B_R \xrightarrow{g} C_R$, we get the exact sequence $0 \to (\ker g)_{\mathfrak{F}} \xrightarrow{\widetilde{\varepsilon}} B_{\mathfrak{F}} \xrightarrow{\widetilde{g}} C_{\mathfrak{F}}$. Therefore $\ker(\widetilde{g}) = \widetilde{\varepsilon}((\ker g)_{\mathfrak{F}}) = \widetilde{\varepsilon}(\widetilde{f'}(A_{\mathfrak{F}})) = \widetilde{f}(A_{\mathfrak{F}})$, as desired. □

As one might expect, it is also possible to prove suitable versions of the 4-lemma, 5-lemma and 3×3-lemma in this setting. Here, we present the 4-lemma with $\mathfrak{F}$-exact sequences. The other statements (and their proofs) are the obvious ones.

Proposition 4.3 (4-lemma with $\mathfrak{F}$-exact sequences)**.** *Let $\mathfrak{F}$ be a perfect stable Gabriel topology on a ring R, and consider the following commutative diagram of right R-module morphisms with $\mathfrak{F}$-exact rows*:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \downarrow{\scriptstyle t_4} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 \end{array}$$

(1) *If t_1, t_3 are $\mathfrak{F}$-epis, t_4 is monic and B_2 is $\mathfrak{F}$-torsionfree, then t_2 is $\mathfrak{F}$-epi.*
(2) *If t_1 is $\mathfrak{F}$–epi and t_2, t_4 are monic, then* $\ker(t_3)$ *is an $\mathfrak{F}$-torsion R-module. In particular, if A_3 is a $\mathfrak{F}$-torsionfree R-module, then t_3 is monic.*

Proof. Apply the functor $-\otimes R_{\mathfrak{F}}$ to the commutative diagram, getting a commutative diagram of $R_{\mathfrak{F}}$-modules with exact rows. The classical 4-lemma and Lemma 4.1 allow us to conclude. □

Remark 4.4. Let R be a non-singular ring and let $\mathfrak{G}$ be the right Goldie topology, that is, the Gabriel topology given by the essential right ideals of R. In this case, the $\mathfrak{G}$-torsion R-modules are precisely the singular R-modules and the notion of $\mathfrak{G}$-exactness can be compared with that of e-exactness. Recall that, given a right R-module extension $A \leq B$, if $A \leq_e B$, then B/A is a singular module, but the converse may fail to be true (even for abelian groups; see [7, pp. 31–32]). This means that the notion of e-exactness is stronger than that of $\mathfrak{G}$-exactness, that is, given a cochain complex

$$\cdots \longrightarrow M^{i-1} \xrightarrow{f^{i-1}} M^i \xrightarrow{f^i} M^{i+1} \xrightarrow{f^{i+1}} \cdots$$

of right R-modules, if the cochain if e-exact, then it is also $\mathfrak{G}$-exact, but the converse is not true in general.

References

[1] I. Akray and A. Zebari, *Essential exact sequences*, Commun. Korean Math. Soc. **35** (2020), no. 2, 469–480. https://doi.org/10.4134/CKMS.c190243

[2] M. J. Arroyo Paniagua, A. Facchini, M. Gran, and G. Janelidze, *What is the spectral category?*, in "Rings and Factorizations", A. Facchini, M. Fontana, A. Geroldinger, and B. Olberding Eds., Springer, New York, 2020, pp. 135–152.
[3] A. Facchini, *Module theory*, Progress in Mathematics, 167, Birkhäuser Verlag, Basel, 1998.
[4] T. Fritz, *Categories of fractions revisited*, Morfismos **15** (2011), no. 2, 19–38.
[5] P. Gabriel and U. Oberst, *Spektralkategorien und reguläre Ringe im von-Neumannschen Sinn*, Math. Z. **92** (1966), 389–395. `https://doi.org/10.1007/BF01112218`
[6] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergeb. Math. Grenzgeb, 35, Springer-Verlag New York, Inc., New York, 1967.
[7] K. R. Goodearl, *Ring Theory*, Marcel Dekker, Inc., New York, 1976.
[8] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, 189, Springer-Verlag, New York, 1999. `https://doi.org/10.1007/978-1-4612-0525-8`
[9] B. Stenström, *Rings of Quotients*, Springer-Verlag, New York, 1975.

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