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AN EXTENSION OF ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with unity. The extension of annihilating-ideal graph of R, $\overline{\mathbb{AG}}(R)$, is the graph whose vertices are nonzero annihilating ideals of R and two distinct vertices I and J are adjacent if and only if there exist $n, m \in \mathbb{N}$ such that $I^n J^m = (0)$ with $I^n, J^m \neq (0)$. First, we differentiate when $\mathbb{AG}(R)$ and $\overline{\mathbb{AG}}(R)$ coincide. Then, we have characterized the diameter and the girth of $\overline{\mathbb{AG}}(R)$ when R is a finite direct products of rings. Moreover, we show that $\overline{\mathbb{AG}}(R)$ contains a cycle, if $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$.

1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. The set of zero-divisors, the annihilator of an element x, the nilradical and the set of all ideals of a ring R are denoted by $Z(R)$, $Ann(x)$, $Nil(R)$ and $\mathbb{I}(R)$, respectively. For a nonzero nilpotent element x of R, n_x denotes the index of nilpotency of x. A graph G is *connected* if there is a path between every two distinct vertices. For every positive integer n, we denote a path of length n, by P_n. Also, $d(x, y)$ is the length of the shortest path between x and y for two distinct vertices $x, y \in V(G)$. The *girth* of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G and $gr(G) = \infty$, if G contains no cycles. A *complete r-partite graph* is a graph whose vertex set is partitioned into r separated subsets such that each vertex is joined to every other vertex that is not in the same subset. For positive integers m and n, the *complete graph* with n vertices is denoted by K^n and the *complete bipartite graph* with parts of sizes m and n will be denoted by $K^{m,n}$.

Recall from [3], an ideal I of R is said to be an *annihilating ideal* if there exists a nonzero ideal J such that $IJ = (0)$. As in [3], we denote sets of all annihilating ideals and nonzero annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)^\star$ respectively. M. Behboodi and Z. Rakeei [3] introduced the concept of the

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annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$. This graph is an undirected simple graph whose vertex set is $\mathbb{A}(R)^\star$ and distinct vertices I and J are adjacent in $\mathbb{AG}(R)$ if and only if $IJ = (0)$. Also, D. Bennis, J. Mikram and F. Taraza [5] introduced the concept of the *extended zero divisor graph* of R, denoted by $\overline{\Gamma}(R)$, such that its vertex set consists of all its nonzero zero divisors and that two distinct vertices x and y are adjacent if and only if there exist $m, n \in \mathbb{N}$ such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$.

In this paper, we follow the same approach of [5] and we introduce the extension of the annihilating-ideal graph of R, $\mathbb{AG}(R)$, which we call, following the terminology of [5], the *extended of the annihilating-ideal graph* of R, $\overline{\mathbb{AG}}(R)$. This is a graph whose vertex set is $\mathbb{A}(R)^\star$ and two distinct vertices I and J are adjacent if and only if there exist $n, m \in \mathbb{N}$ such that $I^n J^m = (0)$ with $I^n, J^m \neq (0)$. It is obvious that $\mathbb{AG}(R)$ is a subgraph of $\overline{\mathbb{AG}}(R)$. Also, note that $\overline{\mathbb{AG}}(R)$ is the empty graph if and only if R is an integral domain.

In Section 2, we discuss about the question of when $\overline{\mathbb{AG}}(R)$ and $\mathbb{AG}(R)$ coincide. Among other results, we investigate some conditions on rings such that the annihilating-ideal graph of finite direct product of rings and extended graph of them are the same. Section 3 is devoted to study the diameter of the extended graph. In Theorem 3.2, we study when a vertex in $\overline{\mathbb{AG}}(R)$ is adjacent to every other vertex. Using this theorem we determine when $Z(R)$ is an ideal. Finally, in Section 4, we study the girth of $\overline{\mathbb{AG}}(R)$. It is shown in Theorem 4.2, when $\overline{\mathbb{AG}}(R)$ contains a cycle. Moreover, in Theorem 4.5, by considering a ring with some properties is proven that $gr(\overline{\mathbb{AG}}(R)) \neq 4$. Among other results, in Theorem 4.6, under some conditions is shown that $\overline{\mathbb{AG}}(R)$ is a complete bipartite graph.

2. When do $\overline{\mathbb{AG}}(R)$ and $\mathbb{AG}(R)$ coincide?

Let R be a commutative ring with unity and $\mathbb{I}(R)$ be the set of ideals of R. We set

$$N_{\mathbb{I}}(R) = \{J \in \mathbb{I}(R) : \exists n \in \mathbb{N} \setminus \{1\},\ J^n = (0)\},$$
$$\mathbb{A}(R) = \{J \in \mathbb{I}(R) : \exists K \in \mathbb{I}(R)^\star,\ \ JK = (0)\} \text{ and}$$
$$\mathbb{A}(R)^\star = \mathbb{A}(R) \setminus \{(0)\}.$$

Definition. Extension of annihilating-ideal graph $\mathbb{AG}(R)$, denoted by $\overline{\mathbb{AG}}(R)$, is the simple graph whose vertex set is $\mathbb{A}(R)^\star$ and two distinct vertices I and J are adjacent if and only if there exist $n, m \in \mathbb{N}$ such that $I^n J^m = (0)$ with $I^n, J^m \neq (0)$.

We are interested in the study of a ring R with nonzero ideals, whose $\overline{\mathbb{AG}}(R)$ and $\mathbb{AG}(R)$ coincide. The following lemma extends [5, Lemma 2.2] and has a key role in the proof of the main results of this section. Recall that, for a nonzero nilpotent ideal J of R, n_J is the index of nilpotency of J.

Lemma 2.1. *Let R be a ring and $J \in \mathbb{I}(R)^\star$. Then the following statements hold.*

(1) *If J is a nilpotent ideal, then $Ann_R(J) \subset Ann_R(J^n)$, $n \geq 2$.*
(2) *If J is not a nilpotent ideal, then*

$$Ann_R(J^2) = Ann_R(J) \Longleftrightarrow Ann_R(J^n) = Ann_R(J),\ n \geq 2.$$

Proof. (1) Let J be a nilpotent ideal of $\mathbb{A}(R)$. If $n_J = 2$, then for every integer $n \geq 2$,

$$Ann_R(J) \subset R = Ann_R((0)) = Ann_R(J^n),$$

as required. Now, let $n_J \geq 3$ and also suppose that there is $n \geq 2$ such that $Ann_R(J^n) = Ann_R(J)$. We claim that $2 \leq n \leq n_J - 1$ (for $n \geq n_J$, $Ann_R(J^n) = Ann_R((0)) = R$, a contradiction). Thus

$$J^{n_J - n} \subseteq Ann_R(J^n) = Ann_R(J).$$

This implies that $J^{n_J - n+1} = (0)$, which is absurd, since $2 \leq n_J - n + 1 \leq n_J - 1$.

(2) Let J be a nonnilpotent ideal such that $Ann_R(J^2) = Ann_R(J)$. By induction, we have $Ann_R(J^{n-1}) = Ann_R(J)$. Now, if $y \in Ann_R(J^n)$, then

$$yJ \subseteq Ann_R(J^{n-1}) = Ann_R(J),$$

and so, $yJ^2 = (0)$. Also, by hypothesis $y \in Ann_R(J)$. Thus, the assertion holds. The converse is straightforward. □

Theorems 2.2 and 2.4 are inspired from [5, Theorem 2.1].

Theorem 2.2. *Let R be a ring such that satisfies the following conditions*:
(i) *For every $J \in N_{\mathbb{I}}(R)$, $n_J = 2$, and*
(ii) *For every $J \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$, $Ann_R(J^2) = Ann_R(J)$.*
Then $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$.

Proof. Let K and J be adjacent vertices in $\overline{\mathbb{AG}}(R)$. Then there exist $n, m \in \mathbb{N}$ such that $K^n, J^m \neq (0)$ and $K^n J^m = (0)$. We consider three cases:

Case (1): If $K, J \in N_{\mathbb{I}}(R)$, then $n_K = n_J = 2$. This means K and J are adjacent vertices in $\mathbb{AG}(R)$.

Case (2): If $K \notin N_{\mathbb{I}}(R)$ and $J \in N_{\mathbb{I}}(R)$, then there is $n \in \mathbb{N}$ such that $K^n J = (0)$. By the assumption and Lemma 2.1, $J \subseteq Ann_R(K)$. Therefore, K and J are adjacent vertices in $\mathbb{AG}(R)$.

Case (3): If K, $J \notin N_{\mathbb{I}}(R)$, then

$$K^n \subseteq Ann_R(J^m) \subseteq Ann_R(J).$$

Thus, $K^n J = (0)$. Also, $J \subseteq Ann_R(K^n) \subseteq Ann_R(K)$. Therefore, K and J are adjacent in $\mathbb{AG}(R)$, as required. □

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let R be a reduced ring. Then $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$.*

Proof. Suppose that $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$. Then by Theorem 2.2, there exists $I \in \mathbb{A}(R)^\star$ such that $Ann_R(I) \neq Ann_R(I^2)$. Thus, there is $z \in Ann_R(I^2)$ such that $zI^2 = (0)$ and $zI \neq (0)$. This yields $zI \in N_{\mathbb{I}}(R)$, which is a contradiction. □

It must be noted that for $R = \mathbb{Z}_{p^3}$, $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$, but R is not a reduced ring. Therefore, the converse of the above corollary is not true.

Theorem 2.4. *Let R be a ring and $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$. Then the following statements hold.*

(1) *For every $J \in N_{\mathbb{I}}(R)$, $n_J \leq 3$.*

(2) *For every $J \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$, $Ann_R(J^2) = Ann_R(J)$.*

Proof. (1) Suppose that there exists $J \in N_{\mathbb{I}}(R)$ such that $n_J \geq 4$. By Lemma 2.1,

$$Ann_R(J) \subset Ann_R(J^n),\ n \geq 2.$$

Thus, there is $y \in Ann_R(J^n) \setminus Ann_R(J)$ such that $RyJ^n = (0)$ and $RyJ \neq (0)$. If $Ry \neq J$, then Ry and J are adjacent in $\overline{\mathbb{AG}}(R)$, but not in $\mathbb{AG}(R)$. Otherwise, Ry^2 and J are adjacent in $\overline{\mathbb{AG}}(R)$ whereas they are not adjacent in $\mathbb{AG}(R)$, both of which contradicted by $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$.

(2) Suppose that $J \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$. We have $Ann_R(J) \subseteq Ann_R(J^2)$, it remains to show the other inclusion. Let y be an element of $Ann_R(J^2)$. Then J and Ry are adjacent in $\overline{\mathbb{AG}}(R)$, which equal to $\mathbb{AG}(R)$. Hence, $RyJ = (0)$ and $y \in Ann_R(J)$, i.e., $Ann_R(J) = Ann_R(J^2)$. □

Theorem 2.5. *Let R be a Noetherian ring and $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$. Then the following assertions hold*:

(1) *For every $J \in N_{\mathbb{I}}(R)$, $n_J \leq 3$.*

(2) *For every $J \in \mathbb{A}(R)$, $\mathbb{I}(\sqrt{Ann_R(J)}) \setminus N_{\mathbb{I}}(R) \subseteq \mathbb{I}(Ann_R(J))$.*

Proof. (1) It follows from part (1) of Theorem 2.4.

(2) Suppose that $K \in \mathbb{I}(\sqrt{Ann_R(J)}) \setminus N_{\mathbb{I}}(R)$. Since R is a Noetherian ring, there is $n \in \mathbb{N}$ such that $K^n J = (0)$. Thus, by hypothesis J and K are adjacent in $\mathbb{AG}(R)$. This implies that $K \in \mathbb{I}(Ann_R(J))$. □

Theorem 2.6. *Let R be a ring such that it satisfies the following conditions*:

(i) *For every $J \in N_{\mathbb{I}}(R)$, $n_J = 2$, and*

(ii) *For every $J \in \mathbb{A}(R)$, $\mathbb{I}(\sqrt{Ann_R(J)}) \setminus N_{\mathbb{I}}(R) \subseteq \mathbb{I}(Ann_R(J))$.*

Then $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$.

Proof. Let K and J be adjacent vertices in $\overline{\mathbb{AG}}(R)$. Then there exist $n, m \in \mathbb{N}$ such that $K^n J^m = (0)$ with $K^n, J^m \neq (0)$. Three cases occur:

Case (1): If $K, J \in N_{\mathbb{I}}(R)$, then $n_K = n_J = 2$, and so, $n = m = 1$. This means K and J are adjacent in $\mathbb{AG}(R)$.

Case (2): If $K \notin N_{\mathbb{I}}(R)$ and $J \in N_{\mathbb{I}}(R)$, then there is $n \in \mathbb{N}$ such that $K^n J = (0)$, and so, $K \subseteq \sqrt{Ann_R(J)}$. Therefore, $K \in \mathbb{I}(\sqrt{Ann_R(J)}) \setminus N_{\mathbb{I}}(R)$. This yields K and J are adjacent in $\mathbb{AG}(R)$.

Case (3): suppose that K, $J \notin N_{\mathbb{I}}(R)$. By hypothesis

$$K \subseteq \sqrt{Ann_R(J^m)} \Rightarrow K \in \mathbb{I}(Ann_R(J^m)).$$

This implies that $J \subseteq \sqrt{Ann_R(K)}$, and so, $J \in \mathbb{I}(\sqrt{Ann_R(K)}) \setminus N_{\mathbb{I}}(R)$. Hence, $KJ = (0)$, as required. □

Proposition 2.7. *Let R be a ring and $J \in \mathbb{A}(R)^{\star} \setminus N_{\mathbb{I}}(R)$ such that the following conditions hold:*

(i) $(0) \subset Nil(R) \subset Z(R)$,
(ii) $Ann_R(J^2) = Ann_R(J)$.

Then for every $K \in N_{\mathbb{I}}(R)$, the following statements hold.

(1) $Ann_R(J) \subseteq Ann_R(K)$.
(2) $Ann_R(J)N_{\mathbb{I}}(R) = (0)$.
(3) *If $n_K \neq 2$, then $K \nsubseteq Ann_R(J)$ and $Ann_R(K) \subseteq Nil(R)$. Moreover, if $x \in Ann_R(J)$, then $x^2 = 0$.*

Proof. (1) Suppose that

$$Ann_R(J) \nsubseteq Ann_R(K).$$

Then, there exists $x \in Ann_R(J) \setminus Ann_R(K)$ such that $x(K + J) \neq (0)$ whereas $x(J + K)^{n_K} = (0)$ (since $K^{n_K} = (0)$). Also, we know $(K + J) \notin N_{\mathbb{I}}(R)$. Therefore,

$$Ann_R(K + J) \neq Ann_R(K + J)^{n_K},$$

this is a contradiction with Lemma 2.1.

(2) Suppose that $K \in N_{\mathbb{I}}(R)$. By (1), $Ann_R(J) \subseteq Ann_R(K)$, and so,

$$Ann_R(J)K \subseteq Ann_R(K)K = (0).$$

This implies that $Ann_R(J)N_{\mathbb{I}}(R) = (0)$.

(3) To the contrary, assume that $K \subseteq Ann_R(J)$. By (1), $K^2 = (0)$, a contradiction. Thus, the first statement is proved.

To prove the second statement, suppose that there is $y \in Ann_R(K)$ such that $y \notin Nil(R)$. By (1),

$$Ann_R(Ry) \subseteq Ann_R(K),$$

and so, $K \subseteq Ann_R(K)$, which is absurd. To prove the "moreover" statement, it is enough to replace Rx with K. □

Proposition 2.8. *Let $(R_i)_{1 \leq i \leq n}$ be a finite family of rings with $n \in \mathbb{N} \setminus \{1\}$. Then $\overline{\mathbb{AG}}(\prod_{i=1}^{n} R_i) = \mathbb{AG}(\prod_{i=1}^{n} R_i)$ if and only if R_i is a reduced ring for every $1 \leq i \leq n$.*

Proof. It suffices to prove the case where $n = 2$. Suppose that R_1 is not a reduced ring. Then there exists $0 \neq x_1 \in R$ such that $x_1^2 = 0$. We have

$$(R_1, (0))(R_1x_1, R_2) = (R_1x_1, (0)) \neq ((0), (0))$$

but

$$(R_1, (0))(R_1x_1, R_2)^2 = ((0), (0)).$$

Then, $\overline{\mathbb{AG}}(R_1 \times R_2) \neq \mathbb{AG}(R_1 \times R_2)$, a contradiction. Also, we know that a direct product of reduced rings is a reduced ring. Thus, the converse follows from Corollary 2.3. □

Example 2.9. Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the prime factorization of an integer n with $k \in \mathbb{N}$. Let $m := \max\{\alpha_i : 1 \leq i \leq k\}$. Then $\overline{\mathbb{AG}}(\mathbb{Z}_n) \neq \mathbb{AG}(\mathbb{Z}_n)$ if and only if $k = 1$ and $m \geq 4$ or $k \geq 2$ and $m \geq 2$. We can conclude $\overline{\mathbb{AG}}(\mathbb{Z}_n) = \mathbb{AG}(\mathbb{Z}_n)$ if and only if $n \in \{p, p^2, p^3\}$ or $m = 1$.

3. Diameter of $\overline{\mathbb{AG}}(R)$

In this section, we study the diameter of the extended annihilating-ideal graph of a ring R.

Theorem 3.1. *Let R be a ring. Then $\overline{\mathbb{AG}}(R)$ is connected and diam$(\overline{\mathbb{AG}}(R)) \leq 3$.*

Proof. Note that $\mathbb{AG}(R)$ is a subgraph of $\overline{\mathbb{AG}}(R)$. Hence the result follows from [3, Theorem 2.1]. $\square$

The following theorem is the first main results of this section, it shows when $\overline{\mathbb{AG}}(R)$ has a vertex adjacent to every other vertex. This is similar to [2, Theorem 2.5].

Theorem 3.2. *Let I be a vertex of $\overline{\mathbb{AG}}(R)$. Then I is adjacent to every other vertex if and only if only one of the following statements hold*:

(1) *$R \cong K \times D$, where K is a field and D is an integral domain.*
(2) $\mathbb{A}(R)^\star = \{J \in \mathbb{I}(R) : \exists m \in \mathbb{N},\ J^m \neq (0), J^m I^{n_I - 1} = (0)\}$.
(3) $\mathbb{A}(R)^\star = \{J \in \mathbb{I}(R) : \exists m \in \mathbb{N},\ J^m \neq (0), J^m a = (0)\}$, *where $0 \neq a \in I$ and $Ra \subset I$.*

Proof. Suppose that I is adjacent to every other vertex of $\overline{\mathbb{AG}}(R)$. If I is a nilpotent ideal, then for each ideal $J \in \mathbb{A}(R)^\star$, there are $n, m \in \mathbb{N}$ such that $I^n J^m = (0)$ with $I^n, J^m \neq (0)$. Thus, $n < n_I$ and $I^{n_I - 1} J^m = (0)$, *i.e.*,

$$\mathbb{A}(R)^\star = \{J \in \mathbb{I}(R) : \exists m \in \mathbb{N}, J^m \neq (0), J^m I^{n_I - 1} = (0)\}.$$

Now, assume that $I \notin N_{\mathbb{I}}(R)$. Then $I^2 = I$. If not, then there are $n, m \in \mathbb{N}$ such that $(I^2)^n I^m = I^{2n+m} = (0)$, a contradiction. Now, let I be a minimal ideal, by Brauer's Lemma, there is an idempotent element e of R such that $R_1 = I = Re$ and $R = R_1 \oplus R_2$. Clearly, $(R_1, (0))$ is adjacent to other vertices of $\overline{\mathbb{AG}}(R)$. Hence, for each $0 \neq c \in R_1$, $(R_1 c, (0))$ is an annihilating ideal of R. If $R_1 c \neq R_1$, then there exists $m \in \mathbb{N}$ such that

$$(R_1, (0))(R_1 c, (0))^m = ((0), (0)),$$

i.e., $R_1 c^m = (0)$, a contradiction. Therefore, R_1 is a field. We claim that R_2 is an integral domain. Suppose that otherwise. Then there is $b \in Z(R_2)^\star$ such that $(R_1, R_2 b) \in \mathbb{A}(R)^\star$ is not adjacent to $(R_1, (0))$, a contradiction. Thus, R_2 must be an integral domain. Now, if I is not a minimal ideal, then there is $0 \neq a \in I$ such that $Ra \subset I$ and it is also adjacent to every other vertex of $\mathbb{A}(R)^\star$. Therefore,

$$\mathbb{A}(R)^\star = \{J \in \mathbb{I}(R) : \exists m \in \mathbb{N},\ J^m \neq (0), J^m a = (0)\}.$$

The converse is straightforward. □

Corollary 3.3. *Let R be a ring. If I is a nilpotent ideal and adjacent to every other vertex of $\overline{\mathbb{AG}}(R)$, then $Z(R)$ is an ideal.*

Proof. According to the proven process Theorem 3.2,

$$\mathbb{A}(R)^\star = \{J \in \mathbb{I}(R) : \exists m \in \mathbb{N}, J^m \neq (0), J^m I^{n_I - 1} = (0)\}.$$

Now, if $x \in Z(R)^\star$, then $Rx \in \mathbb{A}(R)^\star$ and there exists $m \in \mathbb{N}$ such that $(Rx)^m I^{n_I-1} = (0)$. This implies that $x \in \sqrt{Ann_R(I^{n_I-1})}$, and so, $Z(R) = \sqrt{Ann_R(I^{n_I-1})}$ is an ideal. □

Corollary 3.4. *Let R be an Artinian ring and I be a nilpotent ideal which is adjacent to every other vertex of $\overline{\mathbb{AG}}(R)$. Then R is a local ring.*

Proof. By assumption, R is a finite direct sum of commutative local rings. We claim that R is a local ring. Suppose that $R = R_1 \oplus R_2$, where R_i, $i = 1, 2$, are local rings. Then $I = I_1 \oplus I_2$ such that I_i, $i = 1, 2$, are nilpotent ideals of R_i. Without loss of generality, let $n_{I_1} \geq n_{I_2}$. By using part (2) of Theorem 3.2, $I^{n_I-1}(R_1, (0))^m = (I_1^{n_{I_1}-1}, (0))$. That is a contradiction. Therefore, R is a local ring. □

Theorem 3.5. *Let R be a Noetherian ring. Then $\overline{\mathbb{AG}}(R)$ is a complete graph if and only if either $R \cong K_1 \oplus K_2$, where K_1 and K_2 are fields or $\mathbb{A}(R)^\star = N_{\mathbb{I}}(R)$ and $I^{n_I-1}J^{n_J-1} = (0)$, where $I, J \in \mathbb{A}(R)^\star$.*

Proof. The "if" statement holds trivially. Conversely, suppose that $\overline{\mathbb{AG}}(R)$ is complete and $I \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$. By proof of Theorem 3.2, $I = I^2$ and $R \cong K \times D$. Now, for $0 \neq x \in D$, $((0), Rx)$ and $((0), D)$ are adjacent. Therefore, D is a field. □

Theorem 3.6. *Let R be a reduced ring. Then $\overline{\mathbb{AG}}(R)$ is a complete graph if and only if $R \cong K_1 \times K_2$, where K_1 and K_2 are fields.*

Proof. It follows from Corollary 2.3 and [4, Theorem 2.7]. □

We set

$$\overline{\mathbb{A}}(R)^\star = \{I^{n_I-1} \ : \ I \in N_{\mathbb{I}}(R)\} \text{ and}$$
$$\overline{\mathbb{A}}(R)^{\star^2} = \{I^{n_I-1}J^{n_J-1} \ : \ I, J \in N_{\mathbb{I}}(R)\}.$$

Corollary 3.7. *Let R be a Noetherian ring and $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$. Then $\overline{\mathbb{AG}}(R)$ is complete if and only if $\mathbb{A}(R)^\star = N_{\mathbb{I}}(R)$ and $\overline{\mathbb{A}}(R)^{\star^2} = \{(0)\}$.*

Proof. Suppose that $\overline{\mathbb{AG}}(R)$ is complete, by assumption and Theorem 3.5, the assertion holds. The converse is clear. □

Theorem 3.8. *Let R be a ring with $\mathbb{A}(R)^\star = N_{\mathbb{I}}(R) \neq \emptyset$. Then the following statements hold.*

(1) *If* $|\mathbb{A}(R)^{\star}| = 1$*, then* $diam(\overline{\mathbb{AG}}(R)) = 0$.
(2) *If* $|\mathbb{A}(R)^{\star}| \geq 2$*, then*

$$diam(\overline{\mathbb{AG}}(R)) = \begin{cases} 2 & \overline{\mathbb{A}}(R)^{\star^2} \neq \{(0)\}, \\ 1 & otherwise. \end{cases}$$

Proof. (1) If $|\mathbb{A}(R)^{\star}| = 1$, then by [1, Theorem 3.4] either $R \cong \frac{K[x]}{(x^2)}$, where K is a field or $R \cong L$, where L is a coefficient ring of characteristic p^2 and $diam(\overline{\mathbb{AG}}(R)) = 0$.

(2) Suppose that $\overline{\mathbb{A}}(R)^{\star^2} = \{(0)\}$. Clearly, $\overline{\mathbb{AG}}(R)$ is a complete graph, and so, $diam(\overline{\mathbb{AG}}(R)) = 1$. Now, let $\overline{\mathbb{A}}(R)^{\star^2} \neq \{(0)\}$. Then there are distinct vertices $I, J \in \mathbb{A}(R)^{\star}$ such that $I^{n_I-1}J^{n_J-1} \neq (0)$. If $I = IJ$, then $IJ^{n_J-1} = IJ^{n_J} = (0)$, a contradiction. Similarly, $IJ \neq J$. Therefore, $I - IJ - J$ is a path with length 2 between I and J in $\overline{\mathbb{AG}}(R)$. Hence $diam(\overline{\mathbb{AG}}(R)) = 2$. □

The following proposition compares the diameter of the graph $\overline{\mathbb{AG}}(\prod_{i=1}^{n} R_i)$ with the diameter of $\mathbb{AG}(\prod_{i=1}^{n} R_i)$.

Proposition 3.9. *Let* $R = \prod_{i=1}^{n} R_i$*, where* $(R_i)_{1\leq i\leq n}$ *is a finite family of rings with* $n \in \mathbb{N} \setminus \{1\}$*. If* $n \geq 3$*, then* $diam(\overline{\mathbb{AG}}(R)) = diam(\mathbb{AG}(R)) = 3$*. Otherwise, the following statements hold.*

(1) $\mathbb{AG}(R)$ *is complete if and only if* $R_i, i = 1, 2$*, are fields.*
(2) *If* $R_i, i = 1, 2$*, are integral domains and* $|\mathbb{I}(R_1)| \geq 3$ *or* $|\mathbb{I}(R_2)| \geq 3$*, then* $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$ *and* $diam(\mathbb{AG}(R)) = 2$*. In this case* $\mathbb{AG}(R)$ *is a complete bipartite graph.*
(3) *If* $\mathbb{A}(R_i)^{\star} \setminus N_{\mathbb{I}}(R_i) \neq \emptyset$ *for at least one of* $R_i, i = 1, 2$*, then*

$$diam(\overline{\mathbb{AG}}(R)) = diam(\mathbb{AG}(R)) = 3.$$

(4) *If* $\mathbb{A}(R_i)^{\star} = N_{\mathbb{I}}(R_i)$ *for at least one of* $R_i, i = 1, 2$*, then*

$$diam(\mathbb{AG}(R)) = 3 \text{ and } diam(\overline{\mathbb{AG}}(R)) = 2.$$

Proof. For $n \geq 3$,

$$((0), R_2, \ldots, R_n) - (R_1, (0), \ldots, (0)) - ((0), \ldots, (0), R_n) - (R_1, \ldots, R_{n-1}, (0))$$

is the shortest path between $((0), R_2, \ldots, R_n)$ and $(R_1, \ldots, R_{n-1}, (0))$. Hence,

$$diam(\overline{\mathbb{AG}}(R)) = diam(\mathbb{AG}(R)) = 3,$$

as required. Now, let $n = 2$.

(1) Suppose that $\mathbb{AG}(R)$ is complete. Since $(0, 1) \in Z(R)$ and $(0, 1)^2 \neq (0, 0)$, $Z(R)^2 \neq \{0\}$. Also, we know $(0, 1), (1, 0) \in Z(R)$ and $(1, 1) \notin Z(R)$, thus, $Z(R)$ is not an ideal and by [4, Theorem 2.7], $R \cong K_1 \oplus K_2$, where K_1 and K_2 are fields.

(2) It is trivial.

(3) Suppose that R_1 contains a nonnilpotent annihilating ideal I. Then there is an ideal $J \in \mathbb{I}(R_1)$ such that $IJ = (0)$. Therefore,

$$(R_1, (0)) - ((0), R_2) - (J, (0)) - (I, R_2)$$

is a path in both $\mathbb{AG}(R)$ and $\overline{\mathbb{AG}}(R)$ and there is no vertex adjacent to both $(R_1,(0))$ and (I,R_2). Hence, $diam(\overline{\mathbb{AG}}(R)) = diam(\mathbb{AG}(R)) = 3$.

(4) Suppose that $\mathbb{A}(R_1) = N_{\mathbb{I}}(R_1)$ and R_2 is an integral domain. For $I \in \mathbb{A}(R_1)^\star$,

$$(R_1,(0)) - ((0),R_2) - (I^{n_I-1},(0)) - (I,R_2)$$

is only path between $(R_1,(0))$ and (I,R_2) in $\mathbb{AG}(R)$. Thus, $diam(\mathbb{AG}(R)) = 3$. Now, we set $\mathbb{A}(R)^\star = T_1 \cup T_2$, where

$$T_1 = \{(I,(0)) : I \in \mathbb{I}(R_1)^\star\} \text{ and}$$
$$T_2 = \{(L,J) \ : \ L \in \mathbb{A}(R_1),\ J \in \mathbb{I}(R_2)^\star\}.$$

Obviously, the distance between two elements in $\mathbb{A}(R)^\star$ shows that

$$diam(\overline{\mathbb{AG}}(R)) = 2.$$

□

Example 3.10. Let p and q be distinct primes. Then the following assertions hold.

(1) $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 0$, if $n = p^2$.
(2) $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 1$, if $n = p^m$, $m \geq 3$. In this case $\overline{\mathbb{AG}}(\mathbb{Z}_n)$ is a complete graph.
(3) $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 2$, if $n = p^\alpha q^\beta$, $\alpha,\beta \in \mathbb{N}$, $\alpha \geq 2$ or $\beta \geq 2$. Otherwise, $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = diam(\mathbb{AG}(\mathbb{Z}_n)) = 1$. (Graph $\overline{\mathbb{AG}}(\mathbb{Z}_{p^2q})$ is a complete bipartite graph.)
(4) $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 3$, if $\prod_{i=1}^{k} p_i^{\alpha_i}$ is the prime factorization of n, $k \geq 3$.

Proof. The statements (1) and (2) are clear.

(3) By part (4) of Proposition 3.9, $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 2$. Now, if $\alpha = \beta = 1$, then the vertex set of $\overline{\mathbb{AG}}(\mathbb{Z}_{pq})$ is $\{\mathbb{Z}_{pq}p, \mathbb{Z}_{pq}q\}$. This implies that $\overline{\mathbb{AG}}(\mathbb{Z}_{pq})$ is a complete graph and $diam(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 1$.

(4) Follows from Proposition 3.9. □

4. Cycles in $\overline{\mathbb{AG}}(R)$

In this section, we study the girth of the extended of $\mathbb{AG}(R)$.

Theorem 4.1. *Let R be a ring. Then $gr(\overline{\mathbb{AG}}(R)) \in \{3,4,\infty\}$.*

Proof. The assertion follows by applying [1, Theorems 3.3 and 3.4] and [4, Theorem 2.1]. □

Theorem 4.2. *Let R be a ring. If $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$, then $\overline{\mathbb{AG}}(R)$ contains a cycle.*

Proof. Suppose that $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$. By Theorem 2.2, there exists $I \in N_{\mathbb{I}}(R)$ with $n_I \geq 3$ or $I \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$ with $Ann(I) \neq Ann(I^2)$. For the first case, we have $I \neq I^{n_I-1}$. By using connectedness of $\overline{\mathbb{AG}}(R)$, there is $J \in \mathbb{A}(R)^\star$ such that $J \notin \{I, I^{n_I-1}\}$ and $JI^{n_I-1} = (0)$. Then $I - J - I^{n_I-1} - I$ is a cycle of length 3 in $\overline{\mathbb{AG}}(R)$. For the second case, there exists $a \in Z(R)^\star$ such that

$$Ra \subseteq Ann_R(I^2) \text{ and } Ra \not\subseteq Ann_R(I).$$

If $Ra^2 = (0)$, then $Ra - I - Ia - Ra$ is a cycle of length 3. If not,

$$I - Ia - I^2 - Ra - I$$

is a cycle of length 4. Consequently, $\overline{\mathbb{AG}}(R)$ contains a cycle. □

The following corollaries are immediate results from Theorem 4.2.

Corollary 4.3. *Let $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$ and $I \in N_{\mathbb{I}}(R)$ with $n_I \geq 3$. Then $gr(\overline{\mathbb{AG}}(R)) = 3$.*

Corollary 4.4. *Let $I \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$ and $J \in \mathbb{I}(R)$. If $I^2J = (0)$, $IJ \neq (0)$ and $J^2 = (0)$, then $gr(\overline{\mathbb{AG}}(R)) = 3$.*

Theorem 4.5. *Let R be a ring and $\mathbb{A}(R)^\star = N_{\mathbb{I}}(R)$. Then*

$$gr(\overline{\mathbb{AG}}(R)) = \begin{cases} \infty & |\mathbb{A}(R)^\star| \leq 2, \\ 3 & |\mathbb{A}(R)^\star| \geq 3. \end{cases}$$

Proof. If $|\mathbb{A}(R)^\star| = 1$, then either $R \cong \frac{K[x]}{(x^2)}$, where K is a field or $R \cong L$, where L is a coefficient ring of characteristic p^2 and $gr(\overline{\mathbb{AG}}(R)) = \infty$.

If $|\mathbb{A}(R)^\star| = 2$, then by [4, Corollary 2.9], R is an Artinian local ring with exactly two nonzero proper ideals $Z(R)$ and $Z(R)^2$. Hence, $gr(\overline{\mathbb{AG}}(R)) = \infty$.

Now, suppose that $|\mathbb{A}(R)^\star| = 3$. Then by [4, Corollary 2.9], R is an Artinian local ring with exactly three nonzero proper ideals $Z(R), Z(R)^2$ and $Z(R)^3$. Therefore, $\overline{\mathbb{AG}}(R)$ is a complete graph and $gr(\overline{\mathbb{AG}}(R)) = 3$.

Also, if $|\mathbb{A}(R)^\star| \geq 4$, then we consider two cases:

Case (1): Suppose that there is $I \in \mathbb{A}(R)^\star$ with $n_I \geq 3$. Since $\overline{\mathbb{AG}}(R)$ is connected, there is at least an ideal $J \notin \{I, I^2\}$ such that J is adjacent to both I and I^2 in $\overline{\mathbb{AG}}(R)$.

Case (2): Suppose that for each $I \in \mathbb{A}(R)^\star, n_I = 2$. Since $\overline{\mathbb{AG}}(R)$ is connected, for every three distinct vertices $I, J, K \in \mathbb{A}(R)^\star$, there exists a path $J - I - K$. Now, if $K \subseteq J$ or $J \subseteq K$, then $KJ = (0)$. Otherwise, since $(J + K)I = (0)$ we conclude $(J + K) \in \mathbb{A}(R)^\star$. Thus $(J + K)^2 = (0)$ and $JK = (0)$. In the two cases $gr(\overline{\mathbb{AG}}(R)) = 3$. □

Theorem 4.6. *Let R be a ring with $N_{\mathbb{I}}(R) \neq \{(0)\}$ and $gr(\mathbb{AG}(R)) = 4$. Then $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$, $gr(\overline{\mathbb{AG}}(R)) = 4$ and $\overline{\mathbb{AG}}(R)$ is a complete bipartite graph.*

Proof. By [1, Theorem 3.5], $R \cong R_1 \times R_2$, either $R_1 \cong \frac{K[x]}{(x^2)}$, where K is a field or $R_1 \cong L$, where L is a coefficient ring of characteristic p^2 and R_2 is an integral domain, but it is not a field. Then $\mathbb{A}(R)^\star = C \cup D$, where

$$C = \{((x) + (x^2), J), ((0), J) \ : \ J \in \mathbb{I}(R_2)^\star\} \text{ and}$$
$$D = \{(R_1, (0)), ((x) + (x^2), (0))\}.$$

Obviously, all elements of C are adjacent to all elements of D in $\overline{\mathbb{AG}}(R)$. Therefore, $\overline{\mathbb{AG}}(R)$ is a complete bipartite graph and $gr(\overline{\mathbb{AG}}(R)) = 4$. Also, $((x)+(x^2), R_2) - (R_1, (0))$ is an edge in $\overline{\mathbb{AG}}(R)$ whereas, $((x)+(x^2), R_2)$ and $(R_1, (0))$ are not adjacent in $\mathbb{AG}(R)$. Thus, $\overline{\mathbb{AG}}(R) \neq \mathbb{AG}(R)$. $\square$

Theorem 4.7. *Let R be a ring with $N_{\mathbb{I}}(R) \neq \{(0)\}$ and $gr(\mathbb{AG}(R)) = \infty$. Then $gr(\overline{\mathbb{AG}}(R)) \in \{3, 4, \infty\}$.*

Proof. If $\mathbb{AG}(R) = \overline{\mathbb{AG}}(R)$, then $gr(\overline{\mathbb{AG}}(R)) = \infty$. Otherwise, by Theorem 2.2, if there exists $J \in N_{\mathbb{I}}(R)$ with $n_J \geq 3$, then $J^2 \neq (0)$ and $JJ^{n_J-1} = (0)$, by proceeding the proof of [1, Theorem 3.4], we have $J^{n_J-1} = ann_R(Z(R))$. Note that $|\mathbb{A}(R)^\star| \geq 3$ implies that there is $K \in \mathbb{A}(R)^\star$ such that

$$J - K - J^{n_J-1} - J$$

is a cycle in $\overline{\mathbb{AG}}(R)$. This shows that $gr(\overline{\mathbb{AG}}(R)) = 3$.

If not, there exists $J \in \mathbb{A}(R)^\star \setminus N_{\mathbb{I}}(R)$, with $Ann(J) \neq Ann(J^2)$. By [1, Theorem 3.4], we set $R \cong R_1 \times R_2$ such that R_1 is a field and either $R_2 \cong \frac{K[x]}{(x^2)}$, K is a field or $R_2 \cong L$, where L is a coefficient ring of characteristic p^2 and $J = (R_1, (x^2)+(x))$. In this case $\mathbb{AG}(R) = P_4$ and $\overline{\mathbb{AG}}(R)$ with the vertex set

$$\{(R_1, (x^2)), (R_1, (x^2)+(x)), ((0), R_2), ((0), (x^2)+(x))\}$$

is a complete bipartite graph. Thus, $gr(\overline{\mathbb{AG}}(R)) = 4$. $\square$

Proposition 4.8. *Let $R = \prod_{i=1}^n R_i$, where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N} \setminus \{1\}$.*

(1) *If $n = 2$, we have the following assertions*:

(a) *$gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = \infty$ if and only if R_1 and R_2 are integral domains and at least one of them is isomorphic to a field.*

(b) *If R_1 and R_2 are integral domains with $|\mathbb{I}(R_1)| \geq 3$ and $|\mathbb{I}(R_2)| \geq 3$, then $\overline{\mathbb{AG}}(R) = \mathbb{AG}(R)$ and $gr(\mathbb{AG}(R)) = 4$.*

(c) *If R_1 and R_2 are not integral domains, then $gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = 3$.*

(d) *If R_2 is an integral domain and $|\mathbb{A}(R_1)^\star| \geq 2$, then $gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = 3$.*

(e) *If R_2 is an integral domain such that is not a field and $|\mathbb{A}(R_1)^\star| = 1$, then $gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = 4$.*

(f) *If R_1 is a field and $|\mathbb{A}(R_2)^\star| = 1$, then $gr(\overline{\mathbb{AG}}(R)) = 4$ and $gr(\mathbb{AG}(R)) = \infty$.*

(2) *If $n \geq 3$, then $gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = 3$.*

Proof. (1) If $n = 2$, then the proof of parts (a) and (b) is trivial.

(c) Suppose that $|\mathbb{A}(R_i)^\star| = 1$ and $I_i \in \mathbb{A}(R_i)^\star$. Then $I_1^2 = I_2^2 = (0)$ and $((0), I_2) - (I_1, (0)) - (I_1, I_2) - ((0), I_2)$ is a cycle. Now, if $|\mathbb{A}(R_i)^\star| = 2$ and $I_i, J_i \in \mathbb{A}(R_i)^\star$, such that $I_iJ_i = (0)$. Then $((0), I_2) - (I_1, (0)) - (J_1, J_2) - ((0), I_2)$ is a cycle. Therefore, in both cases $gr(\mathbb{AG}(R)) = 3$.

(d) Let $I, J \in \mathbb{A}(R_1)^\star$ with $IJ = (0)$. Then $((0), R_2) - (I, (0)) - (J, (0)) - ((0), R_2)$ is a cycle, as required.

(e) Assume that $I \in \mathbb{A}(R_1)^\star$. Then $I^2 = (0)$ and $(I, (0)) - ((0), J) - (R_1, (0)) - ((0), R_2) - (I, (0))$ is a cycle in $\mathbb{AG}(R)$. Using Theorem 4.6,

$$gr(\overline{\mathbb{AG}}(R)) = 4.$$

(f) It is explicit.

(2) The cycle $(R_1, (0), \dots, (0)) - ((0), R_2, (0), \dots, (0)) - ((0), (0), R_3, (0), \dots, (0)) - (R_1, (0), \dots, (0))$ shows that $gr(\overline{\mathbb{AG}}(R)) = gr(\mathbb{AG}(R)) = 3$. □

Example 4.9. Let n have the prime decomposition $n = \prod_{i=1}^k p_i^{\alpha_i}$. Now, if n be a non-prime natural number, then

$$gr(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = \begin{cases} \infty & n \in \{p^2, p^3, pq\}, \\ 4 & n = p^2 q, \\ 3 & \text{otherwise.} \end{cases}$$

By Proposition 4.8, we deduce that if $n \neq p^2 q$ and $|\mathbb{A}(\mathbb{Z}_n)^\star| \geq 3$, then

$$gr(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 3$$

and for every $n \in \mathbb{N}$ such that $|\mathbb{A}(\mathbb{Z}_n)^\star| \leq 2$, $gr(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = \infty$. Also, if $n = p^2 q$, by part (f) of Proposition 4.8, $gr(\overline{\mathbb{AG}}(\mathbb{Z}_n)) = 4$.

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