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Biharmonic Maps on Doubly Warped Product Manifolds

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Abstract. In this paper, we characterize a class of biharmonic maps from and between doubly product manifolds in terms of theie warping function. Examples are constructed when all of the factors are Euclidean spaces.

1. Introduction

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Such ϕ is said to be harmonic if it is a critical point of the energy functional

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations given as follows :

$$\tau(\phi) = Tr_g \nabla d\phi = 0.$$

Here $\tau(\phi)$ is the tension field of ϕ. We refer one to [5, 6, 7, 8, 9] for background on harmonic maps. As a generalization of harmonic maps, biharmonic maps are defined similarily, as follows : ahe map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

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Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0,$$

where ∇^ϕ is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g \left(\nabla^\phi\right)^2 \tau(\phi) = \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla^M_{e_i}e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. Clearly any harmonic map is biharmonic, therefore it is interesting to construct non-harmonic biharmonic maps (see [1, 2, 3] and [12, 13, 15] for some constructions of non-harmonic biharmonic maps). In [4], the authors studied biharmonic maps between warped products where they gave the condition for the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_f N$ and of the projection $\overline{\pi} : M \times_f N \longrightarrow M$. Moreover, in [10] the authors gave some extensions of the results in [4] together with some further constructions of biharmonic maps. They also gave some characterizations of non-harmonic biharmonic maps using the product of harmonic maps and warping metric. The author in [11] studied the f-harmonicity of some special maps from or into a doubly warped product manifold. He obtained some similar results in [10], such as conditions for the f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps; furthermore, he investigate non-trivial f-harmonicity of the product of two harmonic maps. In [14], the authors study the biharmonic maps between doubly warped product manifolds and they gave some characterizations of non-harmonic biharmonic maps using products of harmonic maps and the warping metric. In this paper, we give a different constructions of biharmonic maps on the doubly warped product manifolds. First, we characterize the biharmonicity of the maps $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ and $\widetilde{\psi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x,y) = \phi(x)$ and $\widetilde{\psi}(x,y) = \psi(y)$. In particular we study the first and the second projection (Theorems 2.3 and 2.7). In this setting we obtain some examples of biharmonic non-harmonic maps. As a second result, we study the biharmonicity of the inclusion maps $i_{y_0} : (M^n, g) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ and $i_{x_0} : (N^n, h) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ (Theorems 2.11 and 2.13). Finally, we determine the conditions of the biharmonicity of the identity maps $\left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \xrightarrow{Id} (M^m \times N^n, G = g \oplus h)$ and $(M^m \times N^n, G = g \oplus h) \xrightarrow{Id} \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ (Theorems 2.15 and 2.19). Some special cases are developed.

2. Main Results

Let (M^m, g) and (N^n, h) two Riemannian manifolds and let $\alpha \in C^\infty(M)$ and $\beta \in C^\infty(N)$ be a positive functions. The doubly warped product $M^m \underset{(\alpha,\beta)}{\times} N^n$ is the product manifolds $M \times N$ endowed with the Riemannian metric $G_{(\alpha,\beta)}$ defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_{(\alpha,\beta)}(X,Y) = (\beta \circ \sigma)^2 g(d\pi(X), d\pi(Y)) + (\alpha \circ \pi)^2 h(d\eta(X), d\eta(Y)),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. For all obtained results in this paper, we consider $\{e_i\}_{1 \le i \le m}$ to be an orthonormal frame on M and $\{f_j\}_{1 \le j \le n}$ to be an orthonormal frame on N. Then, an orthonormal frame on $\left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ is given by $\left\{\frac{1}{\beta}(e_i, 0), \frac{1}{\alpha}(0, f_j)\right\}$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$, the Levi-Civita connection $\widetilde{\nabla}$ of the doubly warped product $M^m \underset{(\alpha,\beta)}{\times} N^n$ is given by (see [14])

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + X_1(\ln\alpha)(0, Y_2) + Y_1(\ln\alpha)(0, X_2)\\ &+ X_2(\ln\beta)(Y_1, 0) + Y_2(\ln\beta)(X_1, 0)\\ &- \beta^2 g(X_1, Y_1)(0, grad \ln\beta) - \alpha^2 h(X_2, Y_2)(grad \ln\alpha, 0).\end{aligned} \tag{2.1}$$

Using equation (2.1), we give some particular cases below.

Proposition 2.1. *Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. The Levi-Civita connection $\widetilde{\nabla}$ of the doubly warped product $M^m \underset{(\alpha,\beta)}{\times} N^n$ satisfies the following equations*

$$\widetilde{\nabla}_{(X_1,0)}(Y_1, 0) = (\nabla_{X_1} Y_1, 0) - \beta^2 g(X_1, Y_1)(0, grad \ln\beta),$$

$$\widetilde{\nabla}_{(0,X_2)}(0, Y_2) = (0, \nabla_{X_2} Y_2) - \alpha^2 h(X_2, Y_2)(grad \ln\alpha, 0),$$

$$\widetilde{\nabla}_{(X_1,0)}(0, Y_2) = X_1(\ln\alpha)(0, Y_2) + Y_2(\ln\beta)(X_1, 0),$$

and

$$\widetilde{\nabla}_{(0,X_2)}(Y_1, 0) = Y_1(\ln\alpha)(0, X_2) + X_2(\ln\beta)(Y_1, 0).$$

In the first, we consider a smooth map $\phi : (M^m, g) \longrightarrow (P^p, k)$ and we define the map $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ by $\widetilde{\phi}(x, y) = \phi(x)$. By calculating the tension field of $\widetilde{\phi}$, we obtain the following result.

Proposition 2.2. *Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ be a smooth map. The tension field of the map $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x, y) = \phi(x)$ is given by*

$$\tau\left(\widetilde{\phi}\right) = \frac{1}{\beta^2}\tau(\phi) + nd\phi(grad \ln \alpha).$$

Proof. By definition of the tension field, we have

$$\begin{aligned}
\tau\left(\widetilde{\phi}\right) &= Tr_{G_{(\alpha,\beta)}} \nabla d\widetilde{\phi} \\
&= \frac{1}{\beta^2}\nabla^{\widetilde{\phi}}_{(e_i,0)} d\widetilde{\phi}(e_i, 0) + \frac{1}{\alpha^2}\nabla^{\widetilde{\phi}}_{(0,f_j)} d\widetilde{\phi}(0, f_j) \\
&- \frac{1}{\beta^2} d\widetilde{\phi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i, 0)\right) - \frac{1}{\alpha^2} d\widetilde{\phi}\left(\widetilde{\nabla}_{(0,f_j)}(0, f_j)\right),
\end{aligned}$$

where we sum over repeated indices. A simple calculation gives

$$\nabla^{\widetilde{\phi}}_{(e_i,0)} d\widetilde{\phi}(e_i, 0) = \nabla^{\phi}_{e_i} d\phi(e_i)$$

and

$$\nabla^{\widetilde{\phi}}_{(0,f_j)} d\widetilde{\phi}(0, f_j) = 0.$$

Using Proposition 2.1, we deduce that

$$\widetilde{\nabla}_{(e_i,0)}(e_i, 0) = (\nabla_{e_i} e_i, 0) - m\beta^2 (0, grad \ln \beta)$$

and

$$\widetilde{\nabla}_{(0,f_j)}(0, f_j) = \left(0, \nabla_{f_j} f_j\right) - n\alpha^2 (grad \ln \alpha, 0),$$

it follows that

$$\tau\left(\widetilde{\phi}\right) = \frac{1}{\beta^2}\left(\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)\right) + nd\phi(grad \ln \alpha),$$

then

$$\tau\left(\widetilde{\phi}\right) = \frac{1}{\beta^2}\tau(\phi) + nd\phi(grad \ln \alpha).$$ □

In the following, we will calculate the bi-tension field of the map $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$.

Theorem 2.3. *Let $\phi : (M^m, g) \longrightarrow (P^p, k)$ be a smooth map. the bi-tension field of the map $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x, y) = \phi(x)$ is* *given by*

$$
\begin{aligned}
\tau_2\left(\widetilde{\phi}\right) &= \frac{1}{\beta^4}\tau_2(\phi) + \frac{2}{\alpha^2\beta^2}\left(\Delta\ln\beta - 2\left|grad\ln\beta\right|^2\right)\tau(\phi) \\
&- \frac{n}{\beta^2}\nabla^{\phi}_{grad\ln\alpha}\tau(\phi) + \frac{2m}{\beta^2}\left|grad\ln\beta\right|^2\tau(\phi) - n^2\nabla^{\phi}_{grad\ln\alpha}d\phi(grad\ln\alpha) \\
&- \frac{n}{\beta^2}\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi(grad\ln\alpha) + Tr_g R^P\left(d\phi(grad\ln\alpha), d\phi\right)d\phi\right).
\end{aligned}
\tag{2.2}
$$

Proof. By definition of the bi-tension field, we have

$$
\begin{aligned}
\tau_2\left(\widetilde{\phi}\right) &= -Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right) \\
&- Tr_{G_{(\alpha,\beta)}}R^P\left(\tau\left(\widetilde{\phi}\right), d\widetilde{\phi}\right)d\widetilde{\phi}.
\end{aligned}
\tag{2.3}
$$

Using the fact that

$$
\tau\left(\widetilde{\phi}\right) = \frac{1}{\beta^2}\tau(\phi) + nd\phi(grad\ln\alpha),
$$

we get

$$
\begin{aligned}
Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right) &= Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\frac{1}{\beta^2}\tau(\phi) \\
&+ nTr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2 d\phi(grad\ln\alpha).
\end{aligned}
$$

For the term $Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\frac{1}{\beta^2}\tau(\phi)$, we have

$$
\begin{aligned}
&Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\frac{1}{\beta^2}\tau(\phi) \\
&= \frac{1}{\beta^2}\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}\frac{1}{\beta^2}\tau(\phi) - \frac{1}{\beta^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}\frac{1}{\beta^2}\tau(\phi) \\
&+ \frac{1}{\alpha^2}\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}\frac{1}{\beta^2}\tau(\phi) - \frac{1}{\alpha^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}\frac{1}{\beta^2}\tau(\phi) \\
&= \frac{1}{\beta^4}Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) - \frac{2m}{\beta^2}\left|grad\ln\beta\right|^2\tau(\phi) \\
&- \frac{2}{\alpha^2\beta^2}\left(f_j\left(f_j(\ln\beta)\right) - 2\left|grad\ln\beta\right|^2\right)\tau(\phi) \\
&+ \frac{2}{\alpha^2\beta^2}\left(\nabla_{f_j}f_j\right)(\ln\beta)\tau(\phi) + \frac{n}{\beta^2}\nabla^{\phi}_{grad\ln\alpha}\tau(\phi),
\end{aligned}
$$

which will lead to

$$\begin{aligned}
&Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\frac{1}{\beta^2}\tau(\phi)\\
&\quad=\frac{1}{\beta^4}Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi)-\frac{2}{\alpha^2\beta^2}\left(\Delta\ln\beta-2\left|grad\ln\beta\right|^2\right)\tau(\phi)\\
&\quad+\frac{n}{\beta^2}\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)-\frac{2m}{\beta^2}\left|grad\ln\beta\right|^2\tau(\phi).
\end{aligned}$$

A similar calculation gives us

$$\begin{aligned}
&Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2 d\phi\left(grad\ln\alpha\right)\\
&\quad=\frac{1}{\beta^2}\nabla^{\widetilde{\phi}}_{(e_i,0)}\nabla^{\widetilde{\phi}}_{(e_i,0)}d\phi\left(grad\ln\alpha\right)-\frac{1}{\beta^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)}d\phi\left(grad\ln\alpha\right)\\
&\quad+\frac{1}{\alpha^2}\nabla^{\widetilde{\phi}}_{(0,f_j)}\nabla^{\widetilde{\phi}}_{(0,f_j)}d\phi\left(grad\ln\alpha\right)-\frac{1}{\alpha^2}\nabla^{\widetilde{\phi}}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)}d\phi\left(grad\ln\alpha\right)\\
&\quad=\frac{1}{\beta^2}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right)+n\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right),
\end{aligned}$$

it follows that

$$\begin{aligned}
&Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\phi}}\right)^2\tau\left(\widetilde{\phi}\right)\\
&\quad=\frac{1}{\beta^4}Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi)+\frac{n}{\beta^2}Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right)\\
&\quad+\frac{n}{\beta^2}\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)+n^2\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right)\\
&\quad-\frac{2}{\alpha^2\beta^2}\left(\Delta\ln\beta-2\left|grad\ln\beta\right|^2\right)\tau(\phi).
\end{aligned}\tag{2.4}$$

Finally for term $Tr_{G_{(\alpha,\beta)}}R^P\left(\tau\left(\widetilde{\phi}\right),d\widetilde{\phi}\right)d\widetilde{\phi}$, it is very simple to see that

$$\begin{aligned}
Tr_{G_{(\alpha,\beta)}}R^P\left(\tau\left(\widetilde{\phi}\right),d\widetilde{\phi}\right)d\widetilde{\phi}&=\frac{1}{\beta^4}Tr_gR^P\left(\tau(\phi),d\phi\right)d\phi\\
&+\frac{n}{\beta^2}Tr_gR^P\left(d\phi\left(grad\ln\alpha\right),d\phi\right)d\phi.
\end{aligned}\tag{2.5}$$

If we replace (2.4) and (2.5) in (2.3), we deduce that

$$\begin{aligned}
&\tau_2\left(\widetilde{\phi}\right)=\frac{1}{\beta^4}\tau_2(\phi)+\frac{2}{\alpha^2\beta^2}\left(\Delta\ln\beta-2\left|grad\ln\beta\right|^2\right)\tau(\phi)\\
&-\frac{n}{\beta^2}\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)+\frac{2m}{\beta^2}\left|grad\ln\beta\right|^2\tau(\phi)-n^2\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right)\\
&-\frac{n}{\beta^2}\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right)+Tr_gR^P\left(d\phi\left(grad\ln\alpha\right),d\phi\right)d\phi\right).\qquad\square
\end{aligned}$$

As a consequence, if ϕ is harmonic, we have the following.

Corollary 2.4. *Let* $\phi : (M^m, g) \longrightarrow (P^p, k)$ *be a harmonic map. The map* $\widetilde{\phi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ *defined by* $\widetilde{\phi}(x, y) = \phi(x)$ *is biharmonic if and only if*

$$Tr_g \left(\nabla^{\phi}\right)^2 d\phi \left(grad \ln \alpha\right) + Tr_g R^P \left(d\phi \left(grad \ln \alpha\right), d\phi\right) d\phi + n\beta^2 \nabla^{\phi}_{grad \ln \alpha} d\phi \left(grad \ln \alpha\right) = 0.$$

In particular if $\phi = Id_M$*, the first projection* $P_1 : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (M^m, g)$ *defined by* $P_1(x, y) = x$ *is biharmonic if and only if*

$$grad \Delta \ln \alpha + \frac{n}{2}\beta^2 grad \left(\left|grad \ln \alpha\right|^2\right) + 2Ricci \left(grad \ln \alpha\right) = 0.$$

We apply this Corollary to construct an example of biharmonic non-harmonic maps.

Example 2.5. Let the first projection $P_1 : \left(\mathbb{R}^m \underset{(\alpha,\beta)}{\times} \mathbb{R}^n, G_{(\alpha,\beta)}\right) \longrightarrow (\mathbb{R}^m, g_{\mathbb{R}^m})$ be defined by

$$P_1 \left(x = (t, x_2, ..., x_m), y = (s, y_2, ..., y_n)\right) = x = (t, x_2, ..., x_m).$$

We suppose that α depends only on t and β depends only on s and set $(\ln \alpha)' = \alpha_1(t)$, $(\ln \beta)' = \beta_1(s)$. Then by Corollary 1, the first projection $P_1 : \Big(\mathbb{R}^m \underset{(\alpha,\beta)}{\times} \mathbb{R}^n, G_{(\alpha,\beta)}\Big) \longrightarrow (\mathbb{R}^m, g_{\mathbb{R}^m})$ is biharmonic if and only if

$$\alpha_1'' + n\beta^2 \alpha_1 \alpha_1' = 0.$$

As particular solutions of this equation, we have

(1) If the function α_1 is constant, which gives us $\alpha(t) = C \exp(kt)$ $(C > 0)$. Then the equation $\alpha_1'' + n\beta^2\alpha_1\alpha_1' = 0$ can be satisfied for any positive function β. In this case, the first projection $P_1 : \left(\mathbb{R}^m \underset{(\alpha,\beta)}{\times} \mathbb{R}^n, G_{(\alpha,\beta)}\right) \longrightarrow (\mathbb{R}^m, g_{\mathbb{R}^m})$ is biharmonic non-harmonic where $\alpha(t) = C \exp(kt)$ and β is any positive function.

(2) For $\beta(s) = q \in \mathbb{N}^*$ and $\alpha(t) = C \sqrt[p]{(pt + k)^2}$ $(C > 0)$ where $p = nq^2$, the first projection $P_1 : \left(\mathbb{R}^m \underset{(\alpha,\beta)}{\times} \mathbb{R}^n, G_{(\alpha,\beta)}\right) \longrightarrow (\mathbb{R}^m, g_{\mathbb{R}^m})$ is biharmonic non-harmonic.

If we replace $\alpha = 1$ in Theorem 2.3, we get the following result:

Corollary 2.6. *Let* $\phi : (M^m, g) \longrightarrow (P^p, k)$ *be a smooth map. The map* $\widetilde{\phi} : \left(M^m \underset{\beta}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ *defined by* $\widetilde{\phi}(x, y) = \phi(x)$ *is biharmonic if and only if*

$$\tau_2(\phi) + 2\beta^2 \left(\Delta \ln \beta + (m-2)|grad \ln \beta|^2\right) \tau(\phi) = 0.$$

In particular if the map ϕ *is biharmonic non-harmonic, we get two cases:*

(1) *If* $m = 2$, *we deduce that the map* $\widetilde{\phi} : \left(M^m \underset{\beta}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ *defined by* $\widetilde{\phi}(x, y) = \phi(x)$ *is biharmonic non-harmonic if and only if the function* $\ln \beta$ *is harmonic.*

(2) *If* $m \neq 2$ *and by calculating* $\Delta\left(\beta^{m-2}\right)$, *we deduce that the map* $\widetilde{\phi} : \left(M^m \underset{\beta}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ *defined by* $\widetilde{\phi}(x, y) = \phi(x)$ *is biharmonic non-harmonic if and only if the function* β^{m-2} *is harmonic.*

Theorem 2.7. *Let* $\psi : (N^n, h) \longrightarrow (P^p, k)$ *be a smooth map, we define* $\widetilde{\psi} : ({}_{\beta}M^m \times_{\alpha} N^n, G_{\alpha,\beta}) \longrightarrow (P^p, k)$ *by* $\widetilde{\psi}(x, y) = \psi(y)$. *The tension field and the bi-tension field of* $\widetilde{\psi}$ *are given by*

$$\tau\left(\widetilde{\psi}\right) = \frac{1}{\alpha^2}\tau(\psi) + m d\psi(grad \ln \beta) \tag{2.6}$$

and

$$\begin{aligned} \tau_2\left(\widetilde{\psi}\right) &= \tau_2(\psi) - \frac{1}{\alpha^2\beta^2}\left(4|grad \ln \alpha|^2 - 2\Delta \ln \alpha\right)\tau(\psi) \\ &- \frac{m}{\alpha^2}\left(Tr_h\left(\nabla^{\psi}\right)^2 d\psi(grad \ln \beta) + \nabla^{\psi}_{grad \ln \beta}\tau(\psi)\right) \\ &- m\nabla^{\psi}_{grad \ln \beta} d\psi(grad \ln \beta) + \frac{2n}{\alpha^2}|grad \ln \alpha|^2 \tau(\psi) \\ &- \frac{m}{\alpha^2} Tr_h R(d\psi(grad \ln \beta), d\psi) d\psi. \end{aligned} \tag{2.7}$$

Proof. Let the map $\widetilde{\psi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ be defined by $\widetilde{\psi}(x, y) = \psi(y)$ where $\psi : (N^n, h) \longrightarrow (P^p, k)$. Note that in this case we have $d\widetilde{\psi}(X, Y) =$

$d\psi(Y)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. For the tension field of $\widetilde{\psi}$, we have

$$\begin{aligned}
\tau\left(\widetilde{\psi}\right) &= Tr_{G_{(\alpha,\beta)}} \nabla d\widetilde{\psi} \\
&= \frac{1}{\beta^2}\nabla^{\widetilde{\psi}}_{(e_i,0)} d\widetilde{\psi}(e_i,0) - \frac{1}{\beta^2} d\widetilde{\psi}\left(\widetilde{\nabla}_{(e_i,0)}(e_i,0)\right) \\
&+ \frac{1}{\alpha^2}\nabla^{\widetilde{\psi}}_{(0,f_j)} d\widetilde{\psi}(0,f_j) - \frac{1}{\alpha^2} d\widetilde{\psi}\left(\widetilde{\nabla}_{(0,f_j)}(0,f_j)\right) \\
&= m d\widetilde{\psi}(0, grad \ln\beta) + \frac{1}{\alpha^2}\nabla^{\psi}_{f_j} d\psi(f_j) - \frac{1}{\alpha^2} d\widetilde{\psi}\left(0, \nabla_{f_j} f_j\right),
\end{aligned}$$

then

$$\tau\left(\widetilde{\psi}\right) = \frac{1}{\alpha^2}\tau(\psi) + m d\psi(grad \ln\beta).$$

By definition, the bi-tension field of $\widetilde{\psi}$ is given by

$$\tau_2\left(\widetilde{\psi}\right) = -Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\psi}}\right)^2 \tau\left(\widetilde{\psi}\right) - Tr_{G_{(\alpha,\beta)}} R\left(\tau\left(\widetilde{\psi}\right), d\widetilde{\psi}\right) d\widetilde{\psi}.$$

For the first term $Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\psi}}\right)^2 \tau\left(\widetilde{\psi}\right)$, a rigourous calculation gives us

$$\begin{aligned}
&Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\widetilde{\psi}}\right)^2 \tau\left(\widetilde{\psi}\right) \\
&\quad= \frac{1}{\alpha^4} Tr_h \left(\nabla^{\psi}\right)^2 \tau(\psi) + \frac{1}{\alpha^2} Tr_h \left(\nabla^{\psi}\right)^2 d\psi(grad \ln\beta) \\
&\quad+ \frac{m}{\alpha^2}\nabla^{\psi}_{grad \ln\beta}\tau(\psi) + m^2 \nabla^{\psi}_{grad \ln\beta} d\psi(grad \ln\beta) \\
&\quad+ \frac{1}{\alpha^2\beta^2}\left(4\left|grad \ln\alpha\right|^2 - 2\Delta \ln\alpha\right)\tau(\psi) - \frac{2n}{\alpha^2}\left|grad \ln\alpha\right|^2 \tau(\psi).
\end{aligned}$$

Finally, it is very easy to see that

$$\begin{aligned}
Tr_{G_{(\alpha,\beta)}} R\left(\tau\left(\widetilde{\psi}\right), d\widetilde{\psi}\right) d\widetilde{\psi} &= \frac{1}{\alpha^4} Tr_h R(\tau(\psi), d\psi) d\psi \\
&+ \frac{m}{\alpha^2} Tr_h R(d\psi(grad \ln\beta), d\psi) d\psi.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tau_2\left(\widetilde{\psi}\right) &= \tau_2(\psi) - \frac{1}{\alpha^2\beta^2}\left(4\left|grad \ln\alpha\right|^2 - 2\Delta \ln\alpha\right)\tau(\psi) \\
&- \frac{m}{\alpha^2}\left(Tr_h \left(\nabla^{\psi}\right)^2 d\psi(grad \ln\beta) + \nabla^{\psi}_{grad \ln\beta}\tau(\psi)\right) \\
&- m\nabla^{\psi}_{grad \ln\beta} d\psi(grad \ln\beta) + \frac{2n}{\alpha^2}\left|grad \ln\alpha\right|^2 \tau(\psi) \\
&- \frac{m}{\alpha^2} Tr_h R(d\psi(grad \ln\beta), d\psi) d\psi.
\end{aligned}$$

□

As a consequence, if ψ is harmonic, we have the following.

Corollary 2.8. *Let $\psi : (N^n, h) \longrightarrow (P^p, k)$ be a harmonic map. The map $\widetilde{\psi} : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\psi}(x, y) = \psi(y)$ is biharmonic if and only if*

$$\begin{aligned}&Tr_h \left(\nabla^{\psi}\right)^2 d\psi \left(grad \ln \beta\right) + Tr_h R \left(d\psi \left(grad \ln \beta\right), d\psi\right) d\psi \\ &+ m\alpha^2 \nabla^{\psi}_{grad \ln \beta} d\psi \left(grad \ln \beta\right) = 0.\end{aligned}$$

In particular if $\psi = Id_N$, the second projection $P_2 : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (N^n, h)$ defined by $P_2(x, y) = y$ is biharmonic if and only if

$$grad\Delta \ln \beta + \frac{m}{2}\alpha^2 grad \left(\left|grad \ln \beta\right|^2\right) + 2Ricci \left(grad \ln \beta\right) = 0.$$

If we replace $\beta = 1$ in Theorem 2.7, we get the following result.

Corollary 2.9. *Let $\psi : (N^n, h) \longrightarrow (P^p, k)$ be a smooth map. The map $\widetilde{\psi} : \left(M^m \underset{\alpha}{\times} N^n, G_{\alpha}\right) \longrightarrow (P^p, k)$defined by $\widetilde{\psi}(x, y) = \psi(y)$ is biharmonic if and only if*

$$\tau_2(\psi) + \frac{2}{\alpha^2}\left(\Delta \ln \alpha + (n-2)\left|grad \ln \alpha\right|^2\right)\tau(\psi) = 0.$$

In particular if the map ψ is biharmonic non-harmonic, we distinguish two cases :

(1) *If $n = 2$, we deduce that the map $\widetilde{\psi} : \left(M^m \underset{\alpha}{\times} N^n, G_{\alpha}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\psi}(x, y) = \psi(x)$ is biharmonic non-harmonic if and only if the function $\ln \alpha$ is harmonic.*

(2) *If $n \neq 2$ and by calculating $\Delta\left(\alpha^{n-2}\right)$, we deduce that the map $\widetilde{\psi} : \left(M^m \underset{\alpha}{\times} N^n, G_{\alpha}\right) \longrightarrow (P^p, k)$ defined by $\widetilde{\phi}(x, y) = \phi(x)$ is biharmonic non-harmonic if and only if the function α^{n-2} is harmonic.*

Now, let's study the biharmonicity of inclusion maps $i_{y_0} : (M^n, g) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ and $i_{x_0} : (N^n, h) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$. To study the biharmonicity of these maps, we will give the expression of the curvature tensor $\widetilde{R}$ of the doubly warped product $\left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$. This expression is given by the following theorem.

Theorem 2.10. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\widetilde{\nabla}$ denote the Levi-Civita connection on $\left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ and $\widetilde{R}$ is the curvature* *tensor associated to* $\widetilde{\nabla}$*, then for all* $X_1, Y_1, Z_1 \in \Gamma(TM)$ *and* $X_2, Y_2, Z_2 \in \Gamma(TN)$*, we have*

$$\begin{aligned}
&\widetilde{R}((X_1,0),(Y_1,0))(Z_1,0) = (R(X_1,Y_1)Z_1,0)\\
&+ \beta^2 |grad \ln\beta|^2 \{g(X_1,Z_1)(Y_1,0) - g(Y_1,Z_1)(X_1,0)\}\\
&+ \beta^2 \{g(X_1,Z_1)Y_1(\ln\alpha) - g(Y_1,Z_1)X_1(\ln\alpha)\}(0, grad \ln\beta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}((X_1,0),(0,Y_2))(0,Z_2) = &-\alpha^2 h(Y_2,Z_2)(\nabla_{X_1} grad \ln\alpha, 0)\\
&- \alpha^2 h(Y_2,Z_2) X_1(\ln\alpha)(grad \ln\alpha, 0)\\
&- (Y_2(Z_2(\ln\beta)) - (\nabla_{Y_2}Z_2)(\ln\beta))(X_1,0)\\
&- Z_2(\ln\beta) Y_2(\ln\beta)(X_1,0) - Z_2(\ln\beta) X_1(\ln\alpha)(0,Y_2)\\
&+ \alpha^2\beta^2 h(Y_2,Z_2) X_1(\ln\alpha)(0, grad \ln\beta),
\end{aligned}$$

$$\begin{aligned}
&\widetilde{R}((0,X_2),(Y_1,0))(Z_1,0) = -\beta^2 g(Y_1,Z_1)(0, \nabla_{X_2} grad \ln\beta)\\
&+ \alpha^2\beta^2 g(Y_1,Z_1) X_2(\ln\beta)(grad \ln\alpha, 0) - Z_1(\ln\alpha) X_2(\ln\beta)(Y_1,0)\\
&- \{Y_1(Z_1(\ln\alpha)) - (\nabla_{Y_1}Z_1)(\ln\alpha) + Z_1(\ln\alpha) Y_1(\ln\alpha)\}(0,X_2)\\
&- \beta^2 g(Y_1,Z_1) X_2(\ln\beta)(0, grad \ln\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{R}((0,X_2),(0,Y_2))(0,Z_2) = (0, R(X_2,Y_2)Z_2)\\
&+ \alpha^2 \{h(X_2,Z_2) Y_2(\ln\beta) - h(Y_2,Z_2) X_2(\ln\beta)\}(grad \ln\alpha, 0)\\
&- \alpha^2 h(Y_2,Z_2) |grad \ln\alpha|^2 (0,X_2) + \alpha^2 h(X_2,Z_2) |grad \ln\alpha|^2 (0,Y_2).
\end{aligned}$$

Proof. For the term $\widetilde{R}((X_1,0),(Y_1,0))(Z_1,0)$, we have

$$\begin{aligned}
\widetilde{R}((X_1,0),(Y_1,0))(Z_1,0) = &\widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) - \widetilde{\nabla}_{(Y_1,0)}\widetilde{\nabla}_{(X_1,0)}(Z_1,0)\\
&- \widetilde{\nabla}_{[(X_1,0),(Y_1,0)]}(Z_1,0).
\end{aligned}$$

By Proposition 2.1, we can obtain

$$\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) = (\nabla_{Y_1}Z_1, 0) - \beta^2 g(Y_1,Z_1)(0, grad \ln\beta),$$

then

$$\begin{aligned}
\widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) = &\widetilde{\nabla}_{(X_1,0)}(\nabla_{Y_1}Z_1, 0)\\
&- \widetilde{\nabla}_{(X_1,0)}\beta^2 g(Y_1,Z_1)(0, grad \ln\beta)\\
&= (\nabla_{X_1}\nabla_{Y_1}Z_1, 0) - \beta^2 g(X_1, \nabla_{Y_1}Z_1)(0, grad \ln\beta)\\
&- \beta^2 g(Y_1,Z_1)\widetilde{\nabla}_{(X_1,0)}(0, grad \ln\beta)\\
&- \beta^2 X_1(g(Y_1,Z_1))(0, grad \ln\beta),
\end{aligned}$$

which leads us to the following formula

$$\begin{aligned}\widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) &= (\nabla_{X_1}\nabla_{Y_1}Z_1,0) - \beta^2 g(X_1,\nabla_{Y_1}Z_1)(0, grad\ln\beta) \\ &- \beta^2 g(Y_1,Z_1)X_1(\ln\alpha)(0, grad\ln\beta) \\ &- \beta^2 g(Y_1,Z_1)|grad\ln\beta|^2(X_1,0) \\ &- \beta^2 g(\nabla_{X_1}Y_1,Z_1)(0, grad\ln\beta) \\ &- \beta^2 g(Y_1,\nabla_{X_1}Z_1)(0, grad\ln\beta).\end{aligned}$$

A similar calculation gives us

$$\begin{aligned}\widetilde{\nabla}_{(Y_1,0)}\widetilde{\nabla}_{(X_1,0)}(Z_1,0) &= (\nabla_{Y_1}\nabla_{X_1}Z_1,0) - \beta^2 g(Y_1,\nabla_{X_1}Z_1)(0, grad\ln\beta) \\ &- \beta^2 g(X_1,Z_1)Y_1(\ln\alpha)(0, grad\ln\beta) \\ &- \beta^2 g(X_1,Z_1)|grad\ln\beta|^2(Y_1,0) \\ &- \beta^2 g(\nabla_{Y_1}X_1,Z_1)(0, grad\ln\beta) \\ &- \beta^2 g(X_1,\nabla_{Y_1}Z_1)(0, grad\ln\beta).\end{aligned}$$

and

$$\begin{aligned}\widetilde{\nabla}_{[(X_1,0),(Y_1,0)]}(Z_1,0) &= \widetilde{\nabla}_{([X_1,Y_1],0)}(Z_1,0) \\ &= \left(\nabla_{[X_1,Y_1]}Z_1,0\right) - \beta^2 g([X_1,Y_1],Z_1)(0, grad\ln\beta).\end{aligned}$$

It follows that

$$\begin{aligned}&\widetilde{R}((X_1,0),(Y_1,0))(Z_1,0) = (R(X_1,Y_1)Z_1,0) \\ &+ \beta^2|grad\ln\beta|^2\{g(X_1,Z_1)(Y_1,0) - g(Y_1,Z_1)(X_1,0)\} \\ &+ \beta^2\{g(X_1,Z_1)Y_1(\ln\alpha) - g(Y_1,Z_1)X_1(\ln\alpha)\}(0, grad\ln\beta).\end{aligned}$$

Now let's look at the term $\widetilde{R}((X_1,0),(0,Y_2))(0,Z_2)$, we have

$$\widetilde{R}((X_1,0),(0,Y_2))(0,Z_2) = \widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(0,Y_2)}(0,Z_2) - \widetilde{\nabla}_{(0,Y_2)}\widetilde{\nabla}_{(X_1,0)}(0,Z_2).$$

By Proposition 2.1, we obtain

$$\widetilde{\nabla}_{(0,Y_2)}(0,Z_2) = (0,\nabla_{Y_2}Z_2) - \alpha^2 h(Y_2,Z_2)(grad\ln\alpha,0),$$

then

$$\begin{aligned}\widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(0,Y_2)}(0,Z_2) &= \widetilde{\nabla}_{(X_1,0)}(0,\nabla_{Y_2}Z_2) \\ &- \widetilde{\nabla}_{(X_1,0)}\alpha^2 h(Y_2,Z_2)(grad\ln\alpha,0) \\ &= X_1(\ln\alpha)(0,\nabla_{Y_2}Z_2) + (\nabla_{Y_2}Z_2)(\ln\beta)(X_1,0) \\ &- \alpha^2 h(Y_2,Z_2)\widetilde{\nabla}_{(X_1,0)}(grad\ln\alpha,0) \\ &- X_1\left(\alpha^2\right)h(Y_2,Z_2)(grad\ln\alpha,0),\end{aligned}$$

we deduce that

$$\begin{aligned}\widetilde{\nabla}_{(X_1,0)}\widetilde{\nabla}_{(0,Y_2)}(0,Z_2) &= -\alpha^2 h(Y_2,Z_2)(\nabla_{X_1} grad \ln\alpha, 0)\\ &- 2\alpha^2 h(Y_2,Z_2) X_1(\ln\alpha)(grad\ln\alpha,0)\\ &+ \alpha^2\beta^2 h(Y_2,Z_2) X_1(\ln\alpha)(0, grad\ln\beta)\\ &+ X_1(\ln\alpha)(0,\nabla_{Y_2}Z_2) + (\nabla_{Y_2}Z_2)(\ln\beta)(X_1,0).\end{aligned}$$

The same calculation steps gives us

$$\begin{aligned}\widetilde{\nabla}_{(0,Y_2)}\widetilde{\nabla}_{(X_1,0)}(0,Z_2) &= X_1(\ln\alpha)(0,\nabla_{Y_2}Z_2) + Z_2(\ln\beta) X_1(\ln\alpha)(0,Y_2)\\ &+ Z_2(\ln\beta) Y_2(\ln\beta)(X_1,0) + Y_2(Z_2(\ln\beta))(X_1,0)\\ &- \alpha^2 h(X_2,Y_2) X_1(\ln\alpha)(grad\ln\alpha,0),\end{aligned}$$

it follows that

$$\begin{aligned}\widetilde{R}((X_1,0),(0,Y_2))(0,Z_2) &= -\alpha^2 h(Y_2,Z_2)(\nabla_{X_1} grad\ln\alpha,0)\\ &- \alpha^2 h(Y_2,Z_2) X_1(\ln\alpha)(grad\ln\alpha,0)\\ &- (Y_2(Z_2(\ln\beta)) - (\nabla_{Y_2}Z_2)(\ln\beta))(X_1,0)\\ &+ \alpha^2\beta^2 h(Y_2,Z_2) X_1(\ln\alpha)(0,grad\ln\beta)\\ &- Z_2(\ln\beta) Y_2(\ln\beta)(X_1,0) - Z_2(\ln\beta) X_1(\ln\alpha)(0,Y_2).\end{aligned}$$

For the term $\widetilde{R}((0,X_2),(Y_1,0))(Z_1,0)$, we have

$$\widetilde{R}((0,X_2),(Y_1,0))(Z_1,0) = \widetilde{\nabla}_{(0,X_2)}\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) - \widetilde{\nabla}_{(Y_1,0)}\widetilde{\nabla}_{(0,X_2)}(Z_1,0).$$

Using Proposition 2.1, we get

$$\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) = (\nabla_{Y_1}Z_1,0) - \beta^2 g(Y_1,Z_1)(0,grad\ln\beta),$$

then

$$\begin{aligned}\widetilde{\nabla}_{(0,X_2)}\widetilde{\nabla}_{(Y_1,0)}(Z_1,0) &= \widetilde{\nabla}_{(0,X_2)}(\nabla_{Y_1}Z_1,0) - \widetilde{\nabla}_{(0,X_2)}\beta^2 g(Y_1,Z_1)(0,grad\ln\beta)\\ &= (\nabla_{Y_1}Z_1)(\ln\alpha)(0,X_2) + X_2(\ln\beta)(\nabla_{Y_1}Z_1,0)\\ &- \beta^2 g(Y_1,Z_1)(0,\nabla_{X_2}grad\ln\beta)\\ &+ \alpha^2\beta^2 g(Y_1,Z_1) X_2(\ln\beta)(grad\ln\alpha,0)\\ &- 2\beta^2 X_2(\ln\beta) g(Y_1,Z_1)(0,grad\ln\beta).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\widetilde{\nabla}_{(Y_1,0)}\widetilde{\nabla}_{(0,X_2)}(Z_1,0) &= \widetilde{\nabla}_{(Y_1,0)}Z_1(\ln\alpha)(0,X_2) + \widetilde{\nabla}_{(Y_1,0)}X_2(\ln\beta)(Z_1,0)\\ &= Z_1(\ln\alpha) Y_1(\ln\alpha)(0,X_2)\\ &+ Z_1(\ln\alpha) X_2(\ln\beta)(Y_1,0)\\ &+ Y_1\{Z_1(\ln\alpha)\}(0,X_2) + X_2(\ln\beta)(\nabla_{Y_1}Z_1,0)\\ &- \beta^2 g(Y_1,Z_1) X_2(\ln\beta)(0,grad\ln\beta),\end{aligned}$$

which gives us

$$\begin{aligned}\widetilde{R}\left(\left(0,X_2\right),\left(Y_1,0\right)\right)\left(Z_1,0\right) &= -\beta^2 g\left(Y_1,Z_1\right)\left(0,\nabla_{X_2} grad\ln\beta\right)\\ &+ \alpha^2\beta^2 g\left(Y_1,Z_1\right)X_2\left(\ln\beta\right)\left(grad\ln\alpha,0\right)\\ &- \beta^2 g\left(Y_1,Z_1\right)X_2\left(\ln\beta\right)\left(0,grad\ln\beta\right)\\ &- Z_1\left(\ln\alpha\right)X_2\left(\ln\beta\right)\left(Y_1,0\right) - Z_1\left(\ln\alpha\right)Y_1\left(\ln\alpha\right)\left(0,X_2\right)\\ &- \left\{Y_1\left(Z_1\left(\ln\alpha\right)\right) - \left(\nabla_{Y_1}Z_1\right)\left(\ln\alpha\right)+\right\}\left(0,X_2\right).\end{aligned}$$

To complete the proof of Theorem 2.10, we will calculate the term $\widetilde{R}((0,X_2),(0,Y_2))$ $(0,Z_2)$, we have

$$\begin{aligned}\widetilde{R}\left(\left(0,X_2\right),\left(0,Y_2\right)\right)\left(0,Z_2\right) &= \widetilde{\nabla}_{\left(0,X_2\right)}\widetilde{\nabla}_{\left(0,Y_2\right)}\left(0,Z_2\right) - \widetilde{\nabla}_{\left(0,Y_2\right)}\widetilde{\nabla}_{\left(0,X_2\right)}\left(0,Z_2\right)\\ &- \widetilde{\nabla}_{\left[\left(0,X_2\right),\left(0,Y_2\right)\right]}\left(0,Z_2\right).\end{aligned}$$

Using Proposition 2.1, we get

$$\widetilde{\nabla}_{\left(0,Y_2\right)}\left(0,Z_2\right) = \left(0,\nabla_{Y_2}Z_2\right) - \alpha^2 h\left(Y_2,Z_2\right)\left(grad\ln\alpha,0\right),$$

then

$$\begin{aligned}\widetilde{\nabla}_{\left(0,X_2\right)}\widetilde{\nabla}_{\left(0,Y_2\right)}\left(0,Z_2\right) &= \widetilde{\nabla}_{\left(0,X_2\right)}\left(0,\nabla_{Y_2}Z_2\right)\\ &- \alpha^2\widetilde{\nabla}_{\left(0,X_2\right)}h\left(Y_2,Z_2\right)\left(grad\ln\alpha,0\right)\\ &= \left(0,\nabla_{X_2}\nabla_{Y_2}Z_2\right) - \alpha^2 h\left(X_2,\nabla_{Y_2}Z_2\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(Y_2,Z_2\right)\left|grad\ln\alpha\right|^2\left(0,X_2\right)\\ &- \alpha^2 h\left(Y_2,Z_2\right)X_2\left(\ln\beta\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(\nabla_{X_2}Y_2,Z_2\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(Y_2,\nabla_{X_2}Z_2\right)\left(grad\ln\alpha,0\right).\end{aligned}$$

A similar calculation gives us

$$\begin{aligned}\widetilde{\nabla}_{\left(0,Y_2\right)}\widetilde{\nabla}_{\left(0,X_2\right)}\left(0,Z_2\right) &= \widetilde{\nabla}_{\left(0,Y_2\right)}\left(0,\nabla_{X_2}Z_2\right)\\ &- \alpha^2\widetilde{\nabla}_{\left(0,Y_2\right)}h\left(X_2,Z_2\right)\left(grad\ln\alpha,0\right)\\ &= \left(0,\nabla_{Y_2}\nabla_{X_2}Z_2\right) - \alpha^2 h\left(Y_2,\nabla_{X_2}Z_2\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(X_2,Z_2\right)\left|grad\ln\alpha\right|^2\left(0,Y_2\right)\\ &- \alpha^2 h\left(X_2,Z_2\right)Y_2\left(\ln\beta\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(\nabla_{Y_2}X_2,Z_2\right)\left(grad\ln\alpha,0\right)\\ &- \alpha^2 h\left(X_2,\nabla_{Y_2}Z_2\right)\left(grad\ln\alpha,0\right)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\nabla}_{\left[\left(0,X_2\right),\left(0,Y_2\right)\right]}\left(0,Z_2\right) &= \widetilde{\nabla}_{\left(0,\left[X_2,Y_2\right]\right)}\left(0,Z_2\right)\\ &= \left(0,\nabla_{\left[X_2,Y_2\right]}Z_2\right) - \alpha^2 h\left(\left[X_2,Y_2\right],Z_2\right)\left(grad\ln\alpha,0\right),\end{aligned}$$

we deduce that

$$\begin{aligned}
&\widetilde{R}\left(\left(0, X_2\right),\left(0, Y_2\right)\right)\left(0, Z_2\right)=\left(0, R\left(X_2, Y_2\right) Z_2\right) \\
&+\alpha^2\left\{h\left(X_2, Z_2\right) Y_2(\ln \beta)-h\left(Y_2, Z_2\right) X_2(\ln \beta)\right\}(\operatorname{grad} \ln \alpha, 0) \\
&-\alpha^2 h\left(Y_2, Z_2\right)|\operatorname{grad} \ln \alpha|^2\left(0, X_2\right)+\alpha^2 h\left(X_2, Z_2\right)|\operatorname{grad} \ln \alpha|^2\left(0, Y_2\right) .
\end{aligned}$$ □

For the first case, we have the following result.

Theorem 2.11. *The inclusion map* $i_{y_0}:(M^n, g) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ *defined by* $i_{y_0}(x)=(x, y_0)$ *is biharmonic if and only if*

$$4|\operatorname{grad} \ln \beta|^2 \operatorname{grad} \ln \beta+\operatorname{grad}\left(|\operatorname{grad} \ln \beta|^2\right)=0$$

and

$$|\operatorname{grad} \ln \beta|^2 \operatorname{grad} \ln \alpha=0.$$

Proof. Note that in this case we have $di_{y_0}(X)=(X, 0)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. By definition to the tension field, we have

$$\begin{aligned}
\tau\left(i_{y_0}\right) & =Tr_g \widetilde{\nabla} d i_{y_0} \\
& =\widetilde{\nabla}_{e_i} d i_{y_0}\left(e_i\right)-d i_{y_0}\left(\nabla_{e_i} e_i\right) \\
& =\widetilde{\nabla}_{\left(e_i, 0\right)}\left(e_i, 0\right)-\left(\nabla_{e_i} e_i, 0\right) \\
& =\left(\nabla_{e_i} e_i, 0\right)-m \beta^2(0, \operatorname{grad} \ln \beta)-\left(\nabla_{e_i} e_i, 0\right),
\end{aligned}$$

then

$$\tau\left(i_{y_0}\right)=-m \beta^2(0, \operatorname{grad} \ln \beta) .$$

The inclusion map i_{y_0} is biharmonic if and only if

$$Tr_g\left(\widetilde{\nabla}^{i_{y_0}}\right)^2(0, \operatorname{grad} \ln \beta)+Tr_g \widetilde{R}\left((0, \operatorname{grad} \ln \beta), d i_{y_0}\right) d i_{y_0}=(0,0) .$$

For the term $Tr_g\left(\widetilde{\nabla}^{i_{y_0}}\right)^2(0, \operatorname{grad} \ln \beta)$, we have

$$\begin{aligned}
Tr_g\left(\widetilde{\nabla}^{i_{y_0}}\right)^2(0, \operatorname{grad} \ln \beta) & =\widetilde{\nabla}_{e_i}^{i_{y_0}} \widetilde{\nabla}_{e_i}^{i_{y_0}}(\operatorname{grad} \ln \alpha, 0)-\widetilde{\nabla}_{\nabla_{e_i} e_i}^{i_{y_0}}(\operatorname{grad} \ln \alpha, 0) \\
& =\widetilde{\nabla}_{\left(e_i, 0\right)} \widetilde{\nabla}_{\left(e_i, 0\right)}(0, \operatorname{grad} \ln \beta)-\widetilde{\nabla}_{\left(\nabla_{e_i} e_i, 0\right)}^{i_{y_0}}(0, \operatorname{grad} \ln \beta) \\
& =\widetilde{\nabla}_{\left(e_i, 0\right)} e_i(\ln \alpha)(0, \operatorname{grad} \ln \beta)+|\operatorname{grad} \ln \beta|^2 \widetilde{\nabla}_{\left(e_i, 0\right)}\left(e_i, 0\right) \\
& -\left(\nabla_{e_i} e_i\right)(\ln \alpha)(0, \operatorname{grad} \ln \beta)-|\operatorname{grad} \ln \beta|^2\left(\nabla_{e_i} e_i, 0\right) \\
& =e_i(\ln \alpha)\left\{e_i(\ln \alpha)(0, \operatorname{grad} \ln \beta)+|\operatorname{grad} \ln \beta|^2\left(e_i, 0\right)\right\} \\
& +e_i\left(e_i(\ln \alpha)\right)(0, \operatorname{grad} \ln \beta)-\left(\nabla_{e_i} e_i\right)(\ln \alpha)(0, \operatorname{grad} \ln \beta) \\
& +|\operatorname{grad} \ln \beta|^2\left(\left(\nabla_{e_i} e_i, 0\right)-m \beta^2(0, \operatorname{grad} \ln \beta)\right) \\
& -|\operatorname{grad} \ln \beta|^2\left(\nabla_{e_i} e_i, 0\right),
\end{aligned}$$

then

$$Tr_g\left(\widetilde{\nabla}^{i_{y_0}}\right)^2(0, grad\ln\beta) = |grad\ln\alpha|^2(0, grad\ln\beta) + |grad\ln\beta|^2(grad\ln\alpha, 0)$$
$$+ \Delta\ln\alpha\,(0, grad\ln\beta) - m\beta^2|grad\ln\beta|^2(0, grad\ln\beta).$$

For the second term $Tr_g\widetilde{R}((0, grad\ln\beta), di_{y_0})\,di_{y_0} = (0,0)$, using Theorem 2.10, we can obtain

$$\begin{aligned}
&Tr_g\widetilde{R}((0, grad\ln\beta), di_{y_0})\,di_{y_0} = Tr_g\widetilde{R}((0, grad\ln\beta), di_{y_0}(e_i))\,di_{y_0}(e_i)\\
&= \widetilde{R}((0, grad\ln\beta), (e_i,0))(e_i,0)\\
&= -\beta^2 g(e_i,e_i)(0, \nabla_{grad\ln\beta}grad\ln\beta) + \alpha^2\beta^2 g(e_i,e_i)|grad\ln\beta|^2(grad\ln\alpha,0)\\
&- e_i(\ln\alpha)|grad\ln\beta|^2(e_i,0) - \beta^2 g(e_i,e_i)|grad\ln\beta|^2(0, grad\ln\beta)\\
&- \{e_i(e_i(\ln\alpha)) - (\nabla_{e_i}e_i)(\ln\alpha) + e_i(\ln\alpha)e_i(\ln\alpha)\}(0, grad\ln\beta),
\end{aligned}$$

it follows that

$$\begin{aligned}
Tr_g\widetilde{R}((0, grad\ln\beta), di_{y_0})\,di_{y_0} &= -\frac{m}{2}\beta^2\left(0, grad\left(|grad\ln\beta|^2\right)\right)\\
&+ m\alpha^2\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)\\
&- |grad\ln\beta|^2(grad\ln\alpha,0)\\
&- m\beta^2|grad\ln\beta|^2(0, grad\ln\beta)\\
&- \left\{\Delta\ln\alpha + |grad\ln\alpha|^2\right\}(0, grad\ln\beta).
\end{aligned}$$

We conclude that the inclusion map i_{y_0} is biharmonic if and only if

$$4|grad\ln\beta|^2 grad\ln\beta + grad\left(|grad\ln\beta|^2\right) = 0$$

and

$$|grad\ln\beta|^2 grad\ln\alpha = 0. \qquad \square$$

Corollary 2.12. *For the inclusion map* $i_{y_0} : (M^n, g) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ *defined by* $i_{y_0}(x) = (x, y_0)$, *we have the following cases:*

(1) *If the function* β *is constant, then the inclusion map* i_{x_0} *is biharmonic for any positive function* α.

(2) *If the function* α *is constant, then the inclusion map* i_{x_0} *is biharmonic if and only if*
$$4|grad\ln\beta|^2 grad\ln\beta + grad\left(|grad\ln\beta|^2\right) = 0.$$

(3) *If the functions α and β are not constants, the inclusion map i_{y_0} is never biharmonic.*

Theorem 2.13. *The inclusion map $i_{x_0} : (N^n, h) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ defined by $i_{x_0}(y) = (x_0, y)$ is biharmonic if and only if*

$$4\left|grad\ln\alpha\right|^2 grad\ln\alpha + grad\left(\left|grad\ln\alpha\right|^2\right) = 0.$$

and

$$\left|grad\ln\alpha\right|^2 grad\ln\beta = 0.$$

Proof. Note that in this case we have $di_{x_0}(Y) = (0, Y)$ for any $Y \in \Gamma(TN)$. By definition of the tension field, we have

$$\begin{aligned}
\tau(i_{x_0}) &= Tr_h\widetilde{\nabla} di_{x_0} \\
&= \widetilde{\nabla}^{i_{x_0}}_{f_j} di_{x_0}(f_j) - di_{x_0}\left(\nabla_{f_j} f_j\right) \\
&= \widetilde{\nabla}^{i_{x_0}}_{f_j} di_{x_0}(f_j) - di_{x_0}\left(\nabla_{f_j} f_j\right) \\
&= \widetilde{\nabla}_{(0,f_j)}(0, f_j) - \left(0, \nabla_{f_j} f_j\right) \\
&= \left(0, \nabla_{f_j} f_j\right) - n\alpha^2(grad\ln\alpha, 0) - \left(0, \nabla_{f_j} f_j\right)
\end{aligned}$$

then

$$\tau(i_{x_0}) = -n\alpha^2(grad\ln\alpha, 0).$$

The inclusion map i_{x_0} is biharmonic if and only if

$$Tr_h\left(\widetilde{\nabla}^{i_{x_0}}\right)^2(grad\ln\alpha, 0) + Tr_h\widetilde{R}\left((grad\ln\alpha, 0), di_{x_0}\right)di_{x_0} = (0, 0).$$

For the first term of this equation, we have

$$\begin{aligned}
&Tr_h\left(\widetilde{\nabla}^{i_{x_0}}\right)^2(grad\ln\alpha, 0) \\
&\quad= \widetilde{\nabla}^{i_{x_0}}_{f_j}\widetilde{\nabla}^{i_{x_0}}_{f_j}(grad\ln\alpha, 0) - \widetilde{\nabla}^{i_{x_0}}_{\nabla_{f_j} f_j}(grad\ln\alpha, 0) \\
&\quad= \widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}(grad\ln\alpha, 0) - \widetilde{\nabla}_{\left(0,\nabla_{f_j} f_j\right)}(grad\ln\alpha, 0) \\
&\quad= \left|grad\ln\alpha\right|^2\widetilde{\nabla}_{(0,f_j)}(0, f_j) + \widetilde{\nabla}_{(0,f_j)} f_j(\ln\beta)(grad\ln\alpha, 0) \\
&\quad\quad - \left|grad\ln\alpha\right|^2\left(0, \nabla_{f_j} f_j\right) - \left(\nabla_{f_j} f_j\right)(\ln\beta)(grad\ln\alpha, 0),
\end{aligned}$$

then

$$\begin{aligned} Tr_h \left(\widetilde{\nabla}^{i_{x_0}}\right)^2 & (grad \ln \alpha, 0) \\ &= |grad \ln \alpha|^2 \left(0, \nabla_{f_j} f_j\right) - n\alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\ &+ |grad \ln \alpha|^2 (0, grad \ln \beta) + |grad \ln \beta|^2 (grad \ln \alpha, 0) \\ &+ f_j (f_j (\ln \beta)) (grad \ln \alpha, 0) - |grad \ln \alpha|^2 \left(0, \nabla_{f_j} f_j\right) \\ &- \left(\nabla_{f_j} f_j\right) (\ln \beta) (grad \ln \alpha, 0), \end{aligned}$$

it follows that

$$\begin{aligned} Tr_h \left(\widetilde{\nabla}^{i_{x_0}}\right)^2 (grad \ln \alpha, 0) &= -n\alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\ &+ \left(\Delta \ln \beta + |grad \ln \beta|^2\right) (grad \ln \alpha, 0) \\ &+ |grad \ln \alpha|^2 (0, grad \ln \beta) \end{aligned}$$

For the term $Tr_h \widetilde{R} ((grad \ln \alpha, 0), di_{x_0}) di_{x_0}$, by Theorem 2.10, we have

$$\begin{aligned} Tr_h \widetilde{R} ((grad \ln \alpha, 0), di_{x_0}) di_{x_0} &= \widetilde{R} ((grad \ln \alpha, 0), di_{x_0} (f_j)) di_{x_0} (f_j) \\ &= \widetilde{R} ((grad \ln \alpha, 0), (0, f_j)) (0, f_j) \\ &= -\alpha^2 h (f_j, f_j) (\nabla_{grad \ln \alpha} grad \ln \alpha, 0) \\ &- \alpha^2 h (f_j, f_j) |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\ &- \left(f_j (f_j (\ln \beta)) - \left(\nabla_{f_j} f_j\right) (\ln \beta)\right) (grad \ln \alpha, 0) \\ &- f_j (\ln \beta) f_j (\ln \beta) (grad \ln \alpha, 0) \\ &- f_j (\ln \beta) |grad \ln \alpha|^2 (0, f_j) \\ &+ \alpha^2 \beta^2 h (f_j, f_j) |grad \ln \alpha|^2 (0, grad \ln \beta), \end{aligned}$$

which gives us

$$\begin{aligned} Tr_h \widetilde{R} ((grad \ln \alpha, 0), di_{x_0}) di_{x_0} &= -\frac{n}{2} \alpha^2 \left(grad \left(|grad \ln \alpha|^2\right), 0\right) \\ &- n\alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\ &- \left(\Delta \ln \beta + |grad \ln \beta|^2\right) (grad \ln \alpha, 0) \\ &+ n\alpha^2 \beta^2 |grad \ln \alpha|^2 (0, grad \ln \beta) \\ &- |grad \ln \alpha|^2 (0, grad \ln \beta). \end{aligned}$$

Finally, we deduce that the inclusion map i_{x_0} is biharmonic if and only if

$$4 |grad \ln \alpha|^2 grad \ln \alpha + grad \left(|grad \ln \alpha|^2\right) = 0$$

and

$$\left|grad\ln\alpha\right|^2 grad\ln\beta = 0. \qquad \square$$

Corollary 2.14. *For the inclusion map* $i_{x_0} : (N^n, h) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ *defined by* $i_{x_0}(y) = (x_0, y)$*, we have the following cases:*

(1) *If the function* α *is constant, then the inclusion map* i_{x_0} *is biharmonic for any positive function* β.

(2) *If the function* β *is constant, then the inclusion map* i_{x_0} *is biharmonic if and only if*

$$4\left|grad\ln\alpha\right|^2 grad\ln\alpha + grad\left(\left|grad\ln\alpha\right|^2\right) = 0.$$

(3) *If the functions* α *and* β *are not constants, the inclusion map* i_{x_0} *is never biharmonic.*

Theorem 2.15. *Let* $\phi : \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right) \longrightarrow (M^m \times N^n, G = g \oplus h)$ *be the map defined by* $\phi(x, y) = (x, y)$*. The map* ϕ *is biharmonic if and only if*

$$grad\Delta\ln\alpha + \frac{n}{2}\beta^2 grad\left(\left|grad\ln\alpha\right|^2\right) + 2Ricci\left(grad\ln\alpha\right) = 0$$

and

$$grad\Delta\ln\beta + \frac{m}{2}\alpha^2 grad\left(\left|grad\ln\beta\right|^2\right) + 2Ricci\left(grad\ln\beta\right) = 0.$$

Proof. Note that in this case we have $d\phi(X, Y) = (X, Y)$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$. By definition of the tension field and using the Proposition 2.1, we have

$$\begin{aligned}
\tau(\phi) &= Tr_{G_{(\alpha,\beta)}}\nabla d\phi \\
&= \frac{1}{\beta^2}\left(\nabla^{\phi}_{(e_i,0)} d\phi(e_i, 0) - d\phi\left(\widetilde{\nabla}_{(e_i,0)}(e_i, 0)\right)\right) \\
&+ \frac{1}{\alpha^2}\left(\nabla^{\phi}_{(0,f_j)} d\phi(0, f_j) - d\phi\left(\widetilde{\nabla}_{(0,f_j)}(0, f_j)\right)\right) \\
&= \frac{1}{\beta^2}\left((\nabla_{e_i}e_i, 0) - (\nabla_{e_i}e_i, 0) + m\beta^2(0, grad\ln\beta)\right) \\
&+ \frac{1}{\alpha^2}\left(\left(0, \nabla_{f_j}f_j\right) - \left(0, \nabla_{f_j}f_j\right) + n\alpha^2(grad\ln\alpha, 0)\right),
\end{aligned}$$

then

$$\tau(\phi) = n(grad\ln\alpha, 0) + m(0, grad\ln\beta).$$

The map ϕ is biharmonic if and only if

$$\begin{aligned} &n\left(Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(grad\ln\alpha,0)+Tr_{G_{(\alpha,\beta)}}R\left((grad\ln\alpha,0),d\phi\right)d\phi\right)\\ &+m\left(Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(0,grad\ln\beta)+Tr_{G_{(\alpha,\beta)}}R\left((0,grad\ln\beta),d\phi\right)d\phi\right)=0. \end{aligned} \tag{2.8}$$

We can analyse term by term this expression, for the term $Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(grad\ln\alpha,0)$, we have

$$\begin{aligned} Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(grad\ln\alpha,0) &= \frac{1}{\beta^{2}}\nabla^{\phi}_{(e_i,0)}\nabla^{\phi}_{(e_i,0)}(grad\ln\alpha,0)\\ &- \frac{1}{\beta^{2}}\nabla^{\phi}_{\tilde{\nabla}_{(e_i,0)}(e_i,0)}(grad\ln\alpha,0)\\ &+ \frac{1}{\alpha^{2}}\nabla^{\phi}_{(0,f_j)}\nabla^{\phi}_{(0,f_j)}(grad\ln\alpha,0)\\ &- \frac{1}{\alpha^{2}}\nabla^{\phi}_{\tilde{\nabla}_{(0,f_j)}(0,f_j)}(grad\ln\alpha,0)\\ &= \frac{1}{\beta^{2}}\left(\nabla_{e_i}\nabla_{e_i}grad\ln\alpha,0\right)-\frac{1}{\beta^{2}}\left(\nabla_{\nabla_{e_i}e_i}grad\ln\alpha,0\right)\\ &+ n\nabla^{\phi}_{(grad\ln\alpha,0)}(grad\ln\alpha,0)\\ &= \frac{1}{\beta^{2}}\left(Tr_g\nabla^{2}grad\ln\alpha,0\right)+n\left(\nabla_{grad\ln\alpha}grad\ln\alpha,0\right), \end{aligned}$$

then

$$\begin{aligned} Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(grad\ln\alpha,0) &= \frac{1}{\beta^{2}}\left\{(grad\Delta\ln\alpha,0)+(Ricci\,(grad\ln\alpha)\,,0)\right\}\\ &+\frac{n}{2}\left(grad\left(|grad\ln\alpha|^{2}\right),0\right). \end{aligned} \tag{2.9}$$

A similar calculation leads to

$$\begin{aligned} Tr_{G_{(\alpha,\beta)}}\left(\nabla^{\phi}\right)^{2}(0,grad\ln\beta) &= (0,grad\Delta\ln\beta)+2\,(0,Ricci\,(grad\ln\beta))\\ &+\frac{m}{2}\alpha^{2}\left(0,grad\left(|grad\ln\beta|^{2}\right)\right). \end{aligned} \tag{2.10}$$

Finally, it is very simple to write that

$$Tr_{G_{(\alpha,\beta)}}R\left((grad\ln\alpha,0),d\phi\right)d\phi=\frac{1}{\beta^{2}}\left(Ricci\,(grad\ln\alpha)\,,0\right) \tag{2.11}$$

and

$$Tr_{G_{(\alpha,\beta)}}R\left((0,grad\ln\beta),d\phi\right)d\phi=\frac{1}{\alpha^{2}}\left(0,Ricci\,(grad\ln\beta)\right). \tag{2.12}$$

If we replace (2.9), (2.10), (2.11)) and (2.12) in (2.8), we deduce that the map ϕ is biharmonic if and only if

$$grad\Delta\ln\alpha+\frac{n}{2}\beta^{2}grad\left(|grad\ln\alpha|^{2}\right)+2Ricci\left(grad\ln\alpha\right)=0$$

and

$$grad\Delta\ln\beta+\frac{m}{2}\alpha^{2}grad\left(|grad\ln\beta|^{2}\right)+2Ricci\left(grad\ln\beta\right)=0. \qquad \square$$

Example 2.16. Let the map $\phi:\left(\mathbb{R}^{m}\underset{(\alpha,\beta)}{\times}\mathbb{R}^{n},G_{(\alpha,\beta)}\right)\longrightarrow\left(\mathbb{R}^{m}\times\mathbb{R}^{n},G\right)$ defined by

$$\phi\left(x=(t,x_{2},...,x_{m}),y=(s,y_{2},...,y_{n})\right)=\left(x=(t,x_{2},...,x_{m}),y=(s,y_{2},...,y_{n})\right).$$

We suppose that α depends only on t and β depends only on s and set $(\ln\alpha)'=\alpha_{1}(t)$, $(\ln\beta)'=\beta_{1}(s)$. Then from Theorem 2.11, the map $\phi:\left(\mathbb{R}^{m}\underset{(\alpha,\beta)}{\times}\mathbb{R}^{n},G_{(\alpha,\beta)}\right)\longrightarrow\left(\mathbb{R}^{m}\underset{(\alpha,\beta)}{\times}\mathbb{R}^{n},G\right)$ is biharmonic if and only if

$$\alpha_{1}''+n\beta^{2}\alpha_{1}\alpha_{1}'=0$$

and

$$\beta_{1}''+m\alpha^{2}\beta_{1}\beta_{1}'=0.$$

Particular solutions of this system are given by $\alpha(t)=C_{1}\exp(C_{2}t)$ and $B(s)=K_{1}\exp(K_{2}s)$, where C_{1} and K_{1} are positive constants. For this functions $\alpha(t)=C_{1}\exp(C_{2}t)$ and $B(s)=K_{1}\exp(K_{2}s)$, the map $\phi:\left(\mathbb{R}^{m}\underset{(\alpha,\beta)}{\times}\mathbb{R}^{n},G_{(\alpha,\beta)}\right)\longrightarrow\left(\mathbb{R}^{m}\times\mathbb{R}^{n},G\right)$ is biharmonic nonharmonic.

Finally, we consider the map $\phi:\left(M^{m}\times N^{n},G=g\oplus h\right)\longrightarrow\left(M^{m}\underset{(\alpha,\beta)}{\times}N^{n},G_{(\alpha,\beta)}\right)$ defined by $\phi(x,y)=(x,y)$. To study the biharmonicity of this map, we use two lemmas. In the first lemma, we calculate the term $Tr_{G}\widetilde{\nabla}^{2}\left(grad\ln\alpha,0\right)$.

Lemma 2.17. *Let (M^{m},g) and (N^{n},h) two Riemannian manifolds and let $\alpha\in C^{\infty}(M)$ and $\beta\in C^{\infty}(N)$ be a positive functions. Let $\widetilde{\nabla}$ be the Levi-Civita connection of the doubly warped product $M^{m}\underset{(\alpha,\beta)}{\times}N^{n}$, we have*

$$\begin{aligned}Tr_{G}\widetilde{\nabla}^{2}\left(grad\ln\alpha,0\right)=&\left(grad\Delta\ln\alpha,0\right)-n\alpha^{2}\left|grad\ln\alpha\right|^{2}\left(grad\ln\alpha,0\right)\\&-\beta^{2}\left|grad\ln\beta\right|^{2}\left(grad\ln\alpha,0\right)+\Delta\ln\beta\left(grad\ln\alpha,0\right)\\&+\left|grad\ln\beta\right|^{2}\left(grad\ln\alpha,0\right)+\left|grad\ln\alpha\right|^{2}\left(0,grad\ln\beta\right)\\&-\beta^{2}\left(\left|grad\ln\alpha\right|^{2}+2\Delta\ln\alpha\right)\left(0,grad\ln\beta\right)\\&+\left(Ricci\left(grad\ln\alpha\right),0\right).\end{aligned}$$

Proof. We have

$$
\begin{aligned}
Tr_G\widetilde{\nabla}^2\left(grad\ln\alpha,0\right) &= \widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)}\left(grad\ln\alpha,0\right)\\
&-\widetilde{\nabla}_{\nabla_{(e_i,0)}(e_i,0)}\left(grad\ln\alpha,0\right)\\
&+\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}\left(grad\ln\alpha,0\right)\\
&-\widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}\left(grad\ln\alpha,0\right).
\end{aligned}\tag{2.13}
$$

A simple calculation gives us

$$
\begin{aligned}
&\widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)}\left(grad\ln\alpha,0\right)-\widetilde{\nabla}_{\nabla_{(e_i,0)}(e_i,0)}\left(grad\ln\alpha,0\right)\\
&= \left(\nabla_{e_i}\nabla_{e_i}grad\ln\alpha,0\right)-\beta^2 g\left(e_i,\nabla_{e_i}grad\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&-\beta^2 g\left(e_i,grad\ln\alpha\right)e_i\left(\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&-\beta^2 g\left(e_i,grad\ln\alpha\right)\left|grad\ln\beta\right|^2\left(e_i,0\right)\\
&-\beta^2 g\left(\nabla_{e_i}e_i,grad\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&-\beta^2 g\left(e_i,\nabla_{e_i}grad\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&-\left(\nabla_{\nabla_{e_i}e_i}grad\ln\alpha,0\right)+\beta^2 g\left(\nabla_{e_i}e_i,grad\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&=\left(Tr_g\nabla^2 grad\ln\alpha,0\right)-2\beta^2\Delta\ln\alpha\left(0,grad\ln\beta\right)\\
&-\beta^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)-\beta^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right),
\end{aligned}
$$

then

$$
\begin{aligned}
&\widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)}\left(grad\ln\alpha,0\right)-\widetilde{\nabla}_{\nabla_{(e_i,0)}(e_i,0)}\left(grad\ln\alpha,0\right)\\
&=\left(grad\Delta\ln\alpha,0\right)-\beta^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)\\
&-\beta^2\left(\left|grad\ln\alpha\right|^2+2\Delta\ln\alpha\right)\left(0,grad\ln\beta\right)\\
&+\left(Ricci\left(grad\ln\alpha\right),0\right).
\end{aligned}\tag{2.14}
$$

For the term $\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}\left(grad\ln\alpha,0\right)-\widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}\left(grad\ln\alpha,0\right)$, a similar calculation gives us

$$
\begin{aligned}
&\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}\left(grad\ln\alpha,0\right)-\widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}\left(grad\ln\alpha,0\right)\\
&=\left|grad\ln\alpha\right|^2\widetilde{\nabla}_{(0,f_j)}\left(0,f_j\right)+\widetilde{\nabla}_{(0,f_j)}f_j\left(\ln\beta\right)\left(grad\ln\alpha,0\right)\\
&-\widetilde{\nabla}_{\left(0,\nabla_{f_j}f_j\right)}\left(grad\ln\alpha,0\right)\\
&=\left|grad\ln\alpha\right|^2\left(0,\nabla_{f_j}f_j\right)-n\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\
&+f_j\left(\ln\beta\right)\left|grad\ln\alpha\right|^2\left(0,f_j\right)+f_j\left(\ln\beta\right)f_j\left(\ln\beta\right)\left(grad\ln\alpha,0\right)\\
&+f_j\left(f_j\left(\ln\beta\right)\right)\left(grad\ln\alpha,0\right)-\left|grad\ln\alpha\right|^2\left(0,\nabla_{f_j}f_j\right)\\
&-\left(\nabla_{f_j}f_j\right)\left(\ln\beta\right)\left(grad\ln\alpha,0\right),
\end{aligned}
$$

it follows that

$$\begin{aligned}
&\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}(grad\ln\alpha,0)-\widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}(grad\ln\alpha,0)\\
&\qquad=-n\alpha^2\left|grad\ln\alpha\right|^2(grad\ln\alpha,0)\\
&\qquad+\left|grad\ln\beta\right|^2(grad\ln\alpha,0)\\
&\qquad+\Delta\ln\beta\,(grad\ln\alpha,0)\\
&\qquad+\left|grad\ln\alpha\right|^2(0,grad\ln\beta)\,.
\end{aligned}\tag{2.15}$$

If we replace (2.14) and (2.15) in (2.13) , we deduce that

$$\begin{aligned}
Tr_G\widetilde{\nabla}^2(grad\ln\alpha,0)&=(grad\Delta\ln\alpha,0)-n\alpha^2\left|grad\ln\alpha\right|^2(grad\ln\alpha,0)\\
&-\beta^2\left|grad\ln\beta\right|^2(grad\ln\alpha,0)+\Delta\ln\beta\,(grad\ln\alpha,0)\\
&+\left|grad\ln\beta\right|^2(grad\ln\alpha,0)+\left|grad\ln\alpha\right|^2(0,grad\ln\beta)\\
&-\beta^2\left(\left|grad\ln\alpha\right|^2+2\Delta\ln\alpha\right)(0,grad\ln\beta)\\
&+(Ricci\,(grad\ln\alpha)\,,0)\,.
\end{aligned}$$

□

For the term $Tr_G\widetilde{\nabla}^2(0,grad\ln\beta)$, we can use the following lemma.

Lemma 2.18. *Let (M^m,g) and (N^n,h) two Riemannian manifolds and let $\alpha\in C^\infty(M)$ and $\beta\in C^\infty(N)$ be a positive functions. Let $\widetilde{\nabla}$ be the Levi-Civita connection of the doubly warped product $M^m\underset{(\alpha,\beta)}{\times}N^n$, we have*

$$\begin{aligned}
Tr_G\widetilde{\nabla}^2(0,grad\ln\beta)&=\left|grad\ln\alpha\right|^2(0,grad\ln\beta)+\left|grad\ln\beta\right|^2(grad\ln\alpha,0)\\
&+\Delta\ln\alpha\,(0,grad\ln\beta)-m\beta^2\left|grad\ln\beta\right|^2(0,grad\ln\beta)\\
&+(0,grad\Delta\ln\beta)-\alpha^2\left(2\Delta\ln\beta+\left|grad\ln\beta\right|^2\right)(grad\ln\alpha,0)\\
&-\alpha^2\left|grad\ln\alpha\right|^2(0,grad\ln\beta)+(0,Ricci\,(grad\ln\beta))\,.
\end{aligned}$$

Proof.

$$\begin{aligned}
Tr_G\widetilde{\nabla}^2(0,grad\ln\beta)&=\widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)}(0,grad\ln\beta)-\widetilde{\nabla}_{\nabla_{(e_i,0)}(e_i,0)}(0,grad\ln\beta)\\
&+\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}(0,grad\ln\beta)-\widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}(0,grad\ln\beta)\,.
\end{aligned}$$

The same calculation method gives us

$$\begin{aligned}
&\widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)}(0, grad\ln\beta) - \widetilde{\nabla}_{\nabla_{(e_i,0)}(e_i,0)}(0, grad\ln\beta) \\
&\quad= \widetilde{\nabla}_{(e_i,0)}e_i(\ln\alpha)(0, grad\ln\beta) + |grad\ln\beta|^2\,\widetilde{\nabla}_{(e_i,0)}(e_i,0) \\
&\quad- \widetilde{\nabla}_{\left(\nabla_{e_i}e_i,0\right)}(0, grad\ln\beta) \\
&\quad= e_i(\ln\alpha)\,e_i(\ln\alpha)(0, grad\ln\beta) + e_i(\ln\alpha)\,|grad\ln\beta|^2(e_i,0) \\
&\quad+ e_i(e_i(\ln\alpha))(0, grad\ln\beta) - m\beta^2|grad\ln\beta|^2(0, grad\ln\beta) \\
&\quad- (\nabla_{e_i}e_i)(\ln\alpha)(0, grad\ln\beta) \\
&\quad= |grad\ln\alpha|^2(0, grad\ln\beta) + |grad\ln\beta|^2(grad\ln\alpha, 0) \\
&\quad+ \Delta\ln\alpha\,(0, grad\ln\beta) - m\beta^2|grad\ln\beta|^2(0, grad\ln\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}(0, grad\ln\beta) - \widetilde{\nabla}_{\nabla_{(0,f_j)}(0,f_j)}(0, grad\ln\beta) \\
&= \left(0, \nabla_{f_j}\nabla_{f_j}grad\ln\beta\right) - \alpha^2 h\left(f_j, \nabla_{f_j}grad\ln\beta\right)(grad\ln\alpha, 0) \\
&- \alpha^2 h(f_j, grad\ln\beta)\,|grad\ln\alpha|^2(0, f_j) - \alpha^2 f_j(\ln\beta)(f_j, grad\ln\beta)(grad\ln\alpha, 0) \\
&- \alpha^2 h\left(\nabla_{f_j}f_j, grad\ln\beta\right)(grad\ln\alpha, 0) - \alpha^2 h\left(f_j, \nabla_{f_j}grad\ln\beta\right)(grad\ln\alpha, 0) \\
&- \left(0, \nabla_{\nabla_{f_j}f_j}grad\ln\beta\right) + \alpha^2 h\left(\nabla_{f_j}f_j, grad\ln\beta\right)(grad\ln\alpha, 0) \\
&= \left(0, Tr_h\nabla^2 grad\ln\beta\right) - 2\alpha^2\Delta\ln\beta\,(grad\ln\alpha, 0) \\
&- \alpha^2|grad\ln\alpha|^2(0, grad\ln\beta) - \alpha^2|grad\ln\beta|^2(grad\ln\alpha, 0) \\
&= -2\alpha^2\Delta\ln\beta\,(grad\ln\alpha, 0) - \alpha^2|grad\ln\beta|^2(grad\ln\alpha, 0) \\
&- \alpha^2|grad\ln\alpha|^2(0, grad\ln\beta) + (0, grad\Delta\ln\beta) + (0, Ricci\,(grad\ln\beta))\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
Tr_G\widetilde{\nabla}^2(0, grad\ln\beta) &= |grad\ln\alpha|^2(0, grad\ln\beta) + |grad\ln\beta|^2(grad\ln\alpha, 0) \\
&+ \Delta\ln\alpha\,(0, grad\ln\beta) - m\beta^2|grad\ln\beta|^2(0, grad\ln\beta) \\
&+ (0, grad\Delta\ln\beta) - 2\alpha^2\Delta\ln\beta\,(grad\ln\alpha, 0) \\
&- \alpha^2|grad\ln\alpha|^2(0, grad\ln\beta) - \alpha^2|grad\ln\beta|^2(grad\ln\alpha, 0) \\
&+ (0, Ricci\,(grad\ln\beta))\,.
\end{aligned}$$

□

Theorem 2.19. *Let* $\phi : (M^m \times N^n, G = g \oplus h) \longrightarrow \left(M^m \underset{(\alpha,\beta)}{\times} N^n, G_{(\alpha,\beta)}\right)$ *be the map defined by* $\phi(x, y) = (x, y)$. *The map* ϕ *is biharmonic if and only if*

$$\begin{aligned}
&n\left(grad\Delta\ln\alpha\right)+2n\left(grad\left(\left|grad\ln\alpha\right|^2\right)\right)-\frac{n^2}{2}\alpha^2 grad\left(\left|grad\ln\alpha\right|^2\right)\\
&-2n^2\alpha^2\left|grad\ln\alpha\right|^2 grad\ln\alpha+m^2\beta^4\left|grad\ln\beta\right|^2 grad\ln\alpha\\
&+4n\left(\left|grad\ln\alpha\right|^2\right)grad\ln\alpha+2n\left(\Delta\ln\alpha\right)grad\ln\alpha\\
&-m\beta^2\left(\left(2n+4\right)\left|grad\ln\beta\right|^2+2\Delta\ln\beta\right)grad\ln\alpha+2n\left(Ricci\left(grad\ln\alpha\right)\right)=0
\end{aligned}$$

and

$$\begin{aligned}
&m\left(grad\Delta\ln\beta\right)+2m\left(grad\left(\left|grad\ln\beta\right|^2\right)\right)-\frac{m^2}{2}\beta^2 grad\left(\left|grad\ln\beta\right|^2\right)\\
&-2m^2\beta^2\left|grad\ln\beta\right|^2 grad\ln\beta+n^2\alpha^4\left|grad\ln\alpha\right|^2 grad\ln\beta\\
&+4m\left(\left|grad\ln\beta\right|^2\right)grad\ln\beta+2m\left(\Delta\ln\beta\right)grad\ln\beta\\
&-n\alpha^2\left(\left(2m+4\right)\left|grad\ln\alpha\right|^2+2\Delta\ln\alpha\right)grad\ln\beta+2m\left(Ricci\left(grad\ln\beta\right)\right)=0.
\end{aligned}$$

Proof. By definition of the tension field and using the Proposition 2.1, we have

$$\begin{aligned}
\tau\left(\phi\right)&=Tr_G\nabla d\phi\\
&=\widetilde{\nabla}^{\phi}_{(e_i,0)}d\phi\left(e_i,0\right)-d\phi\left(\nabla_{(e_i,0)}\left(e_i,0\right)\right)\\
&+\widetilde{\nabla}^{\phi}_{(0,f_j)}d\phi\left(0,f_j\right)-d\phi\left(\nabla_{(0,f_j)}\left(0,f_j\right)\right)\\
&=\widetilde{\nabla}^{\phi}_{(e_i,0)}\left(e_i,0\right)-\left(\nabla_{(e_i,0)}\left(e_i,0\right)\right)\\
&+\widetilde{\nabla}^{\phi}_{(0,f_j)}\left(0,f_j\right)-\left(\nabla_{(0,f_j)}\left(0,f_j\right)\right)\\
&=\left(\nabla_{e_i}e_i,0\right)-m\beta^2\left(0,grad\ln\beta\right)-\left(\nabla_{e_i}e_i,0\right)\\
&+\left(0,\nabla_{f_j}f_j\right)-n\alpha^2\left(grad\ln\alpha,0\right)-\left(0,\nabla_{f_j}f_j\right),
\end{aligned}$$

then

$$\tau\left(\phi\right)=-m\beta^2\left(0,grad\ln\beta\right)-n\alpha^2\left(grad\ln\alpha,0\right).$$

The map ϕ is biharmonic if and only if

$$\begin{aligned}
&n\left(Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\alpha^2\left(grad\ln\alpha,0\right)+\alpha^2 Tr_G\widetilde{R}\left(\left(grad\ln\alpha,0\right),d\phi\right)d\phi\right)\\
&+m\left(Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\beta^2\left(0,grad\ln\beta\right)+\beta^2 Tr_G\widetilde{R}\left(\left(0,grad\ln\beta\right),d\phi\right)d\phi\right)\\
&=\left(0,0\right).
\end{aligned}\tag{2.16}$$

We will study term by term the above equation, for the first term $Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\alpha^2$

$(grad\ln\alpha, 0)$, it is clear to see that

$$\begin{aligned}Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\alpha^2\left(grad\ln\alpha,0\right) &= \alpha^2 Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\left(grad\ln\alpha,0\right)+2\alpha^2\left(grad\left(|grad\ln\alpha|^2\right),0\right)\\ &+2\alpha^2\Delta\ln\alpha\left(grad\ln\alpha,0\right)+4\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\ &-4\alpha^2\beta^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)\end{aligned}$$

and using Lemma 2.17, we deduce that

$$\begin{aligned}Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\alpha^2\left(grad\ln\alpha,0\right) &= \alpha^2\left(grad\Delta\ln\alpha,0\right)+2\alpha^2\left(grad\left(|grad\ln\alpha|^2\right),0\right)\\ &-n\alpha^4\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)-\alpha^2\beta^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)\\ &+\alpha^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)+4\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\ &+2\alpha^2\Delta\ln\alpha\left(grad\ln\alpha,0\right)+\alpha^2\Delta\ln\beta\left(grad\ln\alpha,0\right)\\ &-\alpha^2\beta^2\left(5\left|grad\ln\alpha\right|^2+2\Delta\ln\alpha\right)\left(0,grad\ln\beta\right)\\ &+\alpha^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)+\alpha^2\left(Ricci\left(grad\ln\alpha\right),0\right).\end{aligned}$$

By Theorem 2.10, we obtain

$$\begin{aligned}&Tr_G\widetilde{R}\left(\left(grad\ln\alpha,0\right),d\phi\right)d\phi\\ &=\widetilde{R}\left(\left(grad\ln\alpha,0\right),\left(e_i,0\right)\right)\left(e_i,0\right)+\widetilde{R}\left(\left(grad\ln\alpha,0\right),\left(0,f_j\right)\right)\left(0,f_j\right)\\ &=-\frac{n}{2}\alpha^2\left(grad\left(|grad\ln\alpha|^2\right),0\right)-n\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\ &-\Delta\ln\beta\left(grad\ln\alpha,0\right)-\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)\\ &+\left(1-m\right)\beta^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)+\left(1-m\right)\beta^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)\\ &+n\alpha^2\beta^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)-\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)\\ &+\left(Ricci\left(grad\ln\alpha\right),0\right),\end{aligned}$$

it follows that

$$\begin{aligned}&Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\alpha^2\left(grad\ln\alpha,0\right)+\alpha^2 Tr_G\widetilde{R}\left(\left(grad\ln\alpha,0\right),d\phi\right)d\phi\\ &=\alpha^2\left(grad\Delta\ln\alpha,0\right)+2\alpha^2\left(grad\left(|grad\ln\alpha|^2\right),0\right)\\ &-\frac{n}{2}\alpha^4\left(grad\left(|grad\ln\alpha|^2\right),0\right)-2n\alpha^4\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\ &-m\alpha^2\beta^2\left|grad\ln\beta\right|^2\left(grad\ln\alpha,0\right)+4\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right)\\ &+2\alpha^2\Delta\ln\alpha\left(grad\ln\alpha,0\right)+n\alpha^4\beta^2\left|grad\ln\alpha\right|^2\left(0,grad\ln\beta\right)\\ &-\alpha^2\beta^2\left(\left(m+4\right)\left|grad\ln\alpha\right|^2+2\Delta\ln\alpha\right)\left(0,grad\ln\beta\right)\\ &+2\alpha^2\left(Ricci\left(grad\ln\alpha\right),0\right).\end{aligned}$$

A similar calculation gives us

$$\begin{aligned}Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\beta^2(0,grad\ln\beta) &= \beta^2(0,grad\Delta\ln\beta)+2\beta^2\left(0,grad\left(|grad\ln\beta|^2\right)\right)\\ &+4\beta^2|grad\ln\beta|^2(0,grad\ln\beta)+\beta^2\Delta\ln\alpha(0,grad\ln\beta)\\ &-m\beta^4|grad\ln\beta|^2(0,grad\ln\beta)+2\beta^2\Delta\ln\beta(0,grad\ln\beta)\\ &+\beta^2|grad\ln\alpha|^2(0,grad\ln\beta)-\alpha^2\beta^2|grad\ln\alpha|^2(0,grad\ln\beta)\\ &-4\alpha^2\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)-2\alpha^2\beta^2\Delta\ln\beta(grad\ln\alpha,0)\\ &-\alpha^2\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)+\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)\\ &+\beta^2(0,Ricci(grad\ln\beta))\end{aligned}$$

and

$$\begin{aligned}&Tr_G\widetilde{R}((0,grad\ln\beta),d\phi)d\phi\\ &=-\frac{m}{2}\beta^2\left(0,grad\left(|grad\ln\beta|^2\right)\right)-m\beta^2|grad\ln\beta|^2(0,grad\ln\beta)\\ &-\Delta\ln\alpha(0,grad\ln\beta)-|grad\ln\alpha|^2(0,grad\ln\beta)\\ &+(1-n)\alpha^2\beta^2|grad\ln\alpha|^2(0,grad\ln\beta)-|grad\ln\beta|^2(grad\ln\alpha,0)\\ &+m\alpha^2\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)+(1-n)\alpha^2\beta^2|grad\ln\beta|^2(grad\ln\alpha,0)\\ &+\beta^2(0,Ricci(grad\ln\beta)).\end{aligned}$$

Which gives us

$$\begin{aligned}&Tr_G\left(\widetilde{\nabla}^{\phi}\right)^2\beta^2(0,grad\ln\beta)+\beta^2Tr_G\widetilde{R}((0,grad\ln\beta),d\phi)d\phi\\ &=\beta^2(0,grad\Delta\ln\beta)+2\beta^2\left(0,grad\left(|grad\ln\beta|^2\right)\right)\\ &-\frac{m}{2}\beta^4\left(0,grad\left(|grad\ln\beta|^2\right)\right)+4\beta^2|grad\ln\beta|^2(0,grad\ln\beta)\\ &-2m\beta^4|grad\ln\beta|^2(0,grad\ln\beta)+2\beta^2\Delta\ln\beta(0,grad\ln\beta)\\ &-n\alpha^2\beta^2|grad\ln\alpha|^2(0,grad\ln\beta)+2\beta^2(0,Ricci(grad\ln\beta))\\ &-\alpha^2\beta^2\left((n+4)|grad\ln\beta|^2+2\Delta\ln\beta\right)(grad\ln\alpha,0)\\ &+m\alpha^2\beta^4|grad\ln\beta|^2(grad\ln\alpha,0).\end{aligned}$$

Finally, we conclude that the map $\phi:(M^m\times N^n,G=g\oplus h)\longrightarrow(M^m\underset{(\alpha,\beta)}{\times}N^n,G_{(\alpha,\beta)})$ defined by $\phi(x,y)=(x,y)$ is biharmonic if and only if

$$
\begin{aligned}
& n\left(grad \Delta \ln \alpha\right)+2 n\left(grad\left(|grad \ln \alpha|^2\right)\right)-\frac{n^2}{2} \alpha^2 grad\left(|grad \ln \alpha|^2\right) \\
& -2 n^2 \alpha^2|grad \ln \alpha|^2 grad \ln \alpha+m^2 \beta^4|grad \ln \beta|^2 grad \ln \alpha \\
& +4 n\left(|grad \ln \alpha|^2\right) grad \ln \alpha+2 n(\Delta \ln \alpha) grad \ln \alpha \\
& -m \beta^2\left((2 n+4)|grad \ln \beta|^2+2 \Delta \ln \beta\right) grad \ln \alpha+2 n(Ricci(grad \ln \alpha))=0
\end{aligned}
$$

and

$$
\begin{aligned}
& m\left(grad \Delta \ln \beta\right)+2 m\left(grad\left(|grad \ln \beta|^2\right)\right)-\frac{m^2}{2} \beta^2 grad\left(|grad \ln \beta|^2\right) \\
& -2 m^2 \beta^2|grad \ln \beta|^2 grad \ln \beta+n^2 \alpha^4|grad \ln \alpha|^2 grad \ln \beta \\
& +4 m\left(|grad \ln \beta|^2\right) grad \ln \beta+2 m(\Delta \ln \beta) grad \ln \beta \\
& -n \alpha^2\left((2 m+4)|grad \ln \alpha|^2+2 \Delta \ln \alpha\right) grad \ln \beta+2 m(Ricci(grad \ln \beta))=0. \ \square
\end{aligned}
$$

As a consequence of Theorem 2.19, we can summarize the results as follows.

Corollary 2.20. *The map* $\phi : (M^m \times N^n, G = g \oplus h) \longrightarrow \left(M^m \underset{\alpha}{\times} N^n, G_\alpha\right)$ *defined by* $\phi(x,y) = (x,y)$ *is biharmonic if and only if the positive function* $\alpha \in C^\infty(M)$ *satisfies the following equation*

$$
\begin{aligned}
& grad \Delta \ln \alpha+\left(2-\frac{n}{2} \alpha^2\right) grad\left(|grad \ln \alpha|^2\right)+2(\Delta \ln \alpha) grad \ln \alpha \\
& +\left(4-2 n \alpha^2\right)\left(|grad \ln \alpha|^2\right) grad \ln \alpha+2 Ricci(grad \ln \alpha)=0.
\end{aligned}
$$

Corollary 2.21. *The map* $\phi : (M^m \times N^n, G = g \oplus h) \longrightarrow \left(M^m \underset{\beta}{\times} N^n, G_\beta\right)$ *defined by* $\phi(x,y) = (x,y)$ *is biharmonic if and only if the positive function* $\beta \in C^\infty(N)$ *satisfies the following equation*

$$
\begin{aligned}
& (grad \Delta \ln \beta)+\left(2-\frac{m}{2} \beta^2\right)\left(grad\left(|grad \ln \beta|^2\right)\right)+2(\Delta \ln \beta) grad \ln \beta \\
& +\left(4-2 m \beta^2\right)|grad \ln \beta|^2 grad \ln \beta+2 Ricci(grad \ln \beta)=0.
\end{aligned}
$$

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