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On Generalized Ricci Recurrent Spacetimes

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Abstract. The object of the present paper is to characterize generalized Ricci recurrent (GR_4) spacetimes. Among others things, it is proved that a conformally flat GR_4 spacetime is a perfect fluid spacetime. We also prove that a GR_4 spacetime with a Codazzi type Ricci tensor is a generalized Robertson Walker spacetime with Einstein fiber. We further show that in a GR_4 spacetime with constant scalar curvature the energy momentum tensor is semisymmetric. Further, we obtain several corollaries. Finally, we cite some examples which are sufficient to demonstrate that the GR_4 spacetime is non-empty and a GR_4 spacetime is not a trivial case.

1. Introduction

The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector, that is, a vector v satisfying $g(v, v) = 0$, where g is the metric tensor. The signature of the metric g of a Riemannian manifold is $(+, +, +, ...+, +, +)$ and of a semi-Riemannian manifold is $(-, -, -, ...+, +, +)$. Lorentzian manifold is a special case of semi-Riemannian manifold. The signature of the metric of a Lorentzian manifold is $(-, +, +, ...+, +, +)$. In a Lorentzian manifold three types of vectors exist such as timelike, spacelike and null vector. In general, a Lorentzian manifold (M, g) may not have a globally timelike vector field. If (M, g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as spacetime. The foundations of general relativity are based on a 4-dimensional spacetime manifold.

Let (M, g) be an n-dimensional Lorentzian manifold with the Lorentzian metric g. A Lorentzian manifold is said to be recurrent [41] if at a point $x \in M$, there exists a 1-form A on some neighbourhood of x such that $\nabla_X R = A(X)R$, where R denotes the curvature tensor of type $(1, 3)$ and ∇ denotes the covariant differentiation with

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respect to X. In 1952, Patterson [33] introduced the notion of Ricci recurrent manifolds. A Lorentzian manifold (M, g) of dimension n is said to be Ricci recurrent if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z) = B(X)S(Y, Z),$$

where B is a non-zero 1-form. He denotes such a manifold by R_n. Ricci recurrent manifolds have been studied by several authors.

In 1995, De, Guha and Kamilya [11] introduced the notion of generalized Ricci recurrent manifolds. A non-flat Riemannian manifold is called generalized Ricci recurrent realizing the following relation

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z), \tag{1.1}$$

where A and B are two non-zero 1-forms, called associated 1-forms. Such a manifold is denoted by GR_n. If the 1-form B vanishes, then the manifold reduces to a Ricci recurrent manifold R_n. This justifies the name generalized Ricci recurrent manifolds and the symbol GR_n for it. A Lorentzian manifold (M, g) of dimension $n \geq 4$ is named generalized Ricci recurrent spacetimes if the relation (1.1) holds.

To characterize generalized Ricci recurrent (GR_4) spacetimes we assume that the associated vector fields U, V corresponding to the 1-forms A, B respectively are timelike vector fields. In a recent paper [21] Mallick, De and De studied generalized Ricci recurrent manifolds with applications to relativity. In the same paper the authors constructed two examples of GR_n. Also Generalized Ricci recurrent manifolds have been studied by several authors.

A Riemannian or a semi-Riemannian manifold is said to be semisymmetric [40] if its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0,$$

for all X, $Y \in \chi(M)$, where $R(X, Y)$ acts as a derivation on the curvature tensor R. Trivial examples of semisymmetric spaces are locally symmetric spaces and all two-dimensional Riemannian spaces. But a semisymmetric space is not necessarily locally symmetric. A fundamental study on such manifolds was made by Szabo [40]. In this connection we can mention the book of Boeckx, Kowalski and Vanhecke [3] and the References there in.

Also, a Riemannian or a semi-Riemannian manifold is said to be Ricci semisymmetric [31] if the Ricci tensor S of type (0,2) satisfies the condition

$$R(X, Y) \cdot S = 0,$$

for all X, $Y \in \chi(M)$.

On the other hand, generalized Robertson-Walker (GRW) spacetimes were introduced in 1995 by Alias, Romero and Sánchez [1, 2].

A Lorentzian manifold M of dimension $n \geq 3$ is named generalized Robertson-Walker (GRW) spacetime if it is the warped product $M = I \times_{q^2} M^*$ with base

$(I, -dt^2)$, warping function q and the fibre (M^*, g^*) is an (n-1)-dimensional Riemannian manifold [1, 2, 8, 35, 36].

If M^* is a 3-dimensional Riemannian manifold of constant curvature, the spacetime is called a Robertson-Walker (RW) spacetime. Therefore, GRW spacetimes are a wide generalization of RW spacetimes on which standard cosmology is modelled. They include the Einstein-de Sitter spacetime, the static Einstein spacetime, the Friedman cosmological models, the de Sitter spacetime and hence applications as inhomogeneous spacetimes admitting an isotropic radiation [8, 35].

Lorentzian manifolds with Ricci tensor of the form

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.2}$$

where α, β are scalar fields and U is a unit timelike vector field corresponding to the 1-form A(that is, $g(U,U) = -1$), are called perfect fluid spacetimes and are of interest in general relativity. In differential geometry they are named quasi Einstein. Semi-Riemannian quasi Einstein spaces arose in the study of exact solutions of Einstein's field equations and in the investigation of quasi-umbilical hypersurfaces of Pseudo-Euclidean spaces [12, 13]. Form (1.2) of the Ricci tensor is implied by Einstein's equation if the energymomentum tensor of the spacetime is perfect fluid with velocity vector field U. A spacetime is called perfect fluid if the energymomentum tensor is of the form

$$T(X,Y) = (\mu + p)A(X)A(Y) + pg(X,Y),$$

where μ is the energy density, p is the isotropic pressure, U is a unit timelike vector field ($g(U,U) = -1$) metrically equivalent to the 1-form A. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [19].

In addition, p and μ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\mu, T_0)$, where T_0 is the absolute temparature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\mu)$. In this case, the perfect fluid is called isentropic[19]. Moreover, if $p = \mu$, then the perfect fluid is termed as stiff matter(see [39] , page 66). Einstein's field equation is given by $S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y)$, κ is the gravitational constant. Einstein's equation implies that matter determines the geometry of spacetime and conversly, the motion of matter is determined by the metric of the space which is non-flat.

The conformal curvature tensor is defined by [34]:

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z &- \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.3}$$

where R is the curvature tensor of type $(1, 3)$, S is the Ricci tensor of type $(0, 2)$, Q is the Ricci operator given by $g(QX, Y) = S(X, Y)$ and r denotes the scalar curvature.

Perfect fluid spacetimes in four dimensions with divergence free conformal curvature tensor (that is, $divC = 0$) were firstly investigated by Shepley and Taub [38] and successively by Sharma [37] and Coley [9].

Recently in [27] Mantica, Molinari and De extended some results to n-dimensional perfect fluids and proved the following:

Theorem 1.1.([27]) *Let (M, g) be a perfect fluid spacetime, that is, the Ricci tensor is of the form $R_{ij} = \alpha g_{ij} + \beta u_i u_j$. If $\nabla_k u_j = \nabla_j u_k$ and $\nabla_h C^h_{ijk} = 0$, then*

(i) *u_i is a concircular vector field and it is rescalable to a timelike vector X_j such that $\nabla_k X_j = \rho g_{jk}$,*

(ii) *(M, g) is a GRW spacetime with Einstein fiber,*

(iii) *The velocity vector annihilates the Weyl tensor, that is, $u_h C^h_{ijk} = 0$.*

Recently, De et al. [14, 15] studied conformally flat almost pseudo-Ricci symmetric spacetimes and spacetimes with semisymmetric energy momentum tensor respectively. Also in [23] Mallick, Suh and De studied spacetime with pseudo-projective curvature tensor. Also several authors studied spacetimes in different way such as [16, 22, 29] and many others. In [7] Chaki and Ray studied spacetimes with covariant constant energy momentum tensor.

Motivated by the above studies in the present paper we characterize generalized Ricci recurrent spacetimes GR_4. At first we determine the nature of the associated 1-forms of GR_4 spacetimes. Next in Section 3, we consider conformally flat GR_4 spacetimes with an additional restriction and prove that such a spacetime is a perfect fluid spacetime. As a consequence we obtain several corollaries. Section 4 is devoted to study GR_4 spacetimes with Codazzi type of Ricci tensor. In this section we prove that a GR_4 spacetime with Codazzi type of Ricci tensor is a GRW spacetime with Einstein fibre. Also state equation is obtained. In Section 5, it is shown that in a GR_4 spacetime with closed associated 1-forms the energymomentum tensor is semisymmetric and Weyl compatible. Finally, we give some examples of generalized Ricci recurrent spacetimes.

2. Nature of the Associated 1-forms of GR_4 Spacetimes

Putting $Y = Z = e_i$ in (1.1), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq 4$, we get

$$dr(X) = rA(X) + 4B(X), \tag{2.1}$$

where $r = \sum_{i=1}^{4} \varepsilon_i S(e_i, e_i)$, $\varepsilon = g(e_i, e_i)$, from which one obtains

$$\begin{aligned} d^2 r(X, Y) = r(\nabla_Y A)X + A(X)\{rA(Y) + 4B(Y)\} \\ +4(\nabla_Y B)X. \end{aligned} \tag{2.2}$$

Interchanging X and Y in (2.2) and then subtracting we infer that

$$\begin{aligned} r[(\nabla_Y A)X - (\nabla_X A)Y] + 4\{(\nabla_Y B)X - (\nabla_X B)Y \\ +A(X)B(Y) - A(Y)B(X)\} = 0. \end{aligned} \tag{2.3}$$

Suppose the vector fields U and V corresponding to the 1-forms A and B respectively are collinear. Then it follows at once from (2.3) that the 1-form A is closed if and only if B is closed.

However if $r = constant$, then from (2.1) we obtain that the 1-form A is closed if and only if B is closed. If $r = 0$, then from (2.1) we arrive at a contradiction. Therefore in a GR_4 the scalar curvature is non-zero.

From the above discussions we can state the following:

Proposition 2.1. *If the associated vector fields of a GR_4 spacetime are collinear or the scalar curvature is constant, then the associated 1-form A is closed if and only if B is closed. Also the scalar curvature is non-zero in a GR_4 spacetime.*

3. Conformally Flat GR_4 Spacetimes

In this section we characterize conformally flat GR_4 spacetimes. In the paper [21] the authors proved the following:

Theorem 3.1.([21]) *A conformally flat GR_n is a quasi Einstein manifold provided the associated vector fields are collinear.*

It is to be noted that the basic geometric features of GR_4 spacetimes are also being maintained in the Lorentzian manifold which is necessarily a semi-Riemannian manifold. Hence all the results of GR_n Riemannian manifold are true in GR_4 spacetime. Only the form of the Ricci tensor will be changed, because in the spacetime the associated vector field corresponding to the 1-form A is assumed to be a unit timelike vector, that is, $g(U,U) = -1$. In the paper [21] the authors obtained the form of the Ricci tensor as

$$S(X,Y) = \frac{r - \lambda(n-2)}{2(n-1)} g(X,Y) + \frac{(n-2)(r+\lambda n)}{2(n-1)g(U,U)} A(X)A(Y), \tag{3.1}$$

where λ is a scalar defined by $B(X) = \lambda A(X)$.

In the GR_4 spacetime we take U as a unit timelike vector field, that is, $g(U,U) = -1$. Hence (3.1) reduces to

$$S(X,Y) = \frac{r - 2\lambda}{6} g(X,Y) - \frac{(r+4\lambda)}{3} A(X)A(Y), \tag{3.2}$$

which implies that the spacetime is a perfect fluid spacetime.

Thus we obtain the following:

Proposition 3.2. *A conformally flat generalized Ricci recurrent spacetime is a perfect fluid spacetime, provided the associated vector fields are collinear.*

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{3.3}$$

being κ the Einstein's gravitational constant, T is the energymomentum tensor ([39], [32]) describing the matter content of the spacetime.

From (3.2) and (3.3) we obtain

$$\kappa T(X,Y) = -\frac{r+\lambda}{3}g(X,Y) - \frac{r+4\lambda}{3}A(X)A(Y), \tag{3.4}$$

where A is a non-zero 1-form such that $A(X) = g(X,U)$, for all X and U is a unit timelike vector field.
Equation (3.4) is of the form of a perfect fluid spacetime

$$T(X,Y) = (p+\mu)A(X)A(Y) + pg(X,Y),$$

where $\kappa p = -\frac{r+\lambda}{3}$ and $\kappa(p+\mu) = -\frac{r+4\lambda}{3}$ from which it follows that $p = -\frac{r+\lambda}{3\kappa}$ and $\mu = -\frac{\lambda}{\kappa}$, p being the isotropic pressure and μ the energy density.

Therefore the state equation is $p = \frac{1}{3}(\mu - \frac{r}{\kappa})$. But in a generalized Ricci recurrent spacetime the scalar curvature is non-zero. Thus the state equation indicates that the fluid spacetime is not radiative due to the presence of r and κ. Moreover, the values of p and μ are in accordance with the present day observations.

4. GR_4 Spacetimes with Codazzi Type of Ricci Tensor

The Ricci tensor is said to be Codazzi type [17] if $(\nabla_X S)(Y,Z) = (\nabla_Z S)(X,Y)$. Codazzi type of Ricci tensor implies by Bianchi's 2nd identity that the scalar curvature r is constant. Mallick, De and De [21] proved

Theorem 4.1.([21]) *A GR_n with Codazzi type of Ricci tensor is a quasi Einstein manifold whose Ricci tensor is of the form*

$$S(X,Y) = -\frac{s}{t}g(X,Y) + \frac{r}{nt}A(X)A(Y), \tag{4.1}$$

where $t = g(U,U)$ and $s = g(U,V)$.

Since we consider the associated vector field U as a unit timelike vector, the equation (4.1) can be rewritten as

$$S(X,Y) = sg(X,Y) - \frac{r}{4}A(X)A(Y), \tag{4.2}$$

which implies that the spacetime is a perfect fluid. Thus we get the following:

Proposition 4.2. *A GR_4 spacetime with Codazzi type of Ricci tensor is a perfect fluid spacetime.*

In an n-dimensional Lorentzian manifold, we know [24] that,

$$
\begin{aligned}
divC = \frac{n-3}{n-2}[(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) - \frac{1}{2(n-1)}\{g(Y,Z)dr(X) \\
-g(X,Y)dr(Z)\}]. \qquad (4.3)
\end{aligned}
$$

Therefore if the Ricci tensor is of Codazzi type, then from (4.3) it follows that $divC = 0$, that is, the Weyl conformal curvature tensor is divergence free.

Lemma 4.3. *Let (M, g) be a perfect fluid spacetime, that is, $S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y)$, where the vector field U metrically equivalent to the 1-form A is a unit timelike vector field and $\beta \neq 0$. If $div\ C = 0$ and $dr(X) = 0$, then the 1-form A is closed.*

Proof. The divergence of the conformal curvature tensor is given by (4.3). Therefore the conditions $div\ C = 0$ and $dr(X) = 0$, imply $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$. From $S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y)$, by taking a frame field and contracting X and Y, we get $r = 4\alpha - \beta$, so that $4(X\alpha) = (X\beta)$.
Now taking covariant derivative of the Ricci tensor and using $4(X\alpha) = (X\beta)$ from $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$ we infer that

$$
\begin{aligned}
\frac{1}{4}(X\beta)g(Y,Z) + (X\beta)A(Y)A(Z) + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)] \\
= \frac{1}{4}(Y\beta)g(X,Z) + (Y\beta)A(X)A(Z) + \beta[(\nabla_Y A)(X)A(Z) \\
+A(X)(\nabla_Y A)(Z)]. \qquad (4.4)
\end{aligned}
$$

Taking a frame field and contracting over Y and Z we obtain from the above equation

$$(X\beta) + 4(U\beta)A(X) + 4\beta[(\nabla_U A)(X) + A(X)(\nabla_{e_i} A)(e_i)] = 0, \qquad (4.5)$$

where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold. Now putting $X = Y = U$ in (4.4) yields

$$2X\beta = 4(U\beta)A(X) - 4\beta(\nabla_U A)(X). \qquad (4.6)$$

Using (4.6) in (4.5) we get

$$4\beta A(X)(\nabla_{e_i} A)(e_i) = X\beta.$$

Again putting $X = U$ in (4.5) it follows that

$$3(U\beta) + 4\beta(\nabla_{e_i} A)(e_i) = 0. \qquad (4.7)$$

From (4.6) and (4.7) we infer that

$$-3A(X)(U\beta) = (X\beta). \qquad (4.8)$$

Replacing X by U in the above equation and using $A(U) = -1$ gives $U\beta = 0$ and hence from (4.8) we finally get $X\beta = 0$.
Now putting $Z = U$ in (4.4) and using $X\beta = 0$ we obtain

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0,$$

which implies that the 1-form A is closed. Hence the integral curves of the vector field U are geodesic. □

Now using the above Lemma and Theorem 1.1 of [27] we are in a position to state the following:

Theorem 4.4. *A generalized Ricci recurrent spacetime with Codazzi type of Ricci tensor is a GRW spacetime with Einstein fiber. Also the velocity vector field U satisfies the condition $C(X,Y)U = 0$.*

Remark 4.5. For dimension $n = 4$, the condition $C(X,Y)U = 0$ means

$$A(C(X,Y)Z) = 0,$$

where the vector field U is metrically equivalent to the 1-form A.
The above equation implies that [20]

$$A(W)C(X,Y)Z + A(X)C(Y,W)Z + A(Y)C(W,X)Z = 0.$$

Now replacing W by U in the above expression and using $C(X,Y)U = 0$ yields $C(X,Y)Z = 0$. It is known [4] that a *GRW* spacetime M is conformally flat if and only if M is a *RW* spacetime. Thus in $n = 4$ dimension *GRW* spacetime reduces to *RW* spacetime. Therefore a GR_4 spacetime with Codazzi type of Ricci tensor is a *RW* spacetime.

Since a 4-dimensional spacetime with $divC = 0$ and $dr(X) = 0$ is a Yang Pure space [18], Theorem 4.4 can be restated as:

Corollary 4.6. *Any 4-dimensional perfect fluid Yang Pure space with $\beta \neq 0$ is a GRW spacetime with Einstein fiber.*

Remark 4.7. Extensions and modifications of General Relativity have a prominent role in addressing the problem of dark energy and dark matter (the so-called dark side). A generalization of Einstein's theory is the so-called $f(R)$ theory of gravitation. It was introduced by Buchdahl [5] in 1970. In [6], Capozziello et al. proved that an n-dimensional GRW spacetime with divergence free conformal curvature tensor exhibits a perfect fluid stress-energy tensor for any $f(R)$ gravity model. In Theorem 4.4 of our paper, we prove that a generalized Ricci recurrent spacetime with Codazzi type of Ricci tensor is a GRW spacetime with Einstein fiber. Hence, from the result of Capozziello et al., we conclude that the spacetime under consideration in our paper proclaims a perfect fluid stress-energy tensor for any $f(R)$ gravity model.

Now we consider Einstein's equation without cosmological constant, that is,

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{4.9}$$

being κ the Einstein's gravitational constant, T is the energymomentum tensor [32, 39] describing the matter content of the spacetime.

From (4.2) and (4.9) we obtain

$$\kappa T(X,Y) = (s - \frac{r}{2})g(X,Y) - \frac{r}{4}A(X)A(Y), \tag{4.10}$$

where A is a non-zero 1-form such that $A(X) = g(X,U)$, for all X and U is a unit timelike vector field.
The above equation is of the form of a perfect fluid spacetime

$$T(X,Y) = (p+\mu)A(X)A(Y) + pg(X,Y),$$

where $\kappa p = s - \frac{r}{2}$ and $\kappa(p+\mu) = -\frac{r}{4}$ from which it follows that $p = \frac{1}{\kappa}(s - \frac{r}{2})$ and $\mu = \frac{1}{\kappa}[\frac{r}{4} - s]$, p being the isotropic pressure and μ the energy density.

Therefore the state equation is $p = -\mu - \frac{r}{4\kappa}$. Since by hypothesis the Ricci tensor is of Codazzi type, therefore the scalar curvature is constant. Thus the state equation reduces to $p = -\mu + constant$.

5. GR_4 Spacetimes with Constant Scalar Curvature

In this section we characterize generalized Ricci recurrent spacetimes with constant scalar curvature. Taking covariant derivative of (1.1) we get

$$\begin{aligned}(\nabla_X\nabla_Y S)(Z,W) = \nabla_X A(Y)S(Z,W) + A(Y)A(X)S(Z,W)\\ +A(Y)B(X)g(Z,W).\end{aligned} \tag{5.1}$$

Interchanging X and Y in (5.1) we obtain

$$\begin{aligned}(\nabla_Y\nabla_X S)(Z,W) = \nabla_Y A(X)S(Z,W) + A(X)A(Y)S(Z,W)\\ +A(X)B(Y)g(Z,W).\end{aligned} \tag{5.2}$$

Also from (1.1) we have

$$(\nabla_{[X,Y]}S)(Z,W) = A([X,Y])S(Z,W) + B([X,Y])g(Z,W). \tag{5.3}$$

Now subtracting (5.2) and (5.3) from (5.1) and using Ricci identity we get

$$\begin{aligned}(R(X,Y)\cdot S)(Z,W) = \{(\nabla_X A)Y - (\nabla_Y A)X\}S(Z,W)\\ +\{(\nabla_X B)Y - (\nabla_Y B)X\}g(Z,W) + \{A(X)B(Y)\\ -A(Y)B(X)\}g(Z,W).\end{aligned} \tag{5.4}$$

From (2.3) it follows that if the scalar curvature is constant, then $A(X)B(Y) - A(Y)B(X) = 0$, since the 1-forms A and B are closed. Hence from (5.4) we infer that

$$R \cdot S = 0. \tag{5.5}$$

The foregoing equation implies that in such a spacetime under consideration the Ricci tensor is semisymmetric. Therefore from Einstein's field equations we can conclude that the energy momentum tensor is semisymmetric, that is,

$$R \cdot T = 0.$$

In a recent paper [15] De et al. characterize spacetimes with semisymmetric energy momentum tensor. Thus all the results of [15] also hold for GR_4 spacetimes with constant scalar curvature.

Equation (5.5) gives

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0.$$

Now summing cyclically the above equation and applying Bianchi's first identity we get

$$S(R(X,Y)Z,W) + S(R(Y,W)Z,X) + S(R(W,X)Z,Y) = 0. \tag{5.6}$$

Any semi-Riemannian manifold satisfying (5.6) is called Riemannian compatible [25]. Thus we have

Proposition 5.1. *A GR_4 spacetime with constant scalar curvature is Riemann compatible.*

Any semi-Riemannian manifold satisfying

$$S(C(X,Y)Z,W) + S(C(Y,W)Z,X) + S(C(W,X)Z,Y) = 0 \tag{5.7}$$

is called *Weyl-compatible.* Weyl-compatibility have been studied in the Riemannian case by Mantica et al [26].

It is known that both conditions (5.6) and (5.7) are equivalent. Now if we use Einstein's equation in (5.7), we get

$$T(C(X,Y)Z,W) + T(C(Y,W)Z,X) + T(C(W,X)Z,Y) = 0.$$

Therefore we can state the following:

Proposition 5.2. *In a GR_4 spacetime with constant scalar curvature the energy-momentum tensor is Weyl-compatible.*

6. Some Examples of Generalized Ricci Recurrent Spacetimes

Example 6.1.([10]) A generalized concircularly recurrent manifold with constant scalar curvature is a GR_n.

Example 6.2.([28]) A quasi-conformally recurrent manifold with divergence free quasi-conformal curvature tensor is a GR_n, provided the manifold is neither conformally flat nor conformally symmetric.

Example 6.3. The so called $\mathcal{Z}$ tensor is defined by

$$\mathcal{Z}(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y). \tag{6.1}$$

It may be noted that the vanishing of the $\mathcal{Z}$ tensor implies that the manifold to be an Einstein manifold and hence the $\mathcal{Z}$ tensor is a measure of the deviation from an Einstein manifold [30].

$\mathcal{Z}-$recurrent manifold is defined by

$$(\nabla_W \mathcal{Z})(X,Y) = A(W)\mathcal{Z}(X,Y), \tag{6.2}$$

where A is a non-zero 1-form. From (6.1) and (6.2) we obtain

$$(\nabla_W S)(X,Y) = A(W)S(X,Y) + \frac{1}{n}\{dr(W) - A(W)r\}g(X,Y),$$

which implies that the $\mathcal{Z}-$recurrent manifold is a generalized Ricci recurrent manifold.

Conversely, if the manifold is generalized Ricci recurrent, then

$$(\nabla_W \mathcal{Z})(X,Y) = (\nabla_W S)(X,Y) - \frac{dr(W)}{n}g(X,Y),$$

and using (1.1) and (2.1) we get

$$(\nabla_W \mathcal{Z})(X,Y) = A(W)\mathcal{Z}(X,Y).$$

Hence we conclude that a $\mathcal{Z}-$recurrent manifold is a GR_n and conversely.

Remark 6.4. The above examples also hold for GR_4 spacetimes.

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