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Real Hypersurfaces with Invariant Normal Jacobi Operator in the Complex Hyperbolic Quadric

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Abstract. We introduce the notion of Lie invariant normal Jacobi operators for real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_mSO_2$. The invariant normal Jacobi operator implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then in each case, we give a complete classification of real hypersurfaces in $Q^{m*} = SO^o_{m,2}/SO_mSO_2$ with Lie invariant normal Jacobi operators.

1. Introduction

When we consider Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [21, 22, 23]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$.

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to commuting Ricci tensor were investigated by Kimura [5, 6], Pérez [13] and Pérez and Suh [14, 15] respectively. The classification

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problems of the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ with various geometric conditions were discussed in Jeong, Kim and Suh [4], Pérez [13], and Suh [21, 22, 29], where the classification of *contact hypersurfaces*, *parallel Ricci tensor*, *harmonic curvature* and *Jacobi operator* of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ were extensively studied.

Another example of Hermitian symmetric space with rank 2 having non-compact type different from the above ones, is the complex hyperbolic quadric $SO^0_{2,m}/SO_2SO_m$. It is a simply connected Riemannian manifold whose curvature tensor is the negative of the curvature tensor of the complex quadric Q^m (see Besse [2], Helgason [3], and Knapp [10]). The complex hyperbolic quadric also can be regarded as a kind of real Grassmann manifolds of non-compact type with rank 2 . Accordingly, the complex hyperbolic quadric Q^{m*} admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. For $m \geq 2$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of non-compact type and its maximal sectional curvature is equal to -4 (see Klein [7], Kobayashi and Nomizu [11], and Reckziegel [16]).

Two last examples of different Hermitian symmetric spaces with rank 2 in the class of compact type or non-compact type, are the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ or the complex hyperbolic quadric $Q^{m*} = SO^o_{2,m}/SO_mSO_2$, which are a complex hypersurface in complex projective space CP^{m+1} or in complex hyperbolic space respectively(see Romero [17, 18], Suh [24, 25], and Smyth [19]). The complex quadric Q^m or the complex hyperbolic quadric Q^{m*} can be regarded as a kind of real Grassmann manifold of compact or non-compact type with rank 2 respectively(see Helgason [3], Kobayashi and Nomizu [11]). Accordingly, the complex quadric Q^m and the complex hyperbolic quadric Q^{m*} both admit two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$ (see Klein [7] and Reckziegel [16]).

Now let us introduce a complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_2SO_m$, which can be regarded as a Hermitian symmetric space with rank 2 of noncompact type. Montiel and Romero [12] proved that the complex hyperbolic quadric Q^{m*} can be immersed in the indefinite complex hyperbolic space $CH^{m+1}_1(-c)$, $c > 0$, by interchanging the Kähler metric with its opposite. Changing the Kähler metric of CP^{n+1}_{n-s} with its opposite, we have that Q^n_{n-s} endowed with its opposite metric $g' = -g$ is also an Einstein hypersurface of $CH^{n+1}_{s+1}(-c)$. When $s = 0$, we know that $(Q^n_n, g' = -g)$ can be regarded as the complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_2SO_m$, which is immersed in the indefinite complex hyperbolic quadric $CH^{m+1}_1(-c)$, $c > 0$ as a space-like complex Einstein hypersurface.

Apart from the complex structure J there is another distinguished geometric structure on Q^{m*}, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains a S^1-bundle of real structures. Note that these real structures are complex conjugations A on the tangent spaces of the complex hyperbolic quadric Q^{m*}. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in the complex hyperbolic quadric Q^{m*}.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. There are two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*} as follows:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Here $V(A) = \{X \in T_{[z]}Q^{m*} : AX = X\}$ and $JV(A) = \{X \in T_{[z]}Q^{m*} : AX = -X\}$, $[z] \in Q^{m*}$, are the $(+1)$-eigenspace and (-1)-eigenspace for the involution A on $T_{[z]}Q^{m*}$, $[z] \in Q^{m*}$.

When we consider a hypersurface M in the complex hyperbolic quadric Q^{m*}, under the assumption of some geometric properties the unit normal vector field N of M in Q^{m*} can be divided into two cases depending on whether N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [27, 28, 30, 31]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [27] that M is locally congruent to a tube over a totally geodesic complex hyperbolic space $\mathbb{C}H^k$ in the complex hyperbolic quadric Q^{2k^*}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in the complex hyperbolic quadric Q^{m*} is locally congruent to a tube over a totally geodesic and totally real submanifold $\mathbb{R}H^m$ in Q^{m*} or a horosphere (see Suh [9], and Suh and Hwang [30]).

Usually, Jacobi fields along geodesics of a given Riemannian manifold $\bar{M}$ satisfy a well known differential equation. Naturally the classical differential equation inspires the so-called *Jacobi operator*. That is, if $\bar{R}$ is the curvature operator of $\bar{M}$, the Jacobi operator with respect to X at $z \in M$, is defined by

$$(\bar{R}_X Y)(z) = (\bar{R}(Y, X)X)(z)$$

for any $Y \in T_z\bar{M}$. Then $\bar{R}_X \in \text{End}(T_z\bar{M})$ becomes a symmetric endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. Clearly, each tangent vector field X to $\bar{M}$ provides a Jacobi operator with respect to X.

From such a view point, in the complex hyperbolic quadric Q^{m*} the *normal Jacobi operator* $\bar{R}_N$ is defined by

$$\bar{R}_N = \bar{R}(\cdot, N)N \in End\ (T_zM), \quad z \in M$$

for a real hypersurface M in the complex hyperbolic quadric Q^{m*} with unit normal vector field N, where $\bar{R}$ denotes the curvature tensor of the complex hyperbolic quadric Q^{m*}. Of course, the normal Jacobi opeartor $\bar{R}_N$ is a symmetric endomorphism of M in the complex hyperbolic quadric Q^{m*}.

The normal Jacobi operator $\bar{R}_N$ of M in the complex hyperbolic quadric Q^{m*} is said to be *Lie invariant* if the operator $\bar{R}_N$ satisfies

$$0 = (\mathfrak{L}_X \bar{R}_N)Y$$

for any $X, Y \in T_zM$, $z \in M$, where the Lie derivative $(\mathfrak{L}_X \bar{R}_N)Y$ is defined by

$$\begin{aligned}(\mathfrak{L}_X \bar{R}_N)Y =&[X, \bar{R}_N(Y)] - \bar{R}_N([X,Y]) \\ =&\nabla_X(\bar{R}_N(Y)) - \nabla_{\bar{R}_N(Y)}X - \bar{R}_N(\nabla_X Y - \nabla_Y X) \\ =&(\nabla_X \bar{R}_N)Y - \nabla_{\bar{R}_N(Y)}X + \bar{R}_N(\nabla_Y X).\end{aligned} \tag{1.1}$$

For real hypersurfaces in the complex quadric Q^m we investigated the notions of parallel Ricci tensor, harmonic curvature and commuting Ricci tensor, which are respectively given by $\nabla \text{Ric} = 0$, $\delta \text{R}ic = 0$ and $\text{Ric}\cdot\phi = \phi\cdot\text{Ric}$ (see Suh [25], [26], and Suh and Hwang [29]). But from the assumption of Ricci parallel or harmonic curvature, it was difficult for us to derive the fact that either the unit normal vector field N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [25] and [26] we gave a classification with the further assumption of $\mathfrak{A}$-isotropic. Also in the study of complex hyperbolic quadric Q^{m*} we also have some obstructions to get the fact that the unit normal N is singular.

In the paper due to Suh [27] we investigate this problem of isometric Reeb flow for the complex hyperbolic quadric $Q^{m*} = SO^o_{2,m}/SO_mSO_2$. In view of the previous results, naturally, we expected that the classification might include at least the totally geodesic $Q^{m-1^*} \subset Q^{m*}$. But, the results are quite different from our expectations. The totally geodesic submanifolds of the above type are not included. Now we introduce the classification as follows:

Theorem 1.1. *Let M be a real hypersurface of the complex hyperbolic quadric $Q^{m*} = SO^o_{2,m}/SO_mSO_2$, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

But fortunately, when we consider Lie invariant normal Jacobi operator, that is., $\mathcal{L}_X \bar{R}_N = 0$ for any tangent vector field X on M in Q^{m*}, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows:

Theorem 1.2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with Lie invariant normal Jacobi operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Then motivated by Theorem 1.1 and Theorem 1.2, we can give a complete classification for real hypersurfaces in Q^{m*} with invariant normal Jacobi operator as follows:

Theorem 1.3. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$ with Lie invariant normal normal Jacobi operator. Then M is locally congruent to a tube of radius r over a totally geodesic CH^k in Q^{2k^*} or a horosphere* *whose center at infinity is $\mathfrak{A}$-isotropic singular.*

2. The Complex Hyperbolic Quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric Q^{m*} different from the complex quadric Q^m. This section is due to Klein and Suh [9], and Suh [28].

The m-dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m-dimensional complex quadric Q^m, which is a kind of Hermitian symmetric space of non-compact type with rank 2 (see Besse [2], and Helgason [3]).

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [20, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m, which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m, see [7] and [16]. Another related result by Smyth, [20, Theorem 1], which states that any complex hypersurface $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO^0_{2,m}/SO_2SO_m$. As Q^{1^*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO^0_{1,2}/SO_2$, and Q^{2^*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO^0_{2,m}$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the "origin" $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \to G,\ g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 &&&&& \\ & -1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& \ddots & \\ &&&&& 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\mathrm{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2, and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \left\{ X \in \mathfrak{gl}(m+2, \mathbb{R}) : X^t \cdot s = -s \cdot X \right\}$$

(see [10, p. 59]). In the sequel we will write members of $\mathfrak{g}$ as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} ,$$

where X_{11}, X_{12}, X_{21}, X_{22} are real matrices of the dimension 2×2, $2\times m$, $m\times 2$ and $m\times m$, respectively. Then

$$\mathfrak{g} = \left\{ \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right) \ : \ X_{11}^t = -X_{11},\ X_{12}^t = X_{21},\ X_{22}^t = -X_{22} \right\} .$$

The linearisation $\sigma_L = \mathrm{Ad}(s) : \mathfrak{g} \to \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} =& \mathrm{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} : sXs^{-1} = X\} \\ =& \left\{ \left(\begin{smallmatrix} X_{11} & 0 \\ 0 & X_{22} \end{smallmatrix}\right) \ : \ X_{11}^t = -X_{11},\ X_{22}^t = -X_{22} \right\} \\ \cong& \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K, and the $2m$-dimensional linear subspace

$$\mathfrak{m} = \mathrm{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} : sXs^{-1} = -X\} = \left\{ \left(\begin{smallmatrix} 0 & X_{12} \\ X_{21} & 0 \end{smallmatrix}\right) \ : X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X,Y) = \frac{1}{2}\,\mathrm{tr}(Y^t \cdot X) = \mathrm{tr}(Y_{12} \cdot X_{21}) \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

g is clearly $\mathrm{Ad}(K)$-invariant, and therefore corresponds to an $\mathrm{Ad}(G)$-invariant Riemannian metric on Q^{m*}. The complex structure J of the Hermitian symmetric space is given by

$$JX = \mathrm{Ad}(j)X \quad \text{for} \quad X \in \mathfrak{m}, \quad \text{where} \quad j := \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in K .$$

Because j is in the center of K, the orthogonal linear map J is $\mathrm{Ad}(K)$-invariant, and thus defines an $\mathrm{Ad}(G)$-invariant Hermitian structure on Q^{m*}. By identifying the multiplication with the unit complex number i with the application of the linear map J, the tangent spaces of Q^{m*} thus become m-dimensional complex linear spaces, and we will adopt this point of view in the sequel.

Like for the complex quadric (again compare [7, 8, 16]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1-bundle $\mathfrak{A}$ of real structures. The situation here differs from that of the complex quadric in that for Q^{m*}, the real structures in $\mathfrak{A}$ cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, $\mathfrak{A}$ still plays an important role in the description of the geometry of Q^{m*}.

Let

$$a_0 := \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} .$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2\, SO_m$. However, $\mathrm{Ad}(a_0)$ still leaves $\mathfrak{m}$ invariant, and therefore defines an $\mathbb{R}$-linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0}Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0}Q^{m*}$), and hence a real structure on $T_{p_0}Q^{m*}$. But A_0 commutes with $\mathrm{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \left(\begin{smallmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{smallmatrix}\right)$ with $t \in \mathbb{R}$ (so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \mathrm{Ad}(g) = \mu^{-2} \cdot \mathrm{Ad}(g) \circ A_0\ .$$

This equation shows that the object which is $\mathrm{Ad}(K)$-invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the "circle of real structures"

$$\mathfrak{A}_{p_0} := \{\lambda\, A_0 | \lambda \in S^1\}\ .$$

$\mathfrak{A}_{p_0}$ is $\mathrm{Ad}(K)$-invariant, and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*}. For any $A \in \mathfrak{A}$, the tangent line to the fibre of $\mathfrak{A}$ through A is spanned by JA.

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_pQ^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space T_pQ^{m*}. Here $V(A)$ resp. $JV(A)$ are the $(+1)$-eigenspace resp. the (-1)-eigenspace of A. For every unit vector $Z \in T_pQ^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$Z = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [16, Proposition 3]. Here t is uniquely determined by Z. The vector Z is singular, i.e. contained in more than one Cartan subalgebra of $\mathfrak{m}$, if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called $\mathfrak{A}$-*principal*, whereas the vectors with $t = \frac{\pi}{4}$ are called $\mathfrak{A}$-*isotropic*. If Z is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by Z.

Like for the complex quadric, the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be fully described in terms of the "fundamental geometric structures" g, J and $\mathfrak{A}$. In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X,Y)Z$ corresponds to $-[[X,Y],Z]$ for $X, Y, Z \in \mathfrak{m}$, see [11, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}\bar{R}(X,Y)Z = &- g(Y,Z)X + g(X,Z)Y \\ &- g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ \\ &- g(AY,Z)AX + g(AX,Z)AY \\ &- g(JAY,Z)JAX + g(JAX,Z)JAY \end{aligned} \tag{2.1}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m, compare [16, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

Let M be a real hypersurface in complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{F}$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$. We denote by νM the normal bundle of M.

We first introduce some notations. For a fixed real structure $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$ we decompose AX into its tangential and normal component, that is,

$$AX = BX + \rho(X)N$$

where BX is the tangential component of AX and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

Since $JX = \phi X + \eta(X)N$ and $A\xi = B\xi + \rho(\xi)N$ we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = \eta(B\phi X) + \eta(X)\rho(\xi).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$

At each point $[z] \in M$ we define

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M : AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\},$$

which is the maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$. Then by using the same method for real hypersurfaces in complex hyperbolic quadric Q^{m*} as in Berndt and Suh [1] we get the following

Lemma 2.1. *Let M be a real hypersurface in complex hyperbolic quadric Q^{m*}. Then the following statements are equivalent:*

(i) *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal,*

(ii) $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$,

(iii) *There exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$.*

Assume now that the normal vector $N_{[z]}$ of M is not $\mathfrak{A}$-principal. Then there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 < t \leq \frac{\pi}{4}$. This implies

$$\begin{aligned} AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_{[z]} &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi_{[z]} &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned} \tag{2.2}$$

and therefore $\mathcal{Q}_{[z]} = T_{[z]}Q^m \ominus ([Z_1] \oplus [Z_2])$ is strictly contained in $\mathcal{C}_{[z]}$. Moreover, we have

$$A\xi_{[z]} = B\xi_{[z]} \text{ and } \rho(\xi_{[z]}) = 0.$$

We have

$$\begin{aligned} g(B\xi_{[z]} + \delta\xi_{[z]}, N_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, \xi_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, B\xi_{[z]} + \delta\xi_{[z]}) &= \sin^2(2t), \end{aligned}$$

where the function δ denotes $\delta = -g(\xi, A\xi) = -(\sin^2 t - \cos^2 t) = \cos 2t$. Therefore

$$U_{[z]} = \frac{1}{\sin(2t)}(B\xi_{[z]} + \delta\xi_{[z]})$$

is a unit vector in $\mathcal{C}_{[z]}$ and

$$\mathcal{C}_{[z]} = \mathcal{Q}_{[z]} \oplus [U_{[z]}] \text{ (orthogonal direct sum).}$$

If $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$, then N is not $\mathfrak{A}$-principal in an open neighborhood of $[z]$, and therefore U is a well-defined unit vector field on that open neighborhood. We summarize this in the following

Lemma 2.2. *Let M be a real hypersurface in complex hyperbolic quadric Q^{m*} whose unit normal $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$. Then there exists an open neighborhood of $[z]$ in M and a section A in $\mathfrak{A}$ on that neighborhood consisting of real structures such that*

(i) $A\xi = B\xi$ *and* $\rho(\xi) = 0$,

(ii) $U = (B\xi + \delta\xi)/||B\xi + \delta\xi||$ *is a unit vector field tangent to* $\mathcal{C}$,

(iii) $\mathcal{C} = \mathcal{Q} \oplus [U]$.

3. Some General Equations

Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z{\in}M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM : AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

Lemma 3.1.([24]) *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

From the explicit expression of the Riemannian curvature tensor of the complex hyperbolic quadric Q^{m*} we can easily derive the Codazzi equation for a real hypersurface M in Q^{m*}:

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, Z) \\ &= -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\ &\quad - \rho(X)g(BY, Z) + \rho(Y)g(BX, Z) \\ &\quad + \eta(BX)g(BY, \phi Z) + \eta(BX)\rho(Y)\eta(Z) \\ &\quad - \eta(BY)g(BX, \phi Z) - \eta(BY)\rho(X)\eta(Z). \end{aligned} \tag{3.1}$$

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^{m*} satisfies

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M. Inserting $Z = \xi$ into the Codazzi equation leads to

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = 2g(\phi X, Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX).$$

On the other hand, we have

$$\begin{aligned} &g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned} \tag{3.2}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$d\alpha(Y) = d\alpha(\xi)\eta(Y) + 2\delta\rho(Y),$$

where the function $\delta = -g(A\xi, \xi)$ and $\rho(Y) = g(AN, Y)$ for any vector field Y on M in Q^{m*}.

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = &-2\delta\eta(X)\rho(Y) + 2\delta\rho(X)\eta(Y) \\ &+ \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned} 0 =&2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\ &- 2\delta\rho(X)\eta(Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX) + 2\delta\eta(X)\rho(Y) \\ =&g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\ &- 2\rho(X)\eta(BY + \delta Y) + 2\rho(Y)\eta(BX + \delta X) \\ =&g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y) \\ &- 2\rho(X)g(Y, B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\rho(Y). \end{aligned} \tag{3.3}$$

If $AN = N$ we have $\rho = 0$, otherwise we can use Lemma 2.2 to calculate $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$. Thus we have proved

Lemma 3.2. *Let M be a Hopf hypersurface in Q^{m*}, $m \geq 3$. Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\phi B\xi.$$

If the unit normal vector field N is $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ such that $AN = N$. Then we have $\rho = 0$ and $\phi B\xi = -\phi\xi = 0$, and therefore

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

If N is not $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ as in Lemma 2.2 and get

$$\begin{aligned} &\rho(X)(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi B\xi \\ = &-g(X, \phi(B\xi + \delta\xi))(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi(B\xi + \delta\xi) \\ = &||B\xi + \delta\xi||^2(g(X, U)\phi U - g(X, \phi U)U) \\ = &\sin^2(2t)(g(X, U)\phi U - g(X, \phi U)U), \end{aligned}$$

which is equal to 0 on $\mathcal{Q}$ and equal to $\sin^2(2t)\phi X$ on $\mathcal{C} \ominus \mathcal{Q}$. Altogether we have proved:

Lemma 3.3. *Let M be a Hopf hypersurface in Q^{*m}, $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves $\mathcal{Q}$ *and* $\mathcal{C} \ominus \mathcal{Q}$ *invariant and we have*

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\delta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where $\delta = \cos 2t$ *as in section 3.*

4. Invariant Normal Jacobi Operator and a Key Lemma

By the curvature tensor $\bar{R}$ of (2.1) for a real hypersurface in the complex hyperbolic quadric Q^{m*} in section 2, the normal Jacobi operator $\bar{R}_N$ is defined in such a way that

$$\begin{aligned}\bar{R}_N(X) &= \bar{R}(X,N)N\\ &= -X - g(JN,N)JX + g(JX,N)JN + 2g(JX,N)JN\\ &\quad - g(AN,N)AX + g(AX,N)AN - g(JAN,N)JAX + g(JAX,N)JAN\end{aligned}$$

for any tangent vector field X in T_zM and the unit normal N of M in T_zQ^{m*}, $z\in Q^{m*}$. Then the normal Jacobi operator $\bar{R}_N$ becomes a symmetric operator on the tangent space T_zM, $z\in M$, of Q^{m*}. From this, by the complex structure J and the complex conjugations $A \in \mathfrak{A}$, together with the fact that $g(A\xi, N) = 0$ and $\xi = -JN$ in section 3, the normal Jacobi operator $\bar{R}_N$ is given by

$$\begin{aligned}\bar{R}_N(Y) = &-Y - 3\eta(Y)\xi - g(AN,N)AY\\ &+ g(AY,N)AN + g(AY,\xi)A\xi\end{aligned} \tag{4.1}$$

for any $Y\in T_zM$, $z\in M$. Then the derivative of $\bar{R}_N$ is given by

$$\begin{aligned}(\nabla_X\bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_XY)\\ &= -3(\nabla_X\eta)(Y)\xi - 3\eta(Y)\nabla_X\xi\\ &\quad - \{g(\bar{\nabla}_X(AN),N) + g(AN,\bar{\nabla}_XN)\}AY\\ &\quad - g(AN,N)\{\bar{\nabla}_X(AY) - A\nabla_XY\}\\ &\quad + \{g(\bar{\nabla}_X(AY) - A\nabla_XY,N) + g(AY,\bar{\nabla}_XN)\}AN\\ &\quad + g(AY,N)\bar{\nabla}_X(AN)\\ &\quad + \{g(\bar{\nabla}_X(AY) - A\nabla_XY,\xi)A\xi + g(AY,\bar{\nabla}_X\xi)\}A\xi\\ &\quad + g(AY,\xi)\bar{\nabla}_X(A\xi),\end{aligned} \tag{4.2}$$

where the connection $\bar{\nabla}$ on the complex hyperbolic quadric Q^{m*} is given by

$$\begin{aligned}\bar{\nabla}_X(AY) - A\nabla_XY &= (\bar{\nabla}_XA)Y + A\bar{\nabla}_XY - A\nabla_XY\\ &= q(X)JAY + A\sigma(X,Y)\\ &= q(X)JAY + g(SX,Y)AN.\end{aligned}$$

From this, together with the invariance of $\mathcal{L}_X \bar{R}_N = 0$ in (1.1), it follows that

$$
\begin{aligned}
(4.3)\qquad & \nabla_{\bar{R}_N(Y)}X - \bar{R}_N(\nabla_Y X) \\
&= (\nabla_X \bar{R}_N)Y \\
&= -3g(\phi SX, Y)\xi - 3\eta(Y)\phi SX \\
&\quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}AY \\
&\quad - g(AN, N)\{q(X)JAY + g(SX, Y)AN\} \\
&\quad + \{q(X)g(JAY, N) + g(SX, Y)g(AN, N)\}AN \\
&\quad - g(AY, SX)AN + g(AY, N)\{(\bar{\nabla}_X A)N + A\bar{\nabla}_X N\} \\
&\quad + g((\bar{\nabla}_X A)Y, \xi)A\xi + g(AY, \phi SX + \sigma(X, \xi))A\xi \\
&\quad + g(AY, \xi)\bar{\nabla}_X(A\xi),
\end{aligned}
$$

where we have used the equation of Gauss $\bar{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi)$, $\sigma(X, \xi)$ denotes the normal bundle $T^{\perp}M$ valued second fundament tensor on M in Q^{m*}. From this, putting $Y = \xi$ and using $(\bar{\nabla}_X A)Y = q(X)JAY$, and $\bar{\nabla}_X N = -SX$ we have

$$
\begin{aligned}
(4.4)\qquad & \nabla_{\bar{R}_N(\xi)}X - \bar{R}_N(\nabla_\xi X) \\
&= (\nabla_X \bar{R}_N)\xi = -3\phi SX \\
&\quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}A\xi \\
&\quad - g(AN, N)\{q(X)JA\xi + g(SX, \xi)AN\} \\
&\quad + \{q(X)g(JA\xi, N) + g(SX, \xi)g(AN, N)\}AN \\
&\quad - g(A\xi, SX)AN + g(q(X)JA\xi, \xi)A\xi \\
&\quad + g(A\xi, \phi SX + \sigma(X, \xi))A\xi \\
&\quad + g(A\xi, \xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)\}AN.
\end{aligned}
$$

From this, by taking the inner product with the unit normal N, we have

$$
\begin{aligned}
(4.5)\qquad & -g(A\xi, SX)g(AN, N) + g(A\xi, \xi)\{q(X)g(JA\xi, N) \\
&\quad + g(A\phi SX, N) + g(SX, \xi)g(AN, N)\} = 0.
\end{aligned}
$$

Then by putting $X = \xi$ and using the assumption of Hopf, we have

$$
(4.6)\qquad q(\xi)g(A\xi, \xi)^2 = 0.
$$

This gives that $q(\xi) = 0$ or $g(A\xi, \xi) = 0$. The latter case implies that the unit normal N is $\mathfrak{A}$-isotropic. Now we only consider the case $q(\xi) = 0$.

We put $Y = \xi$ in (4.1). Then it follows that

$$
\bar{R}_N(\xi) = -4\xi - \{g(AN, N) - g(A\xi, \xi)\}A\xi = -4\xi - 2g(AN, N)A\xi,
$$

where we have used that $g(A\xi, \xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N)$.

Differentiating this one, it follows that

$$(4.7)\qquad \begin{aligned} &\nabla_{\bar{R}_N(\xi)}X - \bar{R}_N(\nabla_\xi X) \\ &= (\nabla_X \bar{R}_N)\xi \\ &= -4\nabla_X\xi - 2\{g(\bar{\nabla}_X(AN), N)A\xi + g(AN, \bar{\nabla}_X N)A\xi\} \\ &\quad - 2g(AN, N)\bar{\nabla}_X(A\xi). \end{aligned}$$

Then, by putting $Y = \xi$, and taking the inner product of (4.7) with the unit normal N, we have

$$g(AN, N)\{q(\xi)g(A\xi, \xi) - \alpha g(A\xi, \xi)\} = 0.$$

From this, together with $q(\xi) = 0$, it follows that

$$(4.8)\qquad \alpha g(A\xi, \xi)g(AN, N) = 0.$$

Then from (4.8) we can assert the following lemma.

Lemma 4.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with parallel normal Jacobi operator. Then the unit normal vector field N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Proof. When the Reeb function α is non-vanishing, the unit normal N is $\mathfrak{A}$-isotropic. When the Reeb function α identically vanishes, let us show that N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. In order to do this, from the condition of Hopf, we can differentiate $S\xi = \alpha\xi$ and use the equation of Codazzi (3.1) in section 3, then we get the formula

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

From this, if we put $\alpha = 0$, together with the fact $g(\xi, AN) = 0$ in section 3, we know $g(Y, AN)g(\xi, A\xi) = 0$ for any $Y\in T_zM$, $z\in M$. This gives that the vector AN is normal, that is, $AN = g(AN, N)N$ or $g(A\xi, \xi) = 0$, which implies respectively the unit normal N is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. This completes the proof of our Lemma. □

By virtue of this Lemma, we distinguish between two classes of real hypersurfaces in the complex hyperbolic quadric Q^{m*} with invariant normal Jacobi operator : those that have $\mathfrak{A}$-principal unit normal, and those that have $\mathfrak{A}$-isotropic unit normal vector field N. We treat the respective cases in sections 5 and 6.

5. Invariant Normal Jacobi Operator with $\mathfrak{A}$-principal Normal

In this section let us consider a real hypersurface M in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjugation $A\in\mathfrak{A}$. This also implies that $A\xi = -\xi$ for the Reeb vector field $\xi = -JN$.

Then the normal Jacobi operator $\bar{R}_N$ in section 4 becomes

$$(5.1)\qquad \bar{R}_N(X) = -X - 3\eta(X)\xi - AX + \eta(X)\xi = -X - 2\eta(X)\xi - AX,$$

where we have used that $AN = N$ and

$$\begin{aligned} g(AX,\xi)A\xi &= g(AX,JN)AJN = g(X,AJN)AJN \\ &= g(X,JAN)JAN = g(X,JN)JN \\ &= \eta(X)\xi. \end{aligned}$$

On the other hand, we can put

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY,N) = g(Y,AN) = g(Y,N) = 0$. So it becomes always $AY = BY$ for any vector field Y on M in Q^{m*}. Then by differentiating (5.1) along any direction X, we have

$$\begin{aligned} (\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= -2(\nabla_X\eta)(Y) - 2\eta(Y)\nabla_X\xi - (\nabla_X B)Y. \end{aligned} \tag{5.2}$$

Now let us consider that the normal Jacobi operator $\bar{R}_N$ is invariant, that is, $\mathcal{L}_X\bar{R}_N = 0$. This is given by

$$\begin{aligned} 0 &= (\mathcal{L}_X\bar{R}_N)Y \\ &= \mathcal{L}_X(\bar{R}_N Y) - \bar{R}_N(\mathcal{L}_X Y) \\ &= [X,\bar{R}_N Y] - \bar{R}_N[X,Y] \\ &= \nabla_X(\bar{R}_N Y) - \nabla_{\bar{R}_N(Y)}X - \bar{R}_N(\nabla_X Y - \nabla_Y X) \\ &= \nabla_X(\bar{R}_N Y) - \nabla_{\bar{R}_N(Y)}X + \bar{R}_N(\nabla_Y X). \end{aligned}$$

Then it follows that

$$\begin{aligned} -2g(\phi SX,Y)\xi - 2\eta(Y)\phi SX - (\nabla_X B)Y = &\{\nabla_Y X + 2\eta(\nabla_Y X)\xi + A\nabla_Y X\} \\ &- \{\nabla_Y X + 2\eta(Y)\nabla_\xi X + \nabla_{AY}X\}. \end{aligned}$$

From this putting $Y = \xi$ and using $A\xi = -\xi$, it follows that

$$\begin{aligned} -2\phi SX - (\nabla_X B)\xi &= 2\eta(\nabla_\xi X)\xi + A\nabla_\xi X + \nabla_\xi X \\ &= -2\phi SX - \{q(X)JA\xi - \sigma(X,A\xi) + \eta(SX)N\}. \end{aligned} \tag{5.3}$$

where we have used the following

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= \bar{\nabla}_X(A\xi) - A\nabla_X\xi \\ &= \Big\{(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi\Big\} - A\phi SX \\ &= q(X)JA\xi + A\phi SX - \sigma(X,A\xi) + g(SX,\xi)AN - A\phi SX \\ &= q(X)JA\xi - \sigma(X,A\xi) + \alpha\eta(X)N. \end{aligned}$$

Then by taking the inner product of (5.3) with the unit normal N, we have

$$q(X) = 2\alpha\eta(X).$$

This implies $q(\xi) = 2\alpha$, and the 1-form q is given by

$$q(X) = q(\xi)\eta(X). \tag{5.4}$$

On the other hand, in section 4 from the Lie invariance of the normal Jacobi operator we have calculated the following

$$\begin{aligned}
\nabla_{\bar{R}_N(\xi)}X - \bar{R}_N(\nabla_\xi X) \\
&= (\nabla_X \bar{R}_N)\xi = -3\phi SX \\
&\quad - \{q(X)g(JAN, N) - g(ASX, N) - g(AN, SX)\}A\xi \\
&\quad - g(AN, N)\{q(X)JA\xi + g(SX, \xi)AN\} \\
&\quad + \{q(X)g(JA\xi, N) + g(SX, \xi)g(AN, N)\}AN \\
&\quad - g(A\xi, SX)AN + g(q(X)JA\xi, \xi)A\xi \\
&\quad + g(A\xi, \phi SX + \sigma(X, \xi))A\xi \\
&\quad + g(A\xi, \xi)\{q(X)JA\xi + A\phi SX + g(SX, \xi)AN\}.
\end{aligned} \tag{5.5}$$

From this, by taking the inner product with the unit normal N, we have

$$\begin{aligned}
-g(A\xi, SX)g(AN, N) + g(A\xi, \xi)\{q(X)g(JA\xi, N) \\
+ g(A\xi, \xi)\{g(JA\xi, N) + g(A\phi SX, N) + g(SX, \xi)g(AN, N)\} = 0.
\end{aligned} \tag{5.6}$$

Then by putting $X = \xi$ and using the assumption of Hopf, we have

$$q(\xi)g(A\xi, \xi)^2 = 0. \tag{5.7}$$

From this, together with (5.4) and $A\xi = -\xi$, it follows that the 1-form q vanishes identically on M.

On the other hand, we know that the complex hyperbolic quadric Q^{m*} can be immersed into the indefinite complex hyperbolic space CH_1^{m+1} in C_2^{m+2} (see Montiel and Romero [12], and Kobayashi and Nomizu [11]). Then the same 1-form q appears in the Weingarten formula

$$\tilde{\nabla}_X \bar{z} = -A_{\bar{z}}X + q(X)J\bar{z}$$

for unit normal vector fields $\{\bar{z}, J\bar{z}\}$ on the complex hyperbolic quadric Q^{m*} which can be immersed in indefinite complex hyperbolic space CH_1^{m+1} as a space-like complex hypersurface, where $\tilde{\nabla}$ denotes the Riemannian connection on CH_1^{m+1} induced from the Euclidean connection on C_2^{m+2} (see Smyth [19] and [20]). But the 1-form q never vanishes on Q^{m*}. This gives a contradiction (see Smyth [19]). This means that there do not exist any real hypersurfaces in the complex hyperbolic quadric Q^{m*} with invariant normal Jacobi operator, that is, $\mathcal{L}_X \bar{R}_N = 0$ for the $\mathfrak{A}$-principal unit normal vector field N.

6. Invariant Normal Jacobi Operator with $\mathfrak{A}$-isotropic Normal

In this section let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be put

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \text{and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with (2.2) and the anti-commuting $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0,\ g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

By virtue of these formulas for the $\mathfrak{A}$-isotropic unit normal, the normal Jacobi operator $\bar{R}_N$ in section 4 is given by

$$\bar{R}_N(Y) = -Y - 3\eta(Y)\xi + g(AY, N)AN + g(AY, \xi)A\xi.$$

Then the derivative of the normal Jacobi operator $\bar{R}_N$ on M is given as follows:

$$\begin{aligned}(\nabla_X \bar{R}_N)Y &\\ = &- 3(\nabla_X \eta)(Y)\xi - 3\eta(Y)\nabla_X \xi + g(\nabla_X(AN), Y)AN \\ &+ g(AN, Y)\nabla_X(AN) + g(Y, \nabla_X(A\xi))A\xi + g(A\xi, Y)\nabla_X(A\xi).\end{aligned} \tag{6.1}$$

On the other hand, the Lie invariance (4.1) gives that

$$\begin{aligned}(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N(Y)) - \bar{R}_N(\nabla_X Y) \\ &= \nabla_{\bar{R}_N(Y)} X - \bar{R}_N(\nabla_Y X).\end{aligned} \tag{6.2}$$

Then by putting $Y = \xi$ in (6.1) and (6.2), and using $\bar{R}_N(\xi) = 4\xi$, we have

$$\begin{aligned}-3\phi SX &- g(AN, \phi SX)AN - g(\phi SX, A\xi)A\xi \\ = &- 4\nabla_\xi X + \{\nabla_\xi X + 3\eta(\nabla_\xi X)\xi \\ &- g(A\nabla_\xi X, N)AN - g(A\nabla_\xi X, \xi)A\xi\}\end{aligned} \tag{6.3}$$

From this, taking the inner product (6.3) with the vector field AN, it follows that

$$4g(\phi SX, AN) = 4g(\nabla_\xi X, AN).$$

Then from this, together with (6.3), we get

$$\phi SX = \nabla_\xi X - \eta(\nabla_\xi X)\xi. \tag{6.4}$$

For any $X\in\xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space T_zM, $z\in M$, we know that $\nabla_\xi X$ is orthogonal to the Reeb vector field ξ, that is, $\eta(\nabla_\xi X) = -g(\nabla_\xi \xi, X) = 0$. Then the formula (6.4) becomes for any tangent vector field $X\in\xi^\perp$

$$\phi SX = \nabla_\xi X. \tag{6.5}$$

When we consider the $\mathfrak{A}$-isotropic unit normal, the vector fields $A\xi$ and AN belong to the distribution $\mathcal{C} - \mathcal{Q}$ in section 3.

On the other hand, by virtue of Lemma 3.1, we prove the following for a Hopf hypersurface in Q^{m*} with $\mathfrak{A}$-isotropic unit normal vector field as follows:

Lemma 6.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field. Then*

$$SAN = 0, \quad \textit{and} \quad SA\xi = 0. \tag{6.6}$$

Proof. Let us denote by $\mathcal{C} - \mathcal{Q} = \text{Span}[A\xi, AN]$. Since N is isotropic, $g(AN, N) = 0$ and $g(A\xi, \xi) = 0$. By differentiating $g(AN, N) = 0$, and using $(\bar\nabla_X A)Y = q(X)JAY$ in the introduction and the equation of Weingarten, we know that

$$\begin{aligned} 0 &= g(\bar\nabla_X(AN), N) + g(AN, \bar\nabla_X N) \\ &= g(q(X)JAN - ASX, N) - g(AN, SX) \\ &= -2g(ASX, N). \end{aligned}$$

Then $SAN = 0$. Moreover, by differentiating $g(A\xi, N) = 0$, and using $g(AN, N) = 0$ and $g(A\xi, \xi) = 0$, we have the following formula

$$\begin{aligned} 0 &= g(\bar\nabla_X(A\xi), N) + g(A\xi, \bar\nabla_X N) \\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X) \\ &= -2g(SA\xi, X) \end{aligned}$$

for any $X\in T_zM$, $z\in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

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