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Applications of the Schwarz Lemma and Jack's Lemma for the Holomorphic Functions

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Abstract. We consider a boundary version of the Schwarz Lemma on a certain class of functions which is denoted by $\mathcal{N}$. For the function $f(z) = z + a_2z^2 + a_3z^3 + ...$ which is defined in the unit disc D such that the function $f(z)$ belongs to the class $\mathcal{N}$, we estimate from below the modulus of the angular derivative of the function $\frac{f''(z)}{f(z)}$ at the boundary point c with $f'(c) = 0$. The sharpness of these inequalities is also proved.

1. Introduction

One of the main tool of complex functions theory is the Schwarz Lemma. This lemma is an important result which gives estimates about the values of holomorphic functions defined from the unit disc into itself. It plays an effective role in many fields of analysis, especially in the theory of geometric function hyperbolic geometry. The standard Schwarz Lemma, which is a direct application of the maximum modulus principle, is commonly stated as follows:

Let $f : D \to D$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\sigma}$ for real σ ([6], p.329).

We use the following lemma from [7] which is related to the function $\frac{f''(z)}{f(z)}$ we will investigate.

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Lemma 1.1.(Jack's Lemma) *Let $f(z)$ be a non-constant and holomorphic function in the unit disc D with $f(0)=0$. If*

$$|f(z_0)| = \max\{|f(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that $\frac{z_0 f'(z_0)}{f(z_0)} = k$.

For historical background about the Schwarz Lemma and its applications on the boundary of the unit disc, we refer to [2, 6]. Also, a different application of Jack's Lemma is shown in [7, 14, 18].

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ which are holomorphic in $D = \{z : |z| < 1\}$. Also, let $\mathcal{N}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in D. \tag{1.1}$$

Let $f(z) \in \mathcal{N}$ be a holomorphic function in the unit disc D. Consider the function

$$\varphi(z) = 2\frac{1-m(z)}{2-m(z)}, \tag{1.2}$$

where $m(z) = \frac{zf'(z)}{f(z)}$. The function $\varphi(z)$ is holomorphic in the unit disc and $\varphi(0) = 0$. We show that $|\varphi(z)| < 1$ for $|z| < 1$. We suppose that there exists a point $z_0 \in D$ such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1.$$

From Jack's Lemma, we have

$$\varphi(z_0) = e^{i\theta}, \quad \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = k.$$

Therefore, from (1.2) we obtain

$$\begin{aligned} \Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \Re\left(2\frac{1-\varphi(z_0)}{2-\varphi(z_0)} + \frac{z_0\varphi'(z_0)}{2-\varphi(z_0)} - \frac{z_0\varphi'(z_0)}{1-\varphi(z_0)}\right) \\ &= \Re\left(2\frac{1-e^{i\theta}}{2-e^{i\theta}} + \frac{ke^{i\theta}}{2-e^{i\theta}} - \frac{ke^{i\theta}}{1-e^{i\theta}}\right). \end{aligned}$$

Since

$$\begin{aligned} 2\frac{1-e^{i\theta}}{2-e^{i\theta}} &= 6\frac{1-\cos\theta - i\sin\theta}{5-4\cos\theta}, \\ \frac{ke^{i\theta}}{2-e^{i\theta}} &= k\frac{2\cos\theta - 1 + 2i\sin\theta}{5-4\cos\theta} \end{aligned}$$

and

$$\frac{ke^{i\theta}}{1-e^{i\theta}} = k\frac{\cos\theta - 1 + i\sin\theta}{2\left(1-\cos\theta\right)},$$

we take

$$\begin{aligned}\Re\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) &= 6\frac{1-\cos\theta}{5-4\cos\theta} + k\frac{2\cos\theta - 1}{5-4\cos\theta} + \frac{k}{2} \\ &\geq 6\frac{1-\cos\theta}{5-4\cos\theta} + \frac{2\cos\theta - 1}{5-4\cos\theta} + \frac{1}{2} \\ &= \frac{6-6\cos\theta + 2\cos\theta - 1}{5-4\cos\theta} + \frac{1}{2} \\ &= \frac{3}{2}\end{aligned}$$

and

$$\Re\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) \geq \frac{3}{2}.$$

This contradicts the assumed inequality (1.1). This means that there is no point $z_0 \in D$ such that $|\varphi(z_0)| = 1$ for all $z \in D$. Thus, we obtain $|\varphi(z)| < 1$ for $z \in D$. By the Schwarz Lemma, we obtain

$$|a_2| \leq \frac{1}{2}. \tag{1.3}$$

The result is sharp and the extremal function is

$$f(z) = z + \frac{z^2}{2}.$$

Since the area of applicability of the Schwarz Lemma is quite wide, there exist many studies about it. Among these are the boundary version of the Schwarz Lemma, which is about estimating from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of the Schwarz Lemma is given as follows:

If f extends continuously to some boundary point c with $|c| = 1$, and if $|f(b)| = 1$ and $f'(c)$ exists, then by the classical Schwarz Lemma, it follows that

$$|f'(c)| \geq 1. \tag{1.4}$$

In addition to conditions of the boundary Schwarz Lemma, if f fixes the point zero, that is, $f(0) = 0$, then the inequality

$$|f'(c)| \geq \frac{2}{1+|f'(0)|} \tag{1.5}$$

is obtained [19].

Inequality (1.5) is sharp, with equality possible for each value of $|f'(0)|$. In addition, for $c=1$ in the inequality (1.5), equality occurs for the function $f(z) = z\frac{z+\gamma}{1+\gamma z}$, $\gamma \in [0,1]$. Also, $|f'(c)| > 1$ unless $f(z) = ze^{i\theta}$, θ real. Inequality (1.4), (1.5) and their generalizations have important applications in the geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 5, 8, 11, 13, 15, 16, 17, 18, 19].

For our results, we need the following lemma called the Julia-Wolff Lemma (see [20]).

Lemma 1.2.(Julia-Wolff Lemma) *Let f be a holomorphic function in D, $f(0)=0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(c)$ at $c \in \partial D$, $|f(c)|=1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

Corollary 1.3. *The holomorphic function f has a finite angular derivative $f'(c)$ if and only if f' has the finite angular limit $f'(c)$ at $c \in \partial D$.*

In [5], all zeros of the holomorphic function in the unit disc different from $z=0$ and the holomorphic function which has no zero in the unit disc except $z=0$ have been considered, respectively. Thus, the stronger inequalities have been obtained. M. Jeong found a necessary and sufficient condition for a holomorphic function with fixed points only at the boundary of the unit disc and had some relations with derivatives of the function at these fixed points [8].

D. M. Burns and S. G. Krantz [3] and D. Chelst [4] studied the uniqueness part of the Schwarz Lemma. According to M. Mateljevic's studies, some other types of results which are related to the subject can be found in [9, 10]. In addition, [11] was posed on ResearchGate where is discussed concerning results in more general aspects.

Mercer [12] prove a version of the Schwarz Lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [13].

In this work, we show an application of Jack's Lemma for certain subclasses of holomorphic functions on the unit disc that provide (1.1) inequality. Also, we will give the Schwarz Lemma for this class. Moreover, we will give at the boundary Schwarz Lemma for this class.

2. Main Results

In this section, a boundary version of the Schwarz Lemma for holomorphic functions is investigated. The modulus of the angular derivative of the holomorphic function $\frac{f''(z)}{f(z)}$ that belongs to the class of $\mathcal{A}$ related to the class $\mathcal{N}$ of holomorphic function, on the boundary point of the unit disc has been estimated from below.

Theorem 2.1. *Let $f(z) \in \mathcal{N}$. Suppose that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f'(c)=0$. Then we have the inequality*

$$\left|\frac{f''(c)}{f(c)}\right| \geq 2. \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$f(z) = z + \frac{z^2}{2}.$$

Proof. Let us consider the following function

$$\varphi(z) = 2\frac{1 - m(z)}{2 - m(z)},$$

where $m(z) = \frac{zf'(z)}{f(z)}$. Then $\varphi(z)$ is holomorphic function in the unit disc D, $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for $|z| < 1$. Also, we have $|\varphi(c)| = 1$ for $f'(c) = 0$. It is obviously that

$$\varphi'(z) = -2\frac{m'(z)}{(2 - m(z))^2}.$$

Therefore, from (1.4), we take

$$1 \leq |\varphi'(c)| = 2\frac{|m'(c)|}{|2 - m(c)|^2}$$

and

$$|m'(c)| \geq 2.$$

Since

$$m'(z) = \frac{(f'(z) + f''(z)z)\, f(z) - f'(z)f''(z)z}{(f(z))^2},$$

for $f'(c) = 0$ we have

$$|m'(c)| = \left| \frac{f''(c)}{f(c)} \right|.$$

Therefore, we obtain

$$\left| \frac{f''(c)}{f(c)} \right| \geq 2.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = z + \frac{z^2}{2}.$$

Then, we obtain

$$f''(z) = 1,$$
$$\frac{f''(z)}{f(z)} = \frac{1}{z + \frac{z^2}{2}}$$

and

$$\left|\frac{f''(-1)}{f(-1)}\right| = 2. \qquad \square$$

The inequality (2.1) can be strengthened as below by taking into account a_2 which is second coefficient in the expansion of the function $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$.

Theorem 2.2. *Let $f(z) \in \mathcal{N}$. Suppose that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f'(c) = 0$. Then we have the inequality*

$$\left|\frac{f''(c)}{f(c)}\right| \geq \frac{4}{1 + 2|a_2|}. \tag{2.2}$$

The inequality (2.2) is sharp with extremal function

$$f(z) = z + \frac{z^2}{2}.$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem 2.1. Therefore, we take from (1.5), we obtain

$$\frac{2}{1 + |\varphi'(0)|} \leq |\varphi'(c)| = 2\frac{|m'(c)|}{|2 - m(c)|^2}.$$

Since

$$\begin{aligned} \varphi(z) &= 2\frac{1 - m(z)}{2 - m(z)} = 2\frac{1 - \frac{zf'(z)}{f(z)}}{2 - \frac{zf'(z)}{f(z)}} \\ &= -2\frac{a_2 z + \left(2a_3 - a_2^2\right) z^2 + \ldots}{1 - a_2 z - \left(2a_3 - a_2^2\right) z^2 - \ldots} \end{aligned}$$

and

$$|\varphi'(0)| = 2|a_2|,$$

we take

$$\frac{2}{1 + 2|a_2|} \leq \frac{|m'(c)|}{2},$$

$$|m'(c)| \geq \frac{4}{1 + 2|a_2|}$$

and

$$\left|\frac{f''(c)}{f(c)}\right| \geq \frac{4}{1 + 2|a_2|}.$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = z + \frac{z^2}{2}.$$

Then

$$\left| \frac{f''(-1)}{f(-1)} \right| = 2.$$

On the other hand, from the Taylor expansion of $f(z)$, we take

$$z + a_2 z^2 + a_3 z^3 + ... = z + \frac{z^2}{2},$$
$$1 + a_2 z + a_3 z^2 + ... = 1 + \frac{z}{2},$$
$$a_2 z + a_3 z^2 + ... = \frac{z}{2}$$

and

$$a_2 = \frac{1}{2}.$$

Therefore, we obtain

$$\frac{4}{1 + 2|a_2|} = 2. \qquad \square$$

The inequality (2.2) can be strengthened as below by taking into account a_3 which is third coefficient in the expansion of the function $f(z) = z + a_2 z^2 + a_3 z^3 + ...$.

Theorem 2.3. *Let $f(z) \in \mathcal{N}$. Suppose that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f'(c) = 0$. Then we have the inequality*

$$\left| \frac{f''(c)}{f(c)} \right| \geq 2 \left(1 + \frac{2(1 - 2|a_2|)^2}{1 - 4|a_2|^2 + 4|a_3|} \right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$f(z) = z \left(1 + \frac{z^2}{2} \right)^{\frac{1}{2}}.$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$t(z) = \frac{\varphi(z)}{B(z)},$$

where $B(z)=z$. The function $t(z)$ is holomorphic in D. According to the maximum princible, we have $|\varphi(z)|\leq|B(z)|$ for each $z\in D$. In particular, we have

$$\begin{aligned}|t(z)| &= -2\frac{a_2z+\left(2a_3-a_2^2\right)z^2+...}{\left(1-a_2z-\left(2a_3-a_2^2\right)z^2-...\right)z}\\ &= -2\frac{a_2+\left(2a_3-a_2^2\right)z+...}{1-a_2z-\left(2a_3-a_2^2\right)z^2-...}\\ |t(0)| &= 2|a_2|\leq 1\end{aligned}$$

and

$$|t'(0)|=4|a_3|.$$

Furthermore, it can be seen that

$$\frac{c\varphi'(c)}{\varphi(c)}=|\varphi'(c)|\geq|B'(c)|=\frac{cB'(c)}{B(c)}.$$

Consider the function

$$n(z)=\frac{t(z)-t(0)}{1-\overline{t(0)}t(z)}.$$

This function is holomorphic in D, $|n(z)|\leq 1$ for $|z|<1$, $n(0)=0$, and $|n(c)|=1$ for $c\in\partial D$. From (1.5), we obtain

$$\begin{aligned}\frac{2}{1+|n'(0)|}\leq|n'(b)| &= \frac{1-|t(0)|^2}{\left|1-\overline{t(0)}t(c)\right|^2}|t'(c)|\\ &\leq\frac{1+|t(0)|}{1-|t(0)|}\left\{|\varphi'(c)|-|B'(c)|\right\}.\end{aligned}$$

Since

$$n'(z)=\frac{1-|t(0)|^2}{\left(1-\overline{t(0)}t(z)\right)^2}t'(z)$$

and

$$|n'(0)|=\frac{|t'(0)|}{1-|t(0)|^2}=\frac{4|a_3|}{1-4|a_2|^2},$$

we take

$$\begin{aligned}\frac{2}{1+\frac{4|a_3|}{1-4|a_2|^2}} &\leq\frac{1+2|a_2|}{1-2|a_2|}\left\{\frac{|m'(c)|}{2}-1\right\},\\ \frac{2\left(1-4|a_2|^2\right)}{1-4|a_2|^2+4|a_3|} &\leq\frac{1+2|a_2|}{1-2|a_2|}\left\{\frac{|m'(c)|}{2}-1\right\},\\ \frac{2\left(1-2|a_2|\right)^2}{1-4|a_2|^2+4|a_3|} &\leq\frac{|m'(c)|}{2}-1,\end{aligned}$$

$$|m'(c)| \geq 2\left(1+\frac{2\left(1-2\left|a_2\right|\right)^2}{1-4\left|a_2\right|^2+4\left|a_3\right|}\right)$$

and

$$\left|\frac{f''(c)}{f(c)}\right| \geq 2\left(1+\frac{2\left(1-2\left|a_2\right|\right)^2}{1-4\left|a_2\right|^2+4\left|a_3\right|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z)=z\left(1+\frac{z^2}{2}\right)^{\frac{1}{2}}.$$

Then

$$f'(z)=\sqrt{2}\frac{z^2+1}{\sqrt{z^2+2}},$$

$$f''(z)=\sqrt{2}\frac{z\left(z^2+3\right)}{\left(z^2+2\right)^{\frac{3}{2}}}$$

and

$$\left|\frac{f''(i)}{f(i)}\right|=4.$$

On the other hand, from the Taylor expansion of $f(z)$, we get

$$z+a_2z^2+a_3z^3+...=z\left(1+\frac{z^2}{2}\right)^{\frac{1}{2}},$$

$$1+a_2z+a_3z^2+...=\left(1+\frac{z^2}{2}\right)^{\frac{1}{2}},$$

$$a_2z+a_3z^2+...=\left(1+\frac{z^2}{2}\right)^{\frac{1}{2}}-1$$

and

$$a_2+a_3z+...=\frac{\left(1+\frac{z^2}{2}\right)^{\frac{1}{2}}-1}{z}.$$

Passing to limit in the last equality yields $a_2=0$. Similarly, using straightforward calculations, we take $a_3=\frac{1}{4}$. Therefore, we obtain

$$2\left(1+\frac{2\left(1-2\left|a_2\right|\right)^2}{1-4\left|a_2\right|^2+4\left|a_3\right|}\right)=4.$$ □

Theorem 2.4. *Let $f(z)\in\mathcal{N}$, $1-\frac{zf'(z)}{f(z)}$ has no zeros in D except $z=0$ and $a_2>0$. Suppose that, for some $c\in\partial D$, f has an angular limit $f(c)$ at c, $f'(c)=0$. Then* *we have the inequality*

$$\left|\frac{f''(c)}{f(c)}\right| \geq 2\left(1-\frac{a_2 \ln^2(2a_2)}{a_2 \ln(2a_2)-|a_3|}\right). \tag{2.4}$$

Proof. Let $a_2>0$ and let us consider the function $t(z)$ as in Theorem 2.3. Taking account of the equality $|t(0)|=2|a_2|$, we denote by $\ln t(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln t(0)=\ln(2a_2)=\ln|2a_2|+i\arg(2a_2)<0,\quad a_2>0$$

and

$$\ln(2a_2)<0.$$

Take the following auxiliary function

$$\Phi(z)=\frac{\ln t(z)-\ln t(0)}{\ln t(z)+\ln t(0)}.$$

It is obvious that $\Phi(z)$ is a holomorphic function in D, $\Phi(0)=0$, $|\Phi(z)|<1$ for $z\in D$. Therefore, the function $\Phi(z)$ satisfies the assumptions of the Schwarz Lemma on the boundary.

Since

$$\Phi'(z)=\frac{2\ln t(0)}{(\ln t(z)+\ln t(0))^2}\frac{t'(z)}{t(z)}$$

and

$$\Phi'(c)=\frac{2|\ln t(0)|}{(\ln t(0)+\ln t(c))^2}\frac{t'(c)}{t(c)},$$

we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} &\geq \frac{2|\ln t(0)|}{|\ln t(0)+\ln t(c)|^2}\left|\frac{t'(c)}{t(c)}\right| \\ &= \frac{-2\ln t(0)}{\ln^2 t(0)+\arg^2 t(c)}\left\{|\varphi'(c)|-|B'(c)|\right\}.\end{aligned}$$

Also, since

$$\begin{aligned}\Phi'(0) &= \frac{2\ln t(0)}{(\ln t(0)+\ln t(0))^2}\frac{t'(0)}{t(0)} \\ &= \frac{1}{2\ln t(0)}\frac{t'(0)}{t(0)} \\ &= \frac{1}{2\ln(2a_2)}\frac{4a_3}{2a_2} \\ &= \frac{a_3}{a_2\ln(2a_2)}\end{aligned}$$

and

$$|\Phi'(0)| = -\frac{|a_3|}{a_2 \ln(2a_2)},$$

we obtain

$$\frac{2}{1 - \frac{|a_3|}{a_2 \ln(2a_2)}} \geq \frac{-2}{\ln(2a_2)} \left(\frac{|m'(c)|}{2} - 1 \right),$$

$$1 - \frac{a_2 \ln^2(2a_2)}{a_2 \ln(2a_2) - |a_3|} \leq \frac{|m'(c)|}{2}$$

and

$$|m'(c)| \geq 2 \left(1 - \frac{a_2 \ln^2(2a_2)}{a_2 \ln(2a_2) - |a_3|} \right). \qquad \square$$

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