KYUNGPOOK Math. J. 60(2020), 485-505
https://doi.org/10.5666/KMJ.2020.60.3.485
pISSN 1225-6951 eISSN 0454-8124

Extreme Points, Exposed Points and Smooth Points of the Space $\mathcal{L}_s({}^2l_\infty^3)$

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Abstract. We present a complete description of all the extreme points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ which leads to a complete formula for $\|f\|$ for every $f \in \mathcal{L}_s({}^2l_\infty^3)^*$. We also show that $extB_{\mathcal{L}_s({}^2l_\infty^3)} \subset extB_{\mathcal{L}_s({}^2l_\infty^n)}$ for every $n \geq 4$. Using the formula for $\|f\|$ for every $f \in \mathcal{L}_s({}^2l_\infty^3)^*$, we show that every extreme point of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ is exposed. We also characterize all the smooth points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$.

1. Introduction

We denote by B_E the closed unit ball of a real Banach space E and by E^* the dual space of E. A point $x \in B_E$ is called an *extreme point* of B_E if the equation $x = \frac{1}{2}(y+z)$ for some $y, z \in B_E$ implies $x = y = z$. A point $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. A point $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $extB_E, expB_E$ and smB_E the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that $P(x) = L(x,x)$ for every $x \in E$. We denote by $\mathcal{L}({}^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x,y)|$. The subspace of all continuous symmetric bilinear forms on E is denoted by $\mathcal{L}_s({}^2E)$. We denote by $\mathcal{P}({}^2E)$ the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

In 1998, Choi *et al.* [2, 3] initiated the classification of the extreme points

Received September 5, 2019; revised March 10, 2020; accepted April 21, 2020.
2010 Mathematics Subject Classification: 46A22.
Key words and phrases: symmetric bilinear forms on $\mathbb{R}^3$ with the supremum norm, extreme points, exposed points, smooth points.

of the unit ball of $\mathcal{P}(^2 l_1^2)$ and $\mathcal{P}(^2 l_2^2)$. Kim classified the exposed 2-homogeneous polynomials in $\mathcal{P}(^2 l_p^2)$ $(1 \leq p \leq \infty)$ [11] and the extreme points, exposed points, and smooth points of the unit ball of $\mathcal{P}(^2 d_*(1,w)^2)$ [13, 15, 19], where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight w. Recently, Kim [25, 26, 28] classified all the extreme points, exposed points, and smooth points of the unit ball of $\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})$, where $\mathbb{R}^2_{h(\frac{1}{2})}$ is the plane $\mathbb{R}^2$ with the hexagonal norm of weight $\frac{1}{2}$.

In 2009, Kim [12] initiated the classification of the extreme points, exposed points, and smooth points of the unit ball of $\mathcal{L}_s(^2 l_\infty^2)$. Kim [14, 16, 17, 18, 20, 21, 22, 23, 24] classified the extreme points, exposed points, and smooth points of the unit balls of the spaces

$$\mathcal{L}_s(^3 l_\infty^2), \mathcal{L}(^3 l_\infty^2), \mathcal{L}_s(^2 d_*(1,w)^2), \mathcal{L}(^2 d_*(1,w)^2), \mathcal{L}_s(^2\mathbb{R}^2_{h(w)}) \text{ and } \mathcal{L}(^2\mathbb{R}^2_{h(w)}),$$

where $\mathbb{R}^2_{h(w)}$ is the plane $\mathbb{R}^2$ with the hexagonal norm of weight w, $\|(x,y)\|_{h(w)} = \max\{|y|, |x| + (1-w)|y|\}$. Recently, Kim [25] characterized the extreme points of the spaces $\mathcal{L}_s(^2 l_\infty^n)$ and $\mathcal{L}(^2 l_\infty^n)$ for $n \geq 2$.

Let $n \geq 2$ and $l_\infty^n := \mathbb{R}^n$ with the supremum norm. Given $\{a_{ij}\}_{i,j=1}^n \subset \mathbb{R}$, let $T \in \mathcal{L}(^2 l_\infty^n)$ be defined by the rule

$$T((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{1 \leq i,j \leq n} a_{ij} x_i y_j.$$

If $n = 2$, for simplicity, we will write

$$T = (a_{11}, a_{22}, a_{12}, a_{21})^t$$

as a 4×1 column vector. If $T \in \mathcal{L}_s(^2 l_\infty^2)$, we will write

$$T = (a_{11}, a_{22}, 2a_{12})^t$$

as a 3×1 column vector. If $n = 3$, for simplicity, we will write

$$T = (a_{11}, a_{22}, a_{33}, a_{12}, a_{21}, a_{13}, a_{31}, a_{23}, a_{32})^t$$

as a 9×1 column vector. If $T \in \mathcal{L}_s(^2 l_\infty^3)$, we will write

$$T = (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{13}, 2a_{23})^t$$

as a 6×1 column vector.

In [12] it was shown:

(a) $extB_{\mathcal{L}_s(^2 l_\infty^2)} = \{\pm(1,0,0)^t, \pm(0,1,0)^t, \pm(\frac{1}{2}, -\frac{1}{2}, 1)^t, \pm(\frac{1}{2}, -\frac{1}{2}, -1)^t\}$;

(b) $expB_{\mathcal{L}_s(^2 l_\infty^2)} = extB_{\mathcal{L}_s(^2 l_\infty^2)}$.

We refer to ([1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] for some recent works about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

In this paper we present a complete description of the 42 extreme points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ which leads to a complete formula of $\|f\|$ for every $f \in \mathcal{L}_s({}^2l_\infty^3)^*$. We also show that $extB_{\mathcal{L}_s({}^2l_\infty^3)} \subset extB_{\mathcal{L}_s({}^2l_\infty^n)}$ for every $n \geq 4$. Using the formula of $\|f\|$ for every $f \in \mathcal{L}_s({}^2l_\infty^3)^*$, we show that every extreme point of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$ is exposed. The main result about smooth points is known to "the Mazur density theorem." Recall that the Mazur density theorem says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. Motivated by the Mazur density theorem we characterize all the smooth points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$.

2. The Extreme Points of the Unit Ball of $\mathcal{L}_s({}^2l_\infty^3)$

Recently, Kim [22] showed the following: Let

$$\begin{aligned}\Omega = \{&(1,1,1,1,1,1),(1,-1,1,0,1,0),(1,1,-1,1,0,0),(-1,1,1,0,0,1),\\ &(1,1,1,-1,1,-1),(1,-1,-1,0,0,1),(-1,-1,1,1,0,0),(1,1,1,1,-1,-1),\\ &(-1,1,-1,0,1,0),(1,1,1,-1,-1,1)\}\end{aligned}$$

and

$$\begin{aligned}\Gamma = \{&[(1,1,1),(1,1,1)],[(1,1,1),(1,-1,1)],[(1,1,1),(1,1,-1)],[(1,1,1),(-1,1,1)],\\ &[(1,-1,1),(1,-1,1)],[(1,-1,1),(1,1,-1)],[(1,-1,1),(-1,1,1)],\\ &[(1,1,-1),(1,1,-1)],[(1,1,-1),(-1,1,1)],[(-1,1,1),(-1,1,1)]\}.\end{aligned}$$

The following statements hold true.

(a) Let $T = (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{13}2a_{23})^t \in \mathcal{L}_s({}^2l_\infty^3)$ with $\|T\| = 1$. Then, $T \in extB_{\mathcal{L}_s({}^2l_\infty^3)}$ if and only if there exist at least 6 linearly independent vectors $W_1, \ldots, W_6 \in \Omega$ and $Z_1, \ldots, Z_6 \in \Gamma$ such that

$$W_j \cdot S = S(Z_j) \text{ for all } S \in \mathcal{L}_s({}^2l_\infty^3) \text{ and } |T(Z_j)| = 1 \text{ for } j = 1, \ldots, 6.$$

Let A be the invertible 6×6 matrix such that the j-row vector of A as $[Row(A)]_j = W_j$ for $j = 1, \ldots, 6$. Then, $AS = (S(Z_1), \cdots, S(Z_6))^t$ for all $S \in \mathcal{L}_s({}^2l_\infty^3)$.

(b) $expB_{\mathcal{L}_s({}^2l_\infty^3)} = extB_{\mathcal{L}_s({}^2l_\infty^3)}$.

Theorem 2.1.([22]) *Let* $T = (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{13}, 2a_{23})^t \in \mathcal{L}_s({}^2l_\infty^3)$. *Then,*

$$\begin{aligned}\|T\| = \max\{&2|a_{12}| + |a_{11} + a_{22} - a_{33}|, 2|a_{13}| + |a_{11} - a_{22} + a_{33}|,\\ &2|a_{23}| + |-a_{11} + a_{22} + a_{33}|, 2|a_{12} + a_{13}| + |a_{11} + a_{22} + a_{33} + 2a_{23}|,\\ &2|a_{12} - a_{13}| + |a_{11} + a_{22} + a_{33} - 2a_{23}|\}.\end{aligned}$$

Note that if $\|T\| = 1$, then $|a_{ii}| \leq 1$ for $i = 1, 2, 3$ and $|a_{ij}| \leq \frac{1}{2}$ for $1 \leq i \neq j \leq 3$. With the aid of Wolfram Mathematica 8, we present a complete description of the 42 extreme points of the unit ball of $\mathcal{L}_s(^2 l_\infty^3)$.

Theorem 2.2.

$$\begin{aligned} extB_{\mathcal{L}_s(^2 l_\infty^3)} = \{&\pm(1,0,0,0,0,0)^t, \pm(0,1,0,0,0,0)^t, \pm(0,0,1,0,0,0)^t, \\ &\pm(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0)^t, \pm(\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0)^t, \pm(\frac{1}{2}, 0, -\frac{1}{2}, 0, 1, 0)^t, \\ &\pm(\frac{1}{2}, 0, -\frac{1}{2}, 0, -1, 0)^t, \pm(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 1)^t, \pm(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -1)^t, \\ &\pm(\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t, \pm(\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^t, \\ &\pm(0, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t, \pm(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^t, \\ &\pm(0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^t, \\ &\pm(-\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t\}. \end{aligned}$$

Proof. Claim: $T = (\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0)^t \in extB_{\mathcal{L}_s(^2 l_\infty^3)}$. Let

$$T_1 = (\frac{1}{2} + \epsilon_{11}, -\frac{1}{2} + \epsilon_{22}, \epsilon_{33}, 1 + \epsilon_{12}, \epsilon_{13}, \epsilon_{23})^t$$

and

$$T_2 = (\frac{1}{2} - \epsilon_{11}, -\frac{1}{2} - \epsilon_{22}, -\epsilon_{33}, 1 - \epsilon_{12}, -\epsilon_{13}, -\epsilon_{23})^t$$

with $\|T_1\| = \|T_2\| = 1$ for some $\epsilon_{ij} \in \mathbb{R}$ for $i, j = 1, 2, 3$. By Theorem 2.1, it follows that

$$\begin{aligned} 0 &= \epsilon_{12} = \epsilon_{13} = \epsilon_{23} \\ 0 &= \epsilon_{11} + \epsilon_{22} + \epsilon_{33}, \\ 0 &= -\epsilon_{11} + \epsilon_{22} + \epsilon_{33}, \\ 0 &= \epsilon_{11} - \epsilon_{22} + \epsilon_{33}, \end{aligned}$$

which show that $\epsilon_{ij} = 0$ for $i, j = 1, 2, 3$. Hence, $T \in extB_{\mathcal{L}(^2 l_\infty^3)}$.
Claim: $T = (\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t \in extB_{\mathcal{L}_s(^2 l_\infty^3)}$. Let

$$T_1 = (\frac{1}{2} + \epsilon_{11}, \epsilon_{22}, \epsilon_{33}, -\frac{1}{2} + \epsilon_{12}, \frac{1}{2} + \epsilon_{13}, \frac{1}{2} + \epsilon_{23})^t$$

and

$$T_2 = (\frac{1}{2} = \epsilon_{11}, -\epsilon_{22}, -\epsilon_{33}, -\frac{1}{2} - \epsilon_{12}, \frac{1}{2} - \epsilon_{13}, \frac{1}{2} - \epsilon_{23})^t$$

with $\|T_1\| = \|T_2\| = 1$ for some $\epsilon_{ij} \in \mathbb{R}$ for $i, j = 1, 2, 3$. By Theorem 2.1, it follows that

$$\begin{aligned} 0 &= \epsilon_{12} - \epsilon_{13} = \epsilon_{12} + \epsilon_{13} \\ 0 &= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} - \epsilon_{23}, \\ 0 &= \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \epsilon_{23}, \\ 0 &= -\epsilon_{11} + \epsilon_{22} + \epsilon_{33}, \\ 0 &= \epsilon_{11} - \epsilon_{22} + \epsilon_{33}, \end{aligned}$$

which show that $\epsilon_{ij} = 0$ for $i, j = 1, 2, 3$. Hence, $T \in extB_{\mathcal{L}({}^2l_\infty^3)}$.

The other 40 bilinear forms in the list of Theorem 2.2 can be proved to be extreme in a similar way. We leave this to the reader.

Let $T = (a_{11}, a_{22}, a_{33}, 2a_{12}, 2a_{13}, 2a_{23})^t \in \mathcal{L}_s({}^2l_\infty^3)$ with $\|T\| = 1$. By the comments made right before Theorem 2.1, $T \in extB_{\mathcal{L}_s({}^2l_\infty^3)}$ if and only if $T = A^{-1}(T(Z_1), \cdots, T(Z_6))^t$ for some $Z_1, \ldots, Z_6 \in \Gamma$ with $|T(Z_j)| = 1$ $(j = 1, \ldots, 6)$. Therefore, we may classify all the extreme points of $B_{\mathcal{L}_s({}^2l_\infty^3)}$ by the following steps: There are 210 choices of 6 vectors among the 10 elements of Ω, that is ${}_{10}C_6 = 210$. First, among 210 cases, find $W_1, \ldots, W_6 \in \Omega$ such that the corresponding matrix A with rows $W_1, \ldots, W_6$ is invertible, and next solve A^{-1} and using Theorem 2.1, obtain $T = A^{-1}(b_1, \cdots, b_6)^t$ satisfying

$$\|A^{-1}(b_1, \cdots, b_6)^t\| = 1$$

for some $b_1, \ldots, b_6 = \pm 1$.

Those $T = A^{-1}(b_1, \cdots, b_6)^t$ are all the extreme points of $B_{\mathcal{L}_s({}^2l_\infty^3)}$. With the aid of Wolfram Mathematica 8, we may check the 42 bilinear forms in the list of Theorem 2.2 are all the extreme points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$. □

Theorem 2.3. *We have* $extB_{\mathcal{L}_s({}^2l_\infty^3)} \subset extB_{\mathcal{L}_s({}^2l_\infty^n)}$ *for every* $n \geq 4$.

Proof. Let

$$\begin{aligned} T((x_1, x_2, x_3), (y_1, y_2, y_3)) &= \frac{1}{2}x_3y_3 - \frac{1}{4}(x_1y_2 + x_2y_1) \\ &\quad + \frac{1}{4}(x_1y_3 + x_3y_1) + \frac{1}{4}(x_2y_3 + x_3y_2) \\ &:= (0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t. \end{aligned}$$

Claim 1: for $n \geq 4$, $T \in extB_{\mathcal{L}_s({}^2l_\infty^n)}$. Use induction on n.

Case 1: $n = 4$.

Suppose that $T = \frac{1}{2}(S_1 + S_2)$ for some $S_1, S_2 \in \mathcal{L}_s({}^2l_\infty^4)$ with $\|S_1\| = 1 = \|S_2\|$.

Write

$$\begin{aligned}&S_1((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4))\\&=\epsilon_1x_1y_1+\epsilon_2x_2y_2+(\frac{1}{2}+a_1)x_3y_3+(-\frac{1}{4}+a_2)(x_1y_2+x_2y_1)\\&\quad+(\frac{1}{4}+a_3)(x_1y_3+x_3y_1)+(\frac{1}{4}+a_4)(x_2y_3+x_3y_2)\\&\quad+b_1(x_4y_1+x_1y_4)+b_2(x_2y_4+x_4y_2)+b_3(x_3y_4+x_4y_3)+b_4x_4y_4\end{aligned}$$

and

$$\begin{aligned}&S_2((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4))\\&=\epsilon_1x_1y_1-\epsilon_2x_2y_2+(\frac{1}{2}-a_1)x_3y_3+(-\frac{1}{4}-a_2)(x_1y_2+x_2y_1)\\&\quad+(\frac{1}{4}-a_3)(x_1y_3+x_3y_1)+(\frac{1}{4}-a_4)(x_2y_3+x_3y_2)\\&\quad-b_1(x_4y_1+x_1y_4)-b_2(x_2y_4+x_4y_2)-b_3(x_3y_4+x_4y_3)-b_4x_4y_4\end{aligned}$$

for some $\epsilon_1,\epsilon_2,a_j,b_j\in\mathbb{R}$ for $j=1,\dots,4$.

Note that, for $i=1,2$,

$$S_i((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))\in\mathcal{L}_s(^2l_\infty^3),\|S_i((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))\|\leq 1,$$

and

$$T=\frac{1}{2}(S_1((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))+S_2((x_1,x_2,x_3,0),(y_1,y_2,y_3,0)).$$

Since $T\in extB_{\mathcal{L}_s(^2l_\infty^3)}$, $T((x_1,x_2,x_3),(y_1,y_2,y_3))=S_1((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))$, which shows that $\epsilon_1=\epsilon_2=a_j=0$ for $j=1,\dots,4$. Hence,

$$\begin{aligned}S_1((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4))&=T((x_1,x_2,x_3),(y_1,y_2,y_3))+b_1(x_4y_1+x_1y_4)\\&\quad+b_2(x_2y_4+x_4y_2)+b_3(x_3y_4+x_4y_3)+b_4x_4y_4\end{aligned}$$

and

$$\begin{aligned}S_2((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4))&=T((x_1,x_2,x_3),(y_1,y_2,y_3))-(b_1(x_4y_1+x_1y_4)\\&\quad+b_2(x_2y_4+x_4y_2)+b_3(x_3y_4+x_4y_3)+b_4x_4y_4).\end{aligned}$$

It follows that

$$\begin{aligned}1&\geq\max\{|S_i((1,1,1,x_4),(1,-1,1,y_4))|,|S_i((1,1,1,x_4),(1,1,-1,y_4))|:\\&\qquad|x_4|\leq 1,|y_4|\leq 1,i=1,2\}\\&=\max\{1+|b_1(y_4+x_4)+b_2(y_4-x_4)+b_3(y_4+x_4)+b_4x_4y_4|,\\&\qquad 1+|b_1(y_4+x_4)+b_2(y_4+x_4)+b_3(y_4-x_4)+b_4x_4y_4|:|x_4|\leq 1,|y_4|\leq 1\},\end{aligned}$$

which imply that, for all $|x_4| \le 1, |y_4| \le 1$,

$$\begin{aligned} 0 &= b_1(y_4+x_4)+b_2(y_4-x_4)+b_3(y_4+x_4)+b_4x_4y_4 \\ 0 &= b_1(y_4+x_4)+b_2(y_4+x_4)+b_3(y_4-x_4)+b_4x_4y_4, \end{aligned}$$

which shows that $b_j = 0$ for $j = 1, \ldots, 4$. Therefore, $T \in extB_{\mathcal{L}_s({}^2l^4_\infty)}$.

Suppose that for $n = k$, $T \in extB_{\mathcal{L}_s({}^2l^k_\infty)}$. We will show that $T \in extB_{\mathcal{L}_s({}^2l^{k+1}_\infty)}$. Suppose that $T = \frac{1}{2}(W_1 + W_2)$ for some $W_1, W_2 \in \mathcal{L}_s({}^2l^{k+1}_\infty)$ with $\|W_1\| = 1 = \|W_2\|$. By the above argument, we may assume that

$$\begin{aligned} W_1((x_1,\ldots,x_{k+1}),(y_1,\ldots,y_{k+1})) &= T((x_1,x_2,x_3),(y_1,y_2,y_3)) \\ &\quad + \sum_{1\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1} \end{aligned}$$

and

$$\begin{aligned} W_2((x_1,\ldots,x_{k+1}),(y_1,\ldots,y_{k+1})) &= T((x_1,x_2,x_3),(y_1,y_2,y_3)) \\ &\quad - \sum_{1\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)-dx_{k+1}y_{k+1} \end{aligned}$$

for some $c_j, d \in \mathbb{R}$ $(j = 1, \ldots, k)$. It follows that

$$\begin{aligned} 1 \ge \max\{&|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,-1,1,y_4,\ldots,y_{k+1}))|, \\ &|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,1,-1,y_4,\ldots,y_{k+1}))|, \\ &|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,-1,-1,y_4,\ldots,y_{k+1}))| : |x_j|\le 1, |y_j| \le 1, \\ &j=4,\ldots,k+1, i=1,2\} \\ = \max\{&1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}+x_{k+1}) \\ &+\sum_{4\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}|, \\ &1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}+x_{k+1})+c_3(y_{k+1}-x_{k+1}) \\ &+\sum_{4\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}|, \\ &1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}-x_{k+1}) \\ &+\sum_{4\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}| \\ &: |x_j|\le 1, |y_j|\le 1, j=4,\ldots,k+1\}, \end{aligned}$$

which shows that, for $|x_j| \le 1, |y_j| \le 1, j = 4, \ldots, k+1$,

$$\begin{aligned} 0 &= c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}+x_{k+1}) \\ &\quad +\sum_{4\le j\le k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}, \end{aligned}$$

$$\begin{aligned}0 &= c_1(y_{k+1}+x_{k+1}) + c_2(y_{k+1}+x_{k+1}) + c_3(y_{k+1}-x_{k+1})\\ &\quad + \sum_{4\le j\le k} c_j(x_j y_{k+1} + x_{k+1}y_j) + d x_{k+1}y_{k+1},\\ 0 &= c_1(y_{k+1}+x_{k+1}) + c_2(y_{k+1}-x_{k+1}) + c_3(y_{k+1}-x_{k+1})\\ &\quad + \sum_{4\le j\le k} c_j(x_j y_{k+1} + x_{k+1}y_j) + d x_{k+1}y_{k+1}.\end{aligned}$$

If $x_j = y_j = 0$ for $j = 4, \ldots, k$, then, for $|x_{k+1}| \le 1, |y_{k+1}| \le 1$,

$$\begin{aligned}0 &= c_1(y_{k+1}+x_{k+1}) + c_2(y_{k+1}-x_{k+1}) + c_3(y_{k+1}+x_{k+1}) + d x_{k+1}y_{k+1},\\ 0 &= c_1(y_{k+1}+x_{k+1}) + c_2(y_{k+1}+x_{k+1}) + c_3(y_{k+1}-x_{k+1}) + d x_{k+1}y_{k+1},\\ 0 &= c_1(y_{k+1}+x_{k+1}) + c_2(y_{k+1}-x_{k+1}) + c_3(y_{k+1}-x_{k+1}) + d x_{k+1}y_{k+1}\end{aligned}$$

which shows that $c_j = 0 = d$ for $j = 1, 2, 3$. Hence,

$$(*)\quad 0 = \sum_{4\le j\le k} c_j(x_j y_{k+1} + x_{k+1}y_j) \quad \text{for } |x_j| \le 1, |y_j| \le 1, j = 4, \ldots, k+1.$$

We claim that $c_j = 0$ for all $j = 4, \ldots, k$. Let $4 \le j_0 \le k$ be fixed. Let $x_j = y_j = 1$ for $4 \le j \ne j_0 \le k$ and $y_{k+1} = -y_{j_0} = 1 = -x_{k+1} = x_{j_0}$. By $(*)$, we have $c_{j_0} = 0$. Hence, $W_1 = T = W_2$, so $T \in extB_{\mathcal{L}_s(^2 l_\infty^{k+1})}$. Therefore, we have shown that $T \in extB_{\mathcal{L}_s(^2 l_\infty^n)}$ for $n \ge 4$.

Let

$$\begin{aligned}S((x_1,x_2,x_3),(y_1,y_2,y_3)) &= \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}(x_1y_2 + x_2y_1)\\ &:= (\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0)^t.\end{aligned}$$

Claim 2: for $n \ge 4$, $S \in extB_{\mathcal{L}_s(^2 l_\infty^n)}$. Use induction on n.

Case 1: $n = 4$

Suppose that $S = \frac{1}{2}(T_1 + T_2)$ for some $T_1, T_2 \in \mathcal{L}_s(^2 l_\infty^4)$ with $\|T_1\| = 1 = \|T_2\|$. Write

$$\begin{aligned}T_1((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) &= (\frac{1}{2}+a_1)x_1y_1 + (-\frac{1}{2}+a_2)x_2y_2 + a_3x_3y_3\\ &\quad + (\frac{1}{2}+b_1)(x_1y_2+x_2y_1) + b_2(x_2y_3+x_3y_2)\\ &\quad + b_3(x_3y_1+x_1y_3) + c_1(x_1y_4+x_4y_1)\\ &\quad + c_2(x_2y_4+x_4y_2) + c_3(x_3y_4+x_4y_3) + c_4x_4y_4\end{aligned}$$

and

$$\begin{aligned}T_2((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = &(\frac{1}{2}-a_1)x_1y_1 + (-\frac{1}{2}-a_2)x_2y_2 - a_3x_3y_3\\ &+ (\frac{1}{2}-b_1)(x_1y_2+x_2y_1) - b_2(x_2y_3+x_3y_2)\\ &- b_3(x_3y_1+x_1y_3) - c_1(x_1y_4+x_4y_1)\\ &- c_2(x_2y_4+x_4y_2) - c_3(x_3y_4+x_4y_3) - c_4x_4y_4\end{aligned}$$

for some $a_i, b_i, c_j \in \mathbb{R}$ for $i = 1, 2, 3, j = 1, \ldots, 4$. Note that, for $i = 1, 2$,

$$T_i((x_1,x_2,x_3,0),(y_1,y_2,y_3,0)) \in \mathcal{L}_s({}^2l_\infty^3), \|T_i((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))\| \leq 1$$

and

$$S = \frac{1}{2}(T_1((x_1,x_2,x_3,0),(y_1,y_2,y_3,0)) + T_2((x_1,x_2,x_3,0),(y_1,y_2,y_3,0)).$$

Since $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)}$, $S((x_1,x_2,x_3),(y_1,y_2,y_3)) = T_1((x_1,x_2,x_3,0),(y_1,y_2,y_3,0))$, which shows that $a_i = b_i = 0$ for $i = 1, 2, 3$. Hence,

$$\begin{aligned}T_1((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = &S((x_1,x_2,x_3),(y_1,y_2,y_3)) + c_1(x_1y_4+x_4y_1)\\ &+ c_2(x_2y_4+x_4y_2) + c_3(x_3y_4+x_4y_3) + c_4x_4y_4\end{aligned}$$

and

$$\begin{aligned}S_2((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)) = &T((x_1,x_2,x_3),(y_1,y_2,y_3)) - (c_1(x_1y_4+x_4y_1)\\ &+ c_2(x_2y_4+x_4y_2) + c_3(x_3y_4+x_4y_3) + c_4x_4y_4).\end{aligned}$$

It follows that

$$\begin{aligned}1 \geq \max\{&|T_i((1,1,1,x_4),(1,-1,1,y_4))|, |T_i((1,1,1,x_4),(1,1,-1,y_4))| :\\ &|x_4| \leq 1, |y_4| \leq 1, i = 1,2\}\\ = \max\{&1 + |c_1(y_4+x_4) + c_2(y_4-x_4) + c_3(y_4+x_4) + c_4x_4y_4|,\\ &1 + |c_1(y_4+x_4) + c_2(y_4+x_4) + c_3(y_4-x_4) + c_4x_4y_4| :\\ &|x_4| \leq 1, |y_4| \leq 1\},\end{aligned}$$

which imply that, for all $|x_4| \leq 1, |y_4| \leq 1$,

$$\begin{aligned}0 &= c_1(y_4+x_4) + c_2(y_4-x_4) + c_3(y_4+x_4) + c_4x_4y_4\\ 0 &= c_1(y_4+x_4) + c_2(y_4+x_4) + c_3(y_4-x_4) + c_4x_4y_4,\end{aligned}$$

which shows that $c_j = 0$ for $j = 1, \ldots, 4$. Therefore, $S \in extB_{\mathcal{L}_s({}^2l_\infty^4)}$.

Suppose that for $n = k$, $S \in extB_{\mathcal{L}_s({}^2l_\infty^k)}$. We will show that $S \in extB_{\mathcal{L}_s({}^2l_\infty^{k+1})}$. Suppose that $S = \frac{1}{2}(W_1 + W_2)$ for some $W_1, W_2 \in \mathcal{L}_s({}^2l_\infty^{k+1})$ with $\|W_1\| = 1 =$

$\|W_2\|$. By the above argument, we may assume that

$$\begin{aligned}W_1((x_1,\ldots,x_{k+1}),(y_1,\ldots,y_{k+1})) &= S((x_1,x_2,x_3),(y_1,y_2,y_3))\\ &\quad + \sum_{1\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j) + dx_{k+1}y_{k+1}\end{aligned}$$

and

$$\begin{aligned}W_2((x_1,\ldots,x_{k+1}),(y_1,\ldots,y_{k+1})) &= S((x_1,x_2,x_3),(y_1,y_2,y_3))\\ &\quad - \sum_{1\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j) - dx_{k+1}y_{k+1}\end{aligned}$$

for some $c_j, d \in \mathbb{R}$ $(j=1,\ldots,k)$. It follows that

$$\begin{aligned}1 \geq \max\{&|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,-1,1,y_4,\ldots,y_{k+1}))|,\\ &|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,1,-1,y_4,\ldots,y_{k+1}))|,\\ &|W_i((1,1,1,x_4,\ldots,x_{k+1}),(1,-1,-1,y_4,\ldots,y_{k+1}))| :\\ &|x_j|\leq 1, |y_j|\leq 1, j=4,\ldots,k+1, i=1,2\}\\ = \max\{&1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}+x_{k+1})\\ &+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}|,\\ &1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}+x_{k+1})+c_3(y_{k+1}-x_{k+1})\\ &+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}|,\\ &1+|c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}-x_{k+1})\\ &+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}| :\\ &|x_j|\leq 1, |y_j|\leq 1, j=4,\ldots,k+1\},\end{aligned}$$

which shows that, for $|x_j|\leq 1, |y_j|\leq 1, j=4,\ldots,k+1$,

$$\begin{aligned}0 &= c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}+x_{k+1})\\ &\quad+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1},\\ 0 &= c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}+x_{k+1})+c_3(y_{k+1}-x_{k+1})\\ &\quad+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1},\\ 0 &= c_1(y_{k+1}+x_{k+1})+c_2(y_{k+1}-x_{k+1})+c_3(y_{k+1}-x_{k+1})\\ &\quad+\sum_{4\leq j\leq k} c_j(x_jy_{k+1}+x_{k+1}y_j)+dx_{k+1}y_{k+1}.\end{aligned}$$

By Claim 1, we conclude that $c_j = 0 = d$ for $j = 1, \ldots, k$. Hence, $W_1 = S = W_2$, so $S \in extB_{\mathcal{L}_s({}^2l_\infty^{k+1})}$. Therefore, we have shown that $S \in extB_{\mathcal{L}_s({}^2l_\infty^n)}$ for $n \geq 4$.

Let

$$S((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1y_1 := (1, 0, 0, 0, 0, 0)^t.$$

Claim 3: for $n \geq 4$, $S \in extB_{\mathcal{L}_s({}^2l_\infty^n)}$.

Suppose that $S = \frac{1}{2}(T_1 + T_2)$ for some $T_1, T_2 \in \mathcal{L}_s({}^2l_\infty^n)$ with $\|T_1\| = 1 = \|T_2\|$. Write

$$\begin{aligned} T_1((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= (1 + a_1)x_1y_1 \\ &\quad + \sum_{2\leq j\leq n} a_jx_jy_j + \sum_{1\leq i<j\leq n} b_{ij}(x_iy_j + x_jy_i) \end{aligned}$$

and

$$\begin{aligned} T_2((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= (1 - a_1)x_1y_1 \\ &\quad - \sum_{2\leq j\leq n} a_jx_jy_j - \sum_{1\leq i<j\leq n} b_{ij}(x_iy_j + x_jy_i) \end{aligned}$$

for some $a_i, b_{ij} \in \mathbb{R}$ for $i, j = 1, \ldots, n$. It follows that

$$1 \geq \max\{|T_l(e_1, e_1)| : l = 1, 2\} = 1 + |a_1|,$$

which imply that $a_1 = 0$. Let $2 \leq j \leq n$ be fixed. Since

$$1 \geq \max\{|T_l(e_1 + te_j, e_1 + te_j)| : l = 1, 2, |t| \leq 1\} = 1 + |a_jt^2 + 2b_{ij}t|$$

and

$$0 = a_jt^2 + 2b_{1j}t$$

for every $|t| \leq 1$. Hence, $0 = a_j = b_{1j}$ for every $2 \leq j \leq n$. Let $2 \leq i < j \leq n$ be fixed. Since

$$1 \geq \max\{|T_l(e_1 + e_i + e_j, e_1 + e_i + e_j)| : l = 1, 2\} = 1 + 2|b_{ij}|,$$

$b_{ij} = 0$. Hence, $S = T_1$, so $S \in extB_{\mathcal{L}_s({}^2l_\infty^n)}$ for $n \geq 4$.

Notice that the other 39 cases are similar since, essentially there are three groups of extreme points, those having an 1, those having two $\frac{1}{2}$'s and those having four $\frac{1}{2}$'s. Therefore, the other 39 extreme points in the list of Theorem 2.2 are extreme in the unit ball of $\mathcal{L}_s({}^2l_\infty^n)$. We complete the proof. □

3. The Exposed Points of the Unit Ball of $\mathcal{L}_s({}^2l_\infty^3)$

Theorem 2.2 leads to a complete formula of $\|f\|$ for every $f \in \mathcal{L}_s({}^2l_\infty^3)^*$.

Theorem 3.1. *Let $f \in \mathcal{L}_s({}^2l_\infty^3)^*$ with $\alpha_{11} := f(x_1x_2), \alpha_{22} := f(y_1y_2), \alpha_{33} := f(z_1z_2), \beta_{12} := f(x_1y_2 + x_2y_1), \beta_{13} := f(x_1z_2 + x_2z_1), \beta_{23} := f(y_1z_2 + y_2z_1)$. Then,*

$$\begin{aligned}\|f\| = \max_{j=1,2,3}\{&|\alpha_{jj}|, \frac{1}{2}|\alpha_{11} - \alpha_{22}| + \frac{1}{2}|\beta_{12}|, \frac{1}{2}|\alpha_{22} - \alpha_{33}| + \frac{1}{2}|\beta_{23}|,\\ &\frac{1}{2}|\alpha_{11} - \alpha_{33}| + \frac{1}{2}|\beta_{13}|, \frac{1}{4}(|2\alpha_{jj} + \beta_{12}| + |\beta_{13} - \beta_{23}|),\\ &\frac{1}{4}(|2\alpha_{jj} - \beta_{12}| + |\beta_{13} + \beta_{23}|)\}.\end{aligned}$$

Proof. By the Krein-Milman Theorem, $\|f\| = \sup_{T\in extB_{\mathcal{L}_s({}^2l_\infty^3)}} |f(T)|$. By Theorem 2.2, it follows that

$$\begin{aligned}\|f\| = \max\{&|f((1,0,0,0,0,0)^t)|, |f((0,1,0,0,0,0)^t)|, |f((0,0,1,0,0,0)^t)|,\\ &|f((\tfrac{1}{2},-\tfrac{1}{2},0,1,0,0)^t)|, |f((\tfrac{1}{2},-\tfrac{1}{2},0,-1,0,0)^t)|, |f((\tfrac{1}{2},0,-\tfrac{1}{2},0,1,0)^t)|,\\ &|f((\tfrac{1}{2},0,-\tfrac{1}{2},0,-1,0)^t)|, |f((0,\tfrac{1}{2},-\tfrac{1}{2},0,0,1)^t)|, |f((0,\tfrac{1}{2},-\tfrac{1}{2},0,0,-1)^t)|,\\ &|f((\tfrac{1}{2},0,0,-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((\tfrac{1}{2},0,0,\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})^t)|,\\ &|f((0,\tfrac{1}{2},0,-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,\tfrac{1}{2},0,\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})^t)|,\\ &|f((0,0,\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,0,\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})^t)|,\\ &|f((-\tfrac{1}{2},0,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,-\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|, |f((0,0,-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})^t)|\}\\ = \max_{j=1,2,3}\{&|\alpha_{jj}|, \frac{1}{2}|\alpha_{11} - \alpha_{22}| + \frac{1}{2}|\beta_{12}|, \frac{1}{2}|\alpha_{22} - \alpha_{33}| + \frac{1}{2}|\beta_{23}|,\\ &\frac{1}{2}|\alpha_{11} - \alpha_{33}| + \frac{1}{2}|\beta_{13}|, \frac{1}{4}(|2\alpha_{jj} + \beta_{12}| + |\beta_{13} - \beta_{23}|),\\ &\frac{1}{4}(|2\alpha_{jj} - \beta_{12}| + |\beta_{13} + \beta_{23}|)\}.\end{aligned}$$

□

Note that if $\|f\| = 1$, then $|\alpha_{jj}| \leq 1$ for $j = 1, 2, 3$ and $|\beta_{12}| \leq 2, |\beta_{13}| \leq 2$ and $|\beta_{23}| \leq 2$.

Theorem 3.2.([17]) *Let E be a real finite dimensional Banach space such that $extB_E$ is finite. Suppose that $x \in extB_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in extB_E \backslash \{\pm x\}$. Then, $x \in expB_E$.*

We give another proof of the following theorem which was shown in [22].

Theorem 3.3.([22]) $expB_{\mathcal{L}_s({}^2l_\infty^3)} = extB_{\mathcal{L}_s({}^2l_\infty^3)}$.

Proof. It suffices to show that if $T \in extB_{\mathcal{L}_s({}^2l_\infty^3)}$, then it is exposed.

Claim: $T = (1,0,0,0,0,0)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (1,0,0,0,0,0) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(1,0,0,0,0,0)^t, \pm(0,1,0,0,0,0)^t, \pm(0,0,1,0,0,0)^t$ are exposed.

Claim: $T = (\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, \frac{7}{4}, 0, 0) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0)^t$ is exposed.

Claim: $T = (\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, -\frac{7}{4}, 0, 0) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, -\frac{1}{2}, 0, -1, 0, 0)^t$ is exposed.

Claim: $T = (\frac{1}{2}, 0, -\frac{1}{2}, 0, 1, 0)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, 0, \frac{7}{4}, 0) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, 0, -\frac{1}{2}, 0, 1, 0)^t$ is exposed.

Claim: $T = (\frac{1}{2}, 0, -\frac{1}{2}, 0, -1, 0)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, 0, -\frac{7}{4}, 0) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, 0, -\frac{1}{2}, 0, -1, 0)^t$ is exposed.

Claim: $T = (0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 1)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (0, \frac{1}{4}, 0, 0, 0, \frac{7}{4}) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 1)^t$ is exposed.

Claim: $T = (0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -1)^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (0, \frac{1}{4}, 0, 0, 0, -\frac{7}{4}) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -1)^t$ is exposed.

Claim: $T = (-\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (-\frac{1}{4}, 0, 0, \frac{5}{4}, 1, \frac{5}{4}) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(-\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, -\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t$ are exposed.

Claim: $T = (\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, -\frac{5}{4}, 1, \frac{5}{4}) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t$ are exposed.

Claim: $T = (\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t$ is exposed.

Let $f = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (\frac{1}{4}, 0, 0, \frac{5}{4}, -1, \frac{5}{4}) \in \mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and 3.2, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for $S \in extB_{\mathcal{L}_s({}^2l_\infty^3)} \backslash \{\pm T\}$. Similarly, $-(\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t, \pm(0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t, \pm(0, 0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})^t$ are exposed.

Claim: $T=(\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2})^t$ is exposed.

Let $f=(\alpha_{11},\alpha_{22},\alpha_{33},\beta_{12},\beta_{13},\beta_{23})=(\frac{1}{4},0,0,\frac{5}{4},1,-\frac{5}{4})\in\mathcal{L}_s({}^2l_\infty^3)^*$. Then, by Theorems 3.1 and Theorem 3.2, $\|f\|=1=f(T)$ and $|f(S)|<1$ for $S\in extB_{\mathcal{L}_s({}^2l_\infty^3)}\backslash\{\pm T\}$. Similarly, $-(\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{2})^t,\pm(0,\frac{1}{2},0,\frac{1}{2},\frac{1}{2},-\frac{1}{2})^t,\pm(0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},)^t$ are exposed. □

4. The Smooth Points of the Unit Ball of $\mathcal{L}_s({}^2l_\infty^3)$

In this section we will characterize all the smooth points of the unit ball of $\mathcal{L}_s({}^2l_\infty^3)$.

Theorem 4.1. *Let $T=(a_{11},a_{22},a_{33},2a_{12},2a_{13},2a_{23})^t\in\mathcal{L}_s({}^2l_\infty^3)$. Then, $T\in smB_{\mathcal{L}_s({}^2l_\infty^3)}$ if and only if*

$$\begin{aligned}
&(|T((1,1,1),(1,1,1))|=1, 0<|a_{11}+a_{22}+a_{33}+2a_{23}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,1),(1,1,1)]\})\\
or\quad &(|T((-1,1,1),(-1,1,1))|=1, 0<|a_{11}+a_{22}+a_{33}+2a_{23}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(-1,1,1),(-1,1,1)]\})\\
or\quad &(|T((1,-1,1),(1,-1,1))|=1, 0<|a_{11}+a_{22}+a_{33}+2a_{13}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{(1,-1,1),(1,-1,1)\})\\
or\quad &(|T((1,1,-1),(1,1,-1))|=1, 0<|a_{11}+a_{22}+a_{33}+2a_{12}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,-1),(1,1,-1)]\})\\
or\quad &(|T((1,1,1),(1,-1,1))|=1, 0<|a_{11}-a_{22}+a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,1),(1,-1,1)]\})\\
or\quad &(|T((1,1,1),(1,1,-1))|=1, 0<|a_{11}+a_{22}-a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,1),(1,1,-1)]\})\\
or\quad &(|T((1,1,1),(-1,1,1))|=1, 0<|-a_{11}+a_{22}+a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,1),(-1,1,1)]\})\\
or\quad &(|T((1,-1,1),(1,1,-1))|=1, 0<|-a_{11}+a_{22}+a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,-1,1),(1,1,-1)]\})\\
or\quad &(|T((1,-1,1),(-1,1,1))|=1, 0<|a_{11}+a_{22}-a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,-1,1),(-1,1,1)]\})\\
or\quad &(|T((1,1,-1),(-1,1,1))|=1, 0<|a_{11}-a_{22}+a_{33}|<1,\\
&|T(Y)|<1 \text{ for all } Y\in\Gamma\backslash\{[(1,1,-1),(-1,1,1)]\}).
\end{aligned}$$

Proof. (⇐): **Case 1**: $|T((1,1,1),(1,1,1))|=1, 0<|a_{11}+a_{22}+a_{33}+2a_{23}|<1, |T(Y)|<1$ for all $Y\in\Gamma\backslash\{[(1,1,1),(1,1,1)]\}$.

By Theorem 2.1, $\|T\|=1, |a_{jj}|<1$ for $j=1,2,3, |a_{12}|<\frac{1}{2}, |a_{13}|<\frac{1}{2}, |a_{23}|<\frac{1}{2}$. Let $l:=T((1,1,1),(1,1,1))=a_{11}+a_{22}+a_{33}+2a_{12}+2a_{13}+2a_{23}$ for some $l\in\{1,-1\}$.

Without loss of generality, we may assume that $l = 1$. Obviously,

$$2(a_{12} + a_{13}) > 0 \text{ and } a_{11} + a_{22} + a_{33} + 2a_{23} > 0.$$

Let $f \in \mathcal{L}_s(^2l_\infty^3)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha_{11} := f(x_1x_2), \alpha_{22} := f(y_1y_2), \alpha_{33} := f(z_1z_2), \beta_{12} := f(x_1y_2+x_2y_1), \beta_{13} := f(x_1z_2+x_2z_1), \beta_{23} := f(y_1z_2+y_2z_1)$.

We claim that $\alpha_{jj} = 1$ $(j = 1, 2, 3)$ and $\beta_{12} = \beta_{13} = \beta_{23} = 2$. Let $n_1 \in \mathbb{N}$ be such that, for $j = 1, 2, 3$,

$$|a_{jj}| + \frac{1}{n_1} < 1, |a_{12}| + \frac{1}{n_1} < \frac{1}{2}, |a_{13}| + \frac{1}{n_1} < \frac{1}{2}, |a_{23}| + \frac{1}{n_1} < \frac{1}{2},$$
$$|T(Y)| + \frac{10}{n_1} < 1 \text{ for all } Y \in \Gamma\backslash\{[(1, 1, 1), (1, 1, 1)]\}.$$

By Theorem 2.1, for $n > n_1$,

$$1 = \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22} \mp \frac{1}{n}, 2a_{23}, a_{33})^t\|,$$
$$1 = \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33} \mp \frac{1}{n})^t\|,$$
$$1 = \|(a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13} \mp \frac{1}{n}, a_{22}, 2a_{23}, a_{33})^t\|,$$
$$1 = \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23} \mp \frac{1}{n}, a_{33})^t\|.$$

It follows that for $n > n_1$,

(1) $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22} \mp \frac{1}{n}, 2a_{23}, a_{33})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} - \alpha_{22})|$,

(2) $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33} \mp \frac{1}{n})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} - \alpha_{33})|$,

(3) $1 \geq |f((a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13} \mp \frac{1}{n}, a_{22}, 2a_{23}, a_{33})^t)| = |1 \pm \frac{1}{2n}(\beta_{12} - \beta_{13})|$,

(4) $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23} \mp \frac{1}{n}, a_{33})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} - \frac{1}{2}\beta_{23})|$.

By $(1) - (4)$, $\alpha_{11} = \alpha_{22} = \alpha_{33}, \beta_{12} = \beta_{13}, \alpha_{11} = \frac{1}{2}\beta_{23}$. Let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ and

$$0 < 2(a_{12} + a_{13}) - \frac{1}{n_2} < 2(a_{12} + a_{13}) + \frac{1}{n_2} < 1,$$
$$0 < a_{11} + a_{22} + a_{33} + 2a_{23} - \frac{1}{n_2} < a_{11} + a_{22} + a_{33} + 2a_{23} + \frac{1}{n_2} < 1.$$

By Theorem 2.1, for $n > n_2$,

$$1 = \|(a_{11} \pm \frac{1}{n}, 2a_{12} \mp \frac{1}{2n}, 2a_{13} \mp \frac{1}{2n}, a_{22} \pm \frac{1}{2n}, 2a_{23} \mp \frac{1}{n}, a_{33} \pm \frac{1}{2n})^t\|.$$

Since

$$1 \geq |f((a_{11}\pm\frac{1}{n}, 2a_{12}\mp\frac{1}{2n}, 2a_{13}\mp\frac{1}{2n}, a_{22}\pm\frac{1}{2n}, 2a_{23}\mp\frac{1}{n}, a_{33}\pm\frac{1}{2n})^t)| = |1\pm\frac{1}{n}(\alpha_{11}-\frac{1}{2}\beta_{12})|,$$

so, $\alpha_{11} = \frac{1}{2}\beta_{12}$, hence, $\beta_{12} = \beta_{13} = \beta_{23}$. Therefore,

$$\begin{aligned} 1 = f(T) &= \sum_{j=1}^{3} a_{jj}\alpha_{jj} + a_{12}\beta_{12} + a_{13}\beta_{13} + a_{23}\beta_{23} \\ &= (a_{11} + a_{22} + a_{33} + 2a_{12} + 2a_{13} + 2a_{23})\alpha_{11} \\ &= \alpha_{11}, \end{aligned}$$

hence, $\alpha_{jj} = 1$ for $j = 1, 2, 3$ and $\beta_{12} = \beta_{13} = \beta_{23} = 2$. Hence, T would be smooth in $\mathcal{L}_s(^2l_\infty^3)$.

Case 2: $|T((1,1,1),(1,-1,1))| = 1, 0 < |a_{11} - a_{22} + a_{33}| < 1, |T(Y)| < 1$ for all $Y \in \Gamma\backslash\{[(1,1,1),(1,-1,1)]\}$

By Theorem 2.1, $\|T\| = 1, |a_{jj}| < 1$ for $j = 1, 2, 3, |a_{12}| < \frac{1}{2}, |a_{23}| < \frac{1}{2}$. Let $l := T((1,-1,1),(1,1,1)) = a_{11} - a_{22} + a_{33} + 2a_{13}$ for some $l \in \{1, -1\}$. Without loss of generality, we may assume that $l = 1$. Obviously,

$$2a_{13} > 0, a_{11} - a_{22} + a_{33} > 0.$$

Let $f \in \mathcal{L}_s(^2l_\infty^3)^*$ be such that $f(T) = 1 = \|f\|$ with $\alpha_{11} := f(x_1x_2), \alpha_{22} := f(y_1y_2), \alpha_{33} := f(z_1z_2), \beta_{12} := f(x_1y_2+x_2y_1), \beta_{13} := f(x_1z_2+x_2z_1), \beta_{23} := f(y_1z_2+y_2z_1)$. We claim that $\alpha_{jj} = 1 = -\alpha_{22}$ for $j = 1, 3$ and $\beta_{12} = \beta_{23} = 0, \beta_{13} = 2$. Let $n_1 \in \mathbb{N}$ be such that, for $j = 1, 2, 3$,

$$\begin{aligned} &|a_{jj}| + \frac{1}{n_1} < 1, |a_{12}| + \frac{1}{n_1} < \frac{1}{2}, |a_{23}| + \frac{1}{n_1} < \frac{1}{2}, \\ &|T(Y)| + \frac{10}{n_1} < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,-1,1),(1,1,1)]\}. \end{aligned}$$

Let $n_2 \in \mathbb{N}$ be such that, $n_2 > n_1$ and for $j = 1, 2, 3$,

$$0 < 2a_{13} - \frac{1}{n_2} < 2a_{13} + \frac{1}{n_2} < 1$$

and

$$0 < a_{11} - a_{22} + a_{33} - \frac{1}{n_2} < a_{11} - a_{22} + a_{33} + \frac{1}{n_2} < 1.$$

By Theorem 2.1, for $n > n_2$,

$$\begin{aligned} 1 &= \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22} \pm \frac{1}{n}, 2a_{23}, a_{33})^t\|, \\ 1 &= \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33} \mp \frac{1}{n})^t\|, \end{aligned}$$

$$
\begin{aligned}
1 &= \|(a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13}, a_{22}, 2a_{23} \mp \frac{1}{n}, a_{33})^t\|,\\
1 &= \|(a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13}, a_{22}, 2a_{23} \pm \frac{1}{n}, a_{33})^t\|,\\
1 &= \|(a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13} \mp \frac{1}{n}, a_{22}, 2a_{23}, a_{33})^t\|.
\end{aligned}
$$

It follows that for $n > n_2$,

(1') $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22} \pm \frac{1}{n}, 2a_{23}, a_{33})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} + \alpha_{22})|$,

(2') $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13}, a_{22}, 2a_{23}, a_{33} \mp \frac{1}{n})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} - \alpha_{33})|$,

(3') $1 \geq |f((a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13}, a_{22}, 2a_{23} \mp \frac{1}{n}, a_{33})^t)| = |1 \pm \frac{1}{2n}(\beta_{12} - \beta_{23})|$,

(4') $1 \geq |f((a_{11}, 2a_{12} \pm \frac{1}{n}, 2a_{13}, a_{22}, 2a_{23} \pm \frac{1}{n}, a_{33})^t)| = |1 \pm \frac{1}{2n}(\beta_{12} + \beta_{23})|$,

(5') $1 \geq |f((a_{11} \pm \frac{1}{n}, 2a_{12}, 2a_{13} \mp \frac{1}{n}, a_{22}, 2a_{23}, a_{33})^t)| = |1 \pm \frac{1}{n}(\alpha_{11} - \frac{1}{2}\beta_{13})|$.

By $(1^{'}) - (5^{'})$, $\alpha_{11} = -\alpha_{22} = \alpha_{33}, \beta_{12} = \beta_{23} = 0, \alpha_{11} = \frac{1}{2}\beta_{13}$. It follows that

$$
\begin{aligned}
1 = f(T) &= \sum_{j=1}^{3} a_{jj}\alpha_{jj} + a_{12}\beta_{12} + a_{13}\beta_{13} + a_{23}\beta_{23}\\
&= (a_{11} - a_{22} + a_{33})\alpha_{11} + 2a_{13}\alpha_{11}\\
&= (1 - 2a_{13})\alpha_{11} + 2a_{13}\alpha_{11}\\
&= \alpha_{11},
\end{aligned}
$$

hence, $\beta_{13} = 2$ and $\alpha_{jj} = 1 = -\alpha_{22}$ for $j = 1, 3$. Hence, T would be smooth in $\mathcal{L}_s({}^2l_\infty^3)$. Since the proofs of other cases are similar as those in the cases 1 and 2, we omit the proofs.

($\Rightarrow$): If not, then we have two cases.

Case 1: $Norm(T)$ is a singleton, where

$$Norm(S) := \{X \in \Gamma : |S(X)| = \|S\|\}$$

for $S \in \mathcal{L}_s({}^2l_\infty^3)$.

Notice that

$$
\begin{aligned}
1 &\geq |a_{11} + a_{22} + a_{33} + 2a_{12}|,\\
1 &\geq |a_{11} + a_{22} + a_{33} + 2a_{13}|,\\
1 &\geq |a_{11} + a_{22} + a_{33} + 2a_{23}|,\\
1 &\geq |-a_{11} + a_{22} + a_{33}|,\\
1 &\geq |a_{11} - a_{22} + a_{33}|,\\
1 &\geq |a_{11} + a_{22} - a_{33}|.
\end{aligned}
$$

We have ten subcases as follows:

$$
\begin{aligned}
&(|T((1,1,1),(1,1,1))| = 1, a_{11}+a_{22}+a_{33}+2a_{23} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,1),(1,1,1)]\})\\
\text{or}\quad &(|T((-1,1,1),(-1,1,1))| = 1, a_{11}+a_{22}+a_{33}+2a_{23} = 0, \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(-1,1,1),(-1,1,1)]\})\\
\text{or}\quad &(|T((1,-1,1),(1,-1,1))| = 1, a_{11}+a_{22}+a_{33}+2a_{13} = 0, \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{(1,-1,1),(1,-1,1)\})\\
\text{or}\quad &(|T((1,1,-1),(1,1,-1))| = 1, a_{11}+a_{22}+a_{33}+2a_{12} = 0, \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,-1),(1,1,-1)]\})\\
\text{or}\quad &(|T((1,1,1),(1,-1,1))| = 1, a_{11}-a_{22}+a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,1),(1,-1,1)]\})\\
\text{or}\quad &(|T((1,1,1),(1,1,-1))| = 1, a_{11}+a_{22}-a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,1),(1,1,-1)]\})\\
\text{or}\quad &(|T((1,1,1),(-1,1,1))| = 1, -a_{11}+a_{22}+a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,1),(-1,1,1)]\})\\
\text{or}\quad &(|T((1,-1,1),(1,1,-1))| = 1, -a_{11}+a_{22}+a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,-1,1),(1,1,-1)]\})\\
\text{or}\quad &(|T((1,-1,1),(-1,1,1))| = 1, a_{11}+a_{22}-a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,-1,1),(-1,1,1)]\})\\
\text{or}\quad &(|T((1,1,-1),(-1,1,1))| = 1, a_{11}-a_{22}+a_{33} = 0 \text{ or } \pm 1,\\
&|T(Y)| < 1 \text{ for all } Y \in \Gamma\backslash\{[(1,1,-1),(-1,1,1)]\}).
\end{aligned}
$$

Subcase 1: $|T((1,1,1),(1,1,1))| = 1, a_{11}+a_{22}+a_{33}+2a_{23} = 0$ or $\pm 1, |T(Y)| < 1$ for all $Y \in \Gamma\backslash\{[(1,1,1),(1,1,1)]\}$.

Suppose that $a_{11}+a_{22}+a_{33}+2a_{23} = 0$. Let

$$f_1 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (0,0,0,2,2,0)$$

and

$$f_2 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (1,1,1,2,2,2) \in \mathcal{L}_s({}^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_k\| = 1 = f_k(T)$ for $k = 1, 2$. Hence, $T \notin smB_{\mathcal{L}_s({}^2l_\infty^3)}$. This is a contradiction.

Suppose that $a_{11}+a_{22}+a_{33}+2a_{23} = 1$. For $|t| \leq 2$, let

$$f_t = (1,1,1,t,t,0) \in \mathcal{L}_s({}^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_t\| = 1 = f_t(T)$ for $|t| \leq 2$. Hence, $T \notin smB_{\mathcal{L}_s({}^2l_\infty^3)}$. This is a contradiction.

Suppose that $a_{11} + a_{22} + a_{33} + 2a_{23} = -1$. For $|t| \leq 2$, let

$$f_t = (-1, -1, -1, t, t, 0) \in \mathcal{L}_s(^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_t\| = 1 = f_t(T)$ for $|t| \leq 2$. Hence, $T \notin smB_{\mathcal{L}_s(^2l_\infty^3)}$. This is a contradiction.

Subcase 2: $|T((1,1,-1),(-1,1,1))| = 1, a_{11} - a_{22} + a_{33} = 0$ or $\pm 1, |T(Y)| < 1$ for all $Y \in \Gamma \backslash \{[(1,1,-1),(-1,1,1)]\}$.

Suppose that $a_{11} - a_{22} + a_{33} = 0$. Then $a_{13} = \frac{1}{2}$. Let

$$f_1 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (1, -1, 1, 0, 2, 0)$$

and

$$f_2 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (0, 0, 0, 0, 2, 0) \in \mathcal{L}_s(^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_k\| = 1 = f_k(T)$ for $k = 1, 2$. Hence, $T \notin smB_{\mathcal{L}_s(^2l_\infty^3)}$.

Suppose that $a_{11} - a_{22} + a_{33} = 1$. Let

$$f_1 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (1, -1, 1, 0, 0, 0)$$

and

$$f_2 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = (1, -1, 1, 0, 1, 0) \in \mathcal{L}_s(^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_k\| = 1 = f_k(T)$ for $k = 1, 2$. Hence, $T \notin smB_{\mathcal{L}_s(^2l_\infty^3)}$.

Suppose that $a_{11} - a_{22} + a_{33} = -1$. Let

$$f_1 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = -(1, -1, 1, 0, 0, 0)$$

and

$$f_2 = (\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{12}, \beta_{13}, \beta_{23}) = -(1, -1, 1, 0, 1, 0) \in \mathcal{L}_s(^2l_\infty^3)^*.$$

By Theorem 3.1, $\|f_k\| = 1 = f_k(T)$ for $k = 1, 2$. Hence, $T \notin smB_{\mathcal{L}_s(^2l_\infty^3)}$.

Since the proofs of other subcases are similar as those of the above, we omit the proofs.

Case 2: $|Norm(T)| \geq 2$.

There exist $X_1 \neq X_2 \in \Gamma$ such that $|T(X_k)| = 1$ for $k = 1, 2$. Note that $sign(T(X_k))\delta_{X_k} \in \mathcal{L}_s(^2l_\infty^3)^*$ and

$$\|sign(T(X_k))\delta_{X_k}\| = 1 = sign(T(X_k))\delta_{X_k}(T)$$

for $k = 1, 2$, which shows that T is not smooth. This is a contradiction. Therefore, we complete the proof. □

Acknowledgements

The author is thankful to the referee for the careful reading and considered suggestions leading to a better presented paper.

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