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A NOTE ON q-ANALOGUE OF POLY-EULER POLYNOMIALS AND ARAKAWA-KANEKO TYPE ZETA FUNCTION

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Abstract. In this paper, we define a q-analogue of the poly-Euler numbers and polynomials which is generalization of the poly Euler numbers and polynomials including q-analogue of polylogarithm function. We also give the relations between generalized poly-Euler polynomials. Furthermore, we introduce zeta fuctions of Arakawa-Kaneko type and talk their properties and the relation with q-analogue of poly-Euler polynomials.

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1. Introduction

Many mathematicians are interested in the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials and their applications. They possess many interesting properties and are treated in many areas of mathematics and physics. Due to these reasons, many applications of Euler numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials have been studied, and recently various analogues for the above numbers and polynomials was introduced (see [1-14]).

In this paper, we use the following notations. $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integer, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{C}$ denotes the set of complex numbers, respectively. The ordinary Euler polynomials $E_n(x)$ are given by the generating functions:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{(see [1, 2, 6])}.$$

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When $x = 0, E_n = E_n(0)$ are called Euler numbers. The first few polynomials are $E_0(x) = 1, E_1(x) = x - \frac{1}{2}, E_2(x) = x^2 - x, E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}$.

Let $x > 0$. The Euler zeta function of Hurwitz type (see [15,16]) is define by

$$\zeta_E(s, x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s} \tag{1}$$

for Re(s) > 0. This function is analytically continued to the whole complex s-plane as an entire function. In fact, this follows from the fact that for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$, the lerch zeta function

$$L(\lambda, x, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+x)^s}, \quad \mathrm{Re(s)} > 0$$

is analytically continued to the whole complex splane as an entire function (see [13, 2.2). It is known that for each non-negative integer n, $\zeta_E(-n, x) = E_n(x)$.

The polylogarithm function Li_k is defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \tag{2}$$

for $k \in \mathbb{Z}$ (see [1,2,3,6,7,8,14,15]).

By using polylogarithm function, Kaneko [6] defined a sequence of rational numbers, which is refered to as poly-Bernoulli numbers.

In [3, 13], the k-th q-analogue of polylogarithm function $Li_{k,q}$ is introduced by

$$Li_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k}, \quad (k \in \mathbb{Z}), \tag{3}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

The q-analogue of polylogarithm function for $k < 1$ is represented by a rational function,

$$Li_{k,q}(x) = \frac{1}{(1-q)^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{q^l x}{1-q^l x}. \tag{4}$$

For $k < 1$, the polylogarithm functions are as follows:

$$\begin{aligned}
Li_{0,q}(x) &= \frac{x}{1-x}, \\
Li_{-1,q}(x) &= \frac{x}{(1-x)(1-qx)}, \\
Li_{-2,q}(x) &= \frac{x(1+qx)}{(1-x)(1-qx)(1-q^2x)}, \\
Li_{-3,q}(x) &= \frac{x(1+2xq+2xq^2+x^2q^3)}{(1-x)(1-qx)(1-q^2x)(1-q^3x)}, \\
Li_{-4,q}(x) &= \frac{x(1+3xq+5xq^2+3x(1+x)q^3+5x^2q^4+3x^2q^5+x^3q^6)}{(1-x)(1-qx)(1-q^2x)(1-q^3x)(1-q^4x)}.
\end{aligned}$$

Note that $\lim_{q\to 1}[n]_q = n$ and $\lim_{q\to 1} Li_{k,q}(x) = Li_k(x)$.

In this paper, we consider a q-analogue of the poly-Euler polynomials containing Equation (4). We also find some relations between a q-analogue of poly-Euler polynomials and ordinary Euler polynomials.

2. A relation with q-analogue of ploy-Bernoulli polynomials

2.1. A q-analogue of the poly-Euler polynomials.

In this section, we define a q-analogue of poly-Euler numbers $E_{n,q}^{(k)}$ and polynomials $E_{n,q}^{(k)}(x)$ by the generating functions. From the definition, we get some identities that is similar to the ordinary Euler polynomials.

Definition 2.1. For $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$, we introduce a q-analogue of poly-Euler polynomials by:

$$\frac{2Li_{k,q}(1-e^{-t})}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{5}$$

When $x = 0$, $E_{n,q}^{(k)} = E_{n,q}^{(k)}(0)$ are called a q-analogue of poly-Euler numbers.

Note that $\lim_{q\to 1}[n]_q = n$ and $\lim_{q\to 1} E_{n,q}^{(k)}(x) = E_n^{(k)}(x)$.

Theorem 2.2. *For $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$, we have*

$$E_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(k)} x^{n-l}. \tag{6}$$

Proof. Let $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$. We easily get

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t(e^t+1)}e^{xt}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we have

$$E_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(k)} x^{n-l}.$$

□

Theorem 2.3. *For $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$, we have*

$$E_{n,q}^{(k)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(k)}(x) y^{n-l}. \tag{7}$$

Proof. Let $n \geq 0$ and $n, k \in \mathbb{Z}$. Then we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x+y)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t(e^t+1)}e^{(x+y)t} \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t(e^t+1)}e^{xt}e^{yt} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(x)y^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we get

$$E_{n,q}^{(k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(x)y^{n-l}.$$

□

Theorem 2.4. *For $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$, we have*

$$\frac{d}{dx}E_{n+1,q}^{(k)}(x) = (n+1)E_{n,q}^{(k)}(x). \tag{8}$$

Proof. Let $n \geq 0$ and $n, k \in \mathbb{Z}$. Then we obtain

$$\begin{aligned}
\frac{d}{dx}E_{n+1,q}^{(k)}(x) &= \frac{d}{dx}\left(\sum_{l=1}^{n+1}\binom{n+1}{l}E_{l,q}^{(k)}x^{n-l+1}\right) \\
&= \sum_{l=0}^{n}(n+1)\binom{n}{l}E_{l,q}^{(k)}x^{n-l} \\
&= (n+1)E_{n,q}^{(k)}(x).
\end{aligned}$$

□

By using the definition of the q-analogue of polylogarithm function $Li_{k,q}(x)$ in Equation (4), we have next relation which is connected with the ordinary poly-Euler polynomials.

Theorem 2.5. *Let $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$. We obtain*

$$E_{n,q}^{(k)}(x) = \frac{1}{n+1}\sum_{l=0}^{\infty}\frac{1}{[l+1]_q^k}\sum_{a=0}^{l+1}(-1)^a\binom{l+1}{a}E_{n+1,q}(x-a). \tag{9}$$

Proof. For $n \geq 0, n, k \in \mathbb{Z}$ and $0 < q < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t(e^t+1)}e^{xt} \\
&= \sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k}\frac{e^{xt}}{t(e^t+1)}
\end{aligned}$$

$$= \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{a=0}^{l+1} \binom{l+1}{a} (-1)^a \sum_{n=-1}^{\infty} \frac{1}{n+1} E_{n+1,q}(x-a) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{a=0}^{l+1} (-1)^a \binom{l+1}{a} E_{n+1,q}(x-a) \right) \frac{t^n}{n!}.$$

Comparing the coefficient of the result, we easily get next equation:

$$E_{n,q}^{(k)}(x) = \frac{1}{n+1} \sum_{l=0}^{\infty} \frac{1}{[l+1]_q^k} \sum_{a=0}^{l+1} (-1)^a \binom{l+1}{a} E_{n+1,q}(x-a).$$

□

2.2. A relation with q-analogue of ploy-Bernoulli polynomials.

We briefly review a q-analogue of poly-Bernoulli polynomials ans numbers (see [9,12]). The q-analogue of exponential function is defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}.$$

In [10], the q-analogue of Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e_q(t)-1} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!}.$$

Let k denote a fixed integer. A q-analogue of poly-Bernoulli polynomials $B_{n,q}^{(k)}(x)$ $(n = 0, 1, 2, \cdots)$ are defined by the generating function

$$\frac{Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!}.$$

Moreover, we call $B_{n,q}^{(k)} = B_{n,q}^{(k)}(0)\,(n = 0, 1, 2, \cdots)$ a q-analogue of poly-Bernoulli numbers. If $k = 1$, then

$$(-1)^n B_{n,q}^{(1)}(-x) = B_{n,q}(x)\,(n \geq 0).$$

The following result yields a relation among q-analogue of poly-Bernoulli and ploy-Euler polynomials.

Theorem 2.6. *For $k \in \mathbb{Z}$ and $n \geq 1$, we have*

$$nE_{n-1,q}^{(k)}(x) + nE_{n-1,q}^{(k)}(x+1) = 2B_{n,q}^{(k)}(x+1) - 2B_{n,q}^{(k)}(x). \tag{10}$$

Proof. We compute both sides of

$$\frac{2Li_{k,q}(1-e^{-t})}{t(1+e^t)} e^{xt}(1+e^t)t = \frac{2Li_{k,q}(1-e^{-t})}{e^t - 1} e^{xt}(e^t - 1).$$

The left-hand side is

$$\begin{aligned}
&\frac{2Li_{k,q}(1-e^{-t})}{t(1+e^t)}e^{xt}(1+e^t)t \\
=\ &\frac{2Li_{k,q}(1-e^{-t})}{t(1+e^t)}e^{xt}t+\frac{2Li_{k,q}(1-e^{-t})}{t(1+e^t)}e^{(x+1)t}t \\
=\ &t\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x)\frac{t^n}{n!}+t\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x+1)\frac{t^n}{n!} \\
=\ &\sum_{n=1}^{\infty}\left(nE_{n-1,q}^{(k)}(x)+nE_{n-1,q}^{(k)}(x+1)\right)\frac{t^n}{n!}
\end{aligned}$$

the right-hand side is

$$\begin{aligned}
&\frac{2Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt}(e^t-1) \\
=\ &\frac{2Li_{k,q}(1-e^{-t})}{e^t-1}e^{(x+1)t}-\frac{2Li_{k,q}(1-e^{-t})}{e^t-1}e^{xt} \\
=\ &\sum_{n=0}^{\infty}\left(2B_{n,q}^{(k)}(x+1)-2B_{n,q}^{(k)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we get Theorem 2.6. □

3. A relation with q-analogue of the Arakawa-Kaneko type zeta functions $Z_{E,k,q}(s,x)$

3.1. A q-analogue of Arakawa-Kaneko type zeta functions.

Definition 3.1. For Re(s) > 0, $x > 0$ and $k \in \mathbb{Z}$, set

$$Z_{E,k,q}(s,x)=\frac{2}{\Gamma(s)}\int_0^{\infty}\frac{Li_{k,q}(1-e^{-t})}{e^t+1}e^{-xt}t^{s-2}dt, \tag{11}$$

the Laplace-Mellin integral. We call it the Arakawa-Kaneko type zeta function for q-analogue of poly-Euler polynomials.

Theorem 3.2. *A q-analogue of the zeta function $Z_{E,k,q}(s,x)$ is defined for* Re(s) > 0 *and* $x > 0$ *if* $k \geq 1$*, and for* Re(s) > 0 *and* $x > 0$ *if* $k \leq 0$*.*

Proof. We can prove the convergence of the defined function as follows.

(i) $k \geq 1$: for $t \geq 0$, we have

$$\begin{aligned}
\frac{Li_{k,q}(1-e^{-t})}{e^t+1}e^{-xt}t^{s-2} &\leq Li_{k,q}(1-e^{-t})e^{-xt}t^{s-2} \\
&\leq \frac{e^t-1}{e^t+1}e^{-xt}t^{s-2}
\end{aligned}$$

$$\leq \quad e^{-xt}t^{s-1}.$$

(ii) $k=0$: for $t \geq 0$, we have

$$\begin{aligned}\frac{Li_{k,q}(1-e^{-t})}{e^t+1}e^{-xt}t^{s-2} &= \frac{e^t-1}{e^t+1}e^{-xt}t^{s-2} \\ &\leq e^{-xt}t^{s-1}.\end{aligned}$$

(iii) $k<0$: for $t \geq 0$, using the equation (4), we have

$$\begin{aligned}&\frac{Li_{k,q}(1-e^{-t})}{e^t+1}e^{-xt}t^{s-2} \\ = \;& \frac{1}{(1-q)^{|k|}}\sum_{l=0}^{|k|}(-1)^l\binom{|k|}{l}\frac{q^l(1-e^{-t})}{(1+e^t)(1-q^l(1-e^{-t}))}e^{-xt}t^{s-2} \\ = \;& \frac{1}{(1-q)^{|k|}}\frac{e^t-1}{e^t+1}e^{-xt}t^{s-2} \\ &+\frac{1}{(1-q)^{|k|}}\sum_{l=1}^{|k|}(-1)^l\binom{|k|}{l}\frac{q^l(1-e^{-t})}{(1+e^t)(1-q^l(1-e^{-t}))}e^{-xt}t^{s-2} \\ \leq \;& \frac{1}{(1-q)^{|k|}}\frac{e^t-1}{e^t+1}e^{-xt}t^{s-2} \\ &+\frac{|k|!}{(1-q)^{|k|}}\sum_{l=1}^{|k|}(-1)^l\frac{q^l(1-e^{-t})}{(1+e^t)(1-q^l(1-e^{-t}))}e^{-xt}t^{s-2} \\ \leq \;& \frac{1}{(1-q)^{|k|}}e^{-xt}t^{s-1}+\frac{|k|!}{(1-q)^{|k|}}\sum_{l=1}^{|k|}\frac{(-1)^lq^l}{1-q^l}e^{-xt}t^{s-2} \\ &\qquad\left(\text{since}\sum_{l=1}^{|k|}\frac{(-1)^lq^l}{1-q^l}<0\right) \\ \leq \;& \frac{1}{(1-q)^{|k|}}e^{-xt}t^{s-1}.\end{aligned}$$

We can guarantee the convergence of the function $Z_{E,k,q}(s,x)$ by dividing it into the above three and proving it. $\square$

3.2. A relation with the Arakawa-Kaneko type zeta functions $Z_{E,k,q}(s,x)$.

We review the zeta function investigated in [2]. For $k \in \mathbb{Z}$, we set

$$Z_{B,k}(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{-xt}t^{s-1}dt,$$

the Laplace-Mellin integral. It is defined for Re(s) >0 and $x>0$ if $k\geq 1$, and for Re(s) >0 and $x>|k|+1$ if $k\leq 0$. The function $s\longmapsto Z_{B,k}(s,x)$ has analytic continuation to an entire function on the whole complex s-plane and

$$Z_{B,k}(-n,x)=(-1)^n B_n^{(k)}(-x)$$

holds for $n\geq 0$, $x>0$.

Definition 3.3. For Re(s) >0, $x>0$ and $k\in\mathbb{Z}$, we define

$$Z_{B,k,q}(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1-e^{-t}}e^{-xt}t^{s-1}dt.$$

There is a relation between $Z_{E,k,q}(s,x)$ and $Z_{B,k,q}(s,x)$.

Theorem 3.4. *For $k\in\mathbb{Z}$, we have*

$$Z_{E,k,q}(s+1,x)+Z_{E,k,q}(s+1,x-1)=2Z_{B,k,q}(s,x)-2Z_{B,k,q}(s,x+1). \quad (12)$$

Proof. We compute both sides of

$$\begin{aligned}&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{t(1+e^t)}t(1+e^t)e^{-xt}t^{s-1}dt\\ =\;&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{-xt}t^{s-1}dt.\end{aligned}$$

The left-hand side is

$$\begin{aligned}&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{t(1+e^t)}t(1+e^t)e^{-xt}t^{s-1}dt\\ =\;&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1+e^t}e^{-xt}t^{(s+1)-2}dt\\ &\quad+\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1+e^t}e^{-(x-1)t}t^{(s+1)-2}dt\\ =\;&Z_{E,k,q}(s+1,x)+Z_{E,k,q}(s+1,x-1)\end{aligned}$$

the right-hand side is

$$\begin{aligned}&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1-e^{-t}}(1-e^{-t})e^{-xt}t^{s-1}dt\\ =\;&\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1-e^{-t}}e^{-xt}t^{s-1}dt\\ &\quad-\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-t})}{1-e^{-t}}e^{-(x+1)t}t^{s-1}dt\\ =\;&2Z_{B,k,q}(s,x)-2Z_{B,k,q}(s,x+1)\end{aligned}$$

from which we deduce the result. $\square$

4. Generalized q-analogue of poly-Euler polynomials and Arakawa-Kaneko type L-Functions

4.1. Generalized q-analogue of poly-Euler polynomials.

Let f be a positive integer and χ the Dirichlet character with conductor $f = f_\chi$. As is well-known, generalized Euler polynomials are defined by the generating function

$$2\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{e^{at}}{e^{ft}+1}e^{xt} = \sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!}.$$

Definition 4.1. Let $k \in \mathbb{Z}$. We define generalized a q-analogue of poly-Euler polynomials $E^{(k)}_{n,\chi,q}(x)\,(n = 0, 1, 2, \cdots)$ by

$$\frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{(x+a)t} = \sum_{n=0}^{\infty}E^{(k)}_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{13}$$

We call $E^{(k)}_{n,\chi,q} = E^{(k)}_{n,\chi,q}(0)\,(n = 0, 1, 2, \cdots)$ generalized a q-analogue of poly-Euler numbers.

One can easily prove the following three theorems as in a q-analogue of poly-Euler polynomial case.

Theorem 4.2. *(Addition formula). For $k \in \mathbb{Z}$ and $n \geq 0$, we obtain*

$$E^{(k)}_{n,\chi,q}(x+y) = \sum_{m=0}^{n}\binom{n}{m}E^{(k)}_{m,\chi,q}(x)y^{n-m}.$$

Proof. We calculate

$$\begin{aligned}
E^{(k)}_{n,\chi,q}(x+y) &= \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{(x+y+a)t} \\
&= \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{(x+a)t}e^{yt} \\
&= \sum_{n=0}^{\infty}E^{(k)}_{n,\chi,q}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}y^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}E^{(k)}_{m,\chi,q}(x)y^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}$$

□

Lemma 4.3. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$E^{(k)}_{n,\chi,q}(x) = \sum_{m=0}^{n}\binom{n}{m}E^{(k)}_{m,\chi,q}x^{n-m}.$$

Proof. We calculate

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}(x)\frac{t^n}{n!} &= \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{(x+a)t} \\
&= \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{at}e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}E_{m,\chi,q}^{(k)}x^{n-m}\right)\frac{t^n}{n!}.
\end{aligned}$$

□

Theorem 4.4. *(Appell sequence) For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\frac{d}{dx}E_{n+1,\chi,q}^{(k)}(x) = (n+1)E_{n,\chi,q}^{(k)}(x).$$

Proof. Since Lemma 4.3, we have

$$\begin{aligned}
\frac{d}{dx}E_{n+1,\chi,q}^{(k)}(x) &= \frac{d}{dx}\sum_{m=0}^{n+1}\binom{n}{m}E_{m,\chi,q}^{(k)}x^{n-m+1} \\
&= \sum_{m=0}^{n}(n+1)\binom{n}{m}E_{m,\chi,q}^{(k)}x^{n-m} \\
&= (n+1)E_{n,\chi,q}^{(k)}(x).
\end{aligned}$$

□

Theorem 4.5. *For any $n \geq 0$, we have*

$$\begin{aligned}
E_{n,\chi,q}^{(k)}(x) &= f^n\sum_{a=0}^{f-1}(-1)^a\chi(a)E_{n,q}^{(k)}\left(\frac{x+a}{f}\right), \\
E_{n,\chi,q}^{(k)} &= f^n\sum_{a=0}^{d-1}(-1)^a\chi(a)E_{n,q}^{(k)}\left(\frac{a}{f}\right).
\end{aligned}$$

Proof. To begin with, let us prove that first identity. We make use of the generating function for $E_{n,\chi,q}^{(k)}(x)$. We calculate

$$\begin{aligned}
&\sum_{n=0}^{\infty} f^n\sum_{a=0}^{f-1}(-1)^a\chi(a)E_{n,q}^{(k)}\left(\frac{x+a}{f}\right)\frac{t^n}{n!} \\
= &\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{n=0}^{\infty}E_{n,q}^{(k)}\left(\frac{x+a}{f}\right)\frac{(ft)^n}{n!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{2Li_{k,q}(1-e^{-ft})}{ft(e^t+1)}e^{(\frac{x+a}{f})ft} \\
&= \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{Li_{k,q}(1-e^{-ft})}{t(e^t+1)}e^{(x+a)t} \\
&= \sum_{n=0}^{\infty}E_{n,\chi,q}^{(k)}(x)\frac{t^n}{n!},
\end{aligned}
$$

which yields the generating function for $E_{n,\chi,q}^{(k)}(x)$. The second identity comes from the first identity for $x = 0$. $\square$

Theorem 4.6. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$
\begin{aligned}
&E_{n,\chi,q}^{(k)}(x) \\
= \ &\frac{f^n}{n+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{m=0}^{\infty}\frac{1}{[m+1]_q^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}E_{n+1,q}\left(\frac{x+a-fj}{f}\right).
\end{aligned}
$$

Proof. From Theorem 2.5 and 4.5, we have

$$
\begin{aligned}
&E_{n,\chi,q}^{(k)}(x) \\
= \ &f^n\sum_{a=0}^{f-1}(-1)^a\chi(a)E_{n,q}^{(k)}\left(\frac{x+a}{f}\right) \\
= \ &f^n\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{1}{n+1}\sum_{m=0}^{\infty}\frac{1}{[m+1]_q^k} \\
&\quad\times\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}E_{n+1,q}\left(\frac{x+a-fj}{f}\right) \\
= \ &\frac{f^n}{n+1}\sum_{a=0}^{f-1}(-1)^a\chi(a)\sum_{m=0}^{\infty}\frac{1}{[m+1]_q^k}\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}E_{n+1,q}\left(\frac{x+a-fj}{f}\right).
\end{aligned}
$$

$\square$

4.2. A q-analogue of the Arakawa-Kaneko type L-Functions.

Definition 4.7. For $k \in \mathbb{Z}$, define the L-series attached to χ by the Laplace-Mellin integral

$$
L_{E,k,q}(s,x,\chi) = \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{-(x-a)t}t^{s-1}dt.
$$

It is defined for Re(s) > 1 and $x > f-1$ if $k \geq 1$, and for Re(s) > 1 and $x > f-1$ if $k \leq 0$. We call $L_{E,k,q}(s,x,\chi)$ a q-analogue of the Arakawa Kaneko type L-Function.

Theorem 4.8. *One has*

$$L_{E,k,q}(s,x,\chi) = f^{-s}\sum_{a=0}^{f-1}(-1)^a\chi(a)Z_{E,k,q}\left(s,\frac{x-a}{f}\right).$$

Proof. The identity is proves by

$$\begin{aligned}
& L_{E,k,q}(s,x,\chi) \\
= \; & \frac{2}{f}\sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{1}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-ft})}{t(e^{ft}+1)}e^{-(x-a)t}t^{s-1}dt \\
= \; & \sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-ft})}{ft(e^{ft}+1)}e^{-(\frac{x-a}{f})ft}t^{s-1}dt \\
& \qquad\qquad\qquad\qquad (\text{put } u = dt) \\
= \; & \sum_{a=0}^{f-1}(-1)^a\chi(a)\frac{1}{f^s}\frac{2}{\Gamma(s)}\int_0^\infty \frac{Li_{k,q}(1-e^{-u})}{u(e^{u}+1)}e^{-(\frac{x-a}{d})u}u^{s-1}dt \\
= \; & f^{-s}\sum_{a=0}^{f-1}(-1)^a\chi(a)Z_{E,k,q}\left(s,\frac{x-a}{f}\right).
\end{aligned}$$

□

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