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IDENTITIES INVOLVING THE DEGENERATE GENERALIZED (p,q)-POLY-BERNOULLI NUMBERS AND POLYNOMIALS†

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Abstract. In this paper, we introduce degenerate generalized poly-Bernoulli numbers and polynomials with (p,q)-logarithm function. We find some identities that are concerned with the Stirling numbers of second kind and derive symmetric identities by using generalized falling factorial sum.

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1. Introduction

Throughout this paper, we use the following notations. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the real numbers and $\mathbb{C}$ denotes the complex numbers, respectively.

The ordinary Bernoulli numbers B_n and polynomials $B_n(x)$ are given by the generating functions:

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!} \quad \text{and} \quad \frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \quad |t|<\frac{\pi}{2}.$$

Many mathematicians have studied about the generalizations of these numbers and polynomials. Carlitz introduced the generating functions of the degenerate poly-Bernoulli polynomials (see [2,6,11]). Also, in [1,4,5,6,7,8,9], some properties of the poly Bernoulli numbers and their zeta functions are researched.

Let $\lambda\in\mathbb{R}$ and $\lambda\neq 0$. We note that the generating function of degenerate Bernoulli polynomials $B_{n,\lambda}(x)$ is given by (see [2,6,11]):

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$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n}. \tag{1.1}$$

It is evident that (1.1) reduces to the generating function of the ordinary Bernoulli polynomials as below:

$$\lim_{\lambda\to 0}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

For $n \in \mathbb{C}$, the (p,q)-number $[n]_{p,q}$ is defined as follows

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}. \tag{1.2}$$

Note that $\lim_{p\to 1}[n]_{p,q} = [n]_q$, $(q \neq 1)$ and $\lim_{q\to 1}[n]_q = n$.

In [4], we defined a generalization of the ordinary Bernoulli polynomials $B_n(x;a)$ with variable a by :

$$\frac{Li_k(1-e^{-t})}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!}$$

where

$$Li_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^k}, \quad (k \in \mathbb{Z}) \quad \text{(see [1,4,5,6,7,8,9])}$$

is polylogarithm function. When $k=1$, the polylogarithm function is appeared $Li_1(x) = -\log(1-x)$ and $Li_1(1-e^{-t}) = t$. Thus, the result shows that the poly-Bernoulli polynomials are identical to the ordinary Bernoulli polynomials $B_n(x)$.

From the Equation(1.2), the (p,q)-analogue of the polylogarithm function $Li_{k,p,q}$ is given by

$$Li_{k,p,q}(x) = \sum_{n=1}^{\infty}\frac{x^n}{[n]_{p,q}^k}, (k \in \mathbb{Z}) \quad \text{(see [5,6,8])}.$$

In [5], by using (p,q)-polylogarithm function, we introduced a generalized (p,q)-poly-Bernoulli numbers and polynomials. For $n, k \in \mathbb{Z}$ with $n \geq 0$ and $0 < q < p \leq 1$, we define a generalized (p,q)-poly-Bernoulli polynomials with variable a by

$$\frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}.$$

In the case $x=0$, it leads to $B_{n,p,q}^{(k)}(0;a) = B_{n,p,q}^{(k)}(a)$ that is, the generalized (p,q)-poly-Bernoulli numbers with variable a.

In this paper, we define degenerate generalized (p,q)-poly-Bernoulli polynomials and make a study the analytic properties of them. We also find some identities that are related with the Stirling numbers of the second kind. Furthermore, using special functions and generalized falling factorial sum, we derive some symmetric properties of the degenerate generalized (p,q)-poly Bernoulli numbers and polynomials.

2. Some identities of the degenerate generalized (p,q)-poly-Bernoulli polynomials

In this section, we introduce degenerate generalized (p,q)-poly-Bernoulli polynomials $B^{(k)}_{n,\lambda,p,q}(x;a)$. We also explore some analytic properties that is concerned with the polynomials.

Definition 2.1. For $\lambda \in \mathbb{R}_+$, $n \geq 0, k \in \mathbb{Z}$, and $0 < q < p \leq 1$, we define degenerate generalized (p,q)-poly-Bernoulli polynomials by the following generating function:

$$\frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(x;a)\frac{t^n}{n!}$$

where

$$Li_{k,p,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]^k_{p,q}}$$

is the k-th (p,q)-polylogarithm function. When $x=0$, $B^{(k)}_{n,\lambda,p,q}(0;a) = B^{(k)}_{n,\lambda,p,q}(a)$ are called the degenerate generalized (p,q)-poly-Bernoulli numbers.
Since $\lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}} = e^t$, it is trivial that the degenerate generalized (p,q)-poly-Bernoulli polynomials are reduced the generalized (p,q)-poly-Bernoulli polynomials :

$$\lim_{\lambda\to 0} B^{(k)}_{n,\lambda,p,q}(x;a) = B^{(k)}_{n,p,q}(x;a).$$

Note that $(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \Pi^n_{k=1}(x-(k-1))$ is falling factorial and $(x|\lambda)_n = \Pi^n_{k=1}(x-\lambda(k-1))$ is generalized falling factorial with increment λ.

Theorem 2.2. *For integer n, m and k with $n \geq 0$ and $m \geq 1$, we have*

$$B^{(k)}_{n,\lambda,p,q}(mx;a) = \sum_{l=0}^{n}\binom{n}{l} m^{n-1} B^{(k)}_{l,\lambda,p,q}(a)(x|\lambda)_{n-l}.$$

Proof. Let $n \geq 0$, $m \geq 1$ and $k \in \mathbb{Z}$. We obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(mx;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}(1+\lambda t)^{\frac{mx}{\lambda}} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} B^{(k)}_{l,\lambda,p,q}(a)(mx|\lambda)_{n-l}\frac{t^n}{n!}.\end{aligned}$$

Therefore we obtain the above result. □

When $m=1$, it satisfies

$$B_{n,\lambda,p,q}^{(k)}(x;a)=\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda,p,q}^{(k)}(a)(x|\lambda)_{n-l}. \tag{2.1}$$

Theorem 2.3. *Let n and k be integers with $n\geq 0$ and $x,y\in\mathbb{R}$. We get the addition theorem :*

$$B_{n,\lambda,p,q}^{(k)}(x+y;a)=\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda,p,q}^{(k)}(x;a)(y|\lambda)_{n-l}.$$

Proof. For $n\geq 0$ and $k\in\mathbb{Z}$ and $x,y\in\mathbb{R}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda,p,q}^{(k)}(x+y;a)\frac{t^n}{n!}&=\frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}(1+\lambda t)^{\frac{x+y}{\lambda}}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda,p,q}^{(k)}(x;a)\ (y|\lambda)_{n-l}\frac{t^n}{n!}.\end{aligned}$$

Note that if $x=0$, then the above result induce (2.1). □

Theorem 2.4. *Let $n\geq 0,k\in\mathbb{Z}$ and $x,y\in\mathbb{R}$. We have*

$$B_{n,\lambda,p,q}^{(k)}(x;a)=\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}B_{l,\lambda,p,q}^{(k)}(x-y;a)\left(\frac{y}{\lambda}\right)_{n-l}.$$

Proof. For $x,y\in\mathbb{R}$ and $n,k\in\mathbb{Z}$ with $n\geq 0$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda,p,q}^{(k)}(x;a)\frac{t^n}{n!}&=\sum_{m=0}^{\infty}B_{m,\lambda,p,q}^{(k)}(x-y;a)\frac{t^m}{m!}\ \sum_{n=0}^{\infty}\left(\frac{y}{\lambda}\right)_n\frac{(\lambda t)^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}B_{l,\lambda,p,q}^{(k)}(x-y;a)\left(\frac{y}{\lambda}\right)_{n-l}\frac{t^n}{n!}.\end{aligned}$$

□

The Stirling numbers of the second kind $S_2(n,m)$ are defined by

$$x^n=\sum_{m=0}^{n}S_2(n,m)(x)_m.$$

The generating function of the Stirling numbers of the second kind are given by

$$\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}=\frac{(e^t-1)^m}{m!}\quad\text{(see [2-6])}.$$

In [5], the (p,q)-polylogarithm function is represented as the following form with the stirling numbers of the second kind:

$$\frac{Li_{k,p,q}(1-e^{-t})}{t}=\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+1}}{[l]_{p,q}^k}l!\frac{S_2(n+1,l)}{n+1}\frac{t^n}{n!}. \tag{2.2}$$

From (2.2), we obtain next result.

Theorem 2.5. *For $n,k \in \mathbb{Z}$ with $n \geq 0$, we have*

$$B^{(k)}_{n,\lambda,p,q}(x;a) = \sum_{i=0}^{n} \binom{n}{i} \sum_{l=1}^{i+1} \frac{(-1)^{l+i+1} l! S_2(i+1,l)}{[l]^k_{p,q}(i+1)} B_{n-i,\lambda}(x;a).$$

Proof. Using the Definition 2.1, (1.1) and (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(x;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{t} \frac{t(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{a}{\lambda}}-1} \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=1}^{i+1} \binom{n}{i} \frac{(-1)^{l+i+1} l! S_2(i+1,l)}{[l]^k_{p,q}(i+1)} B_{n-i,\lambda}(x;a)\frac{t^n}{n!}.\end{aligned}$$

□

Theorem 2.6. *For $n \geq 1$, $k \in \mathbb{Z}$, we find a recurrence relation as below*

$$\begin{aligned}&B^{(k)}_{n,\lambda,p,q}(x+a;a) - B^{(k)}_{n,\lambda,p,q}(x;a) \\ &\quad = \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1} \frac{(-1)^{l+1+r}}{[l+1]^k_{p,q}}(l+1)! S_2(r,l+1)\right)(x|\lambda)_{n-r}.\end{aligned}$$

Proof. Let n and k be integers with $n \geq 1$. We have

$$\begin{aligned}&\sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(x+a;a)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(x;a)\frac{t^n}{n!} \\ &\quad = \frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}(1+\lambda t)^{\frac{x+a}{\lambda}} - \frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &\quad = \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]^k_{p,q}}(1+\lambda t)^{\frac{x}{\lambda}} \\ &\quad = \sum_{r=1}^{\infty}\sum_{l=0}^{r-1} \frac{(-1)^{l+1+r}}{[l+1]^k_{p,q}}(l+1)! S_2(r,l+1)\frac{t^r}{r!}\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty}\sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1} \frac{(-1)^{l+1+r}}{[l+1]^k_{p,q}}(l+1)! S_2(r,l+1)\right)(x|\lambda)_{n-r}\frac{t^n}{n!}.\end{aligned}$$

□

By using the sum of geometric sequence, we note that

$$\sum_{i=0}^{d-1}(1+\lambda t)^{\frac{ai+x}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda}} \frac{1-(1+\lambda t)^{\frac{ad}{\lambda}}}{1-(1+\lambda t)^{\frac{a}{\lambda}}}. \tag{2.3}$$

From (2.3), we get next theorem.

Theorem 2.7. *For $n, k \in \mathbb{Z}$ with $n \geq 0$, we obtain*

$$B^{(k)}_{n,\lambda,p,q}(x;a) = \sum_{l=1}^{n} \frac{(-1)^{l+n} l! S_2(n,l)}{[l]^k_{p,q}} \sum_{i=0}^{d-1} \frac{(1+\lambda t)^{\frac{ai+x}{\lambda}}}{(1+\lambda t)^{\frac{ad}{\lambda}} - 1}.$$

Proof. Let $n \geq 0$ and $k \in \mathbb{Z}$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(x;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{a}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{Li_{k,p,q}(1-e^{-t})}{(1+\lambda t)^{\frac{d}{\lambda}} - 1}\sum_{i=0}^{d-1}(1+\lambda t)^{\frac{ai+x}{\lambda}} \\
&= \frac{Li_{k,p,q}(1-e^{-t})}{t}\sum_{i=0}^{d-1}\frac{t}{(1+\lambda t)^{\frac{ad}{\lambda}-1}}(1+\lambda)^{\frac{ai+x}{\lambda}} \\
&= \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{(-1)^{l+n} l! S_2(n,l)}{[l]^k_{p,q}} \sum_{i=0}^{d-1} \frac{(1+\lambda t)^{\frac{ai+x}{\lambda}}}{(1+\lambda t)^{\frac{ad}{\lambda}} - 1}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{l=1}^{n} \frac{(-1)^{l+n} l! S_2(n,l)}{[l]^k_{p,q}} \sum_{i=0}^{d-1} \frac{(1+\lambda t)^{\frac{ai+x}{\lambda}}}{(1+\lambda t)^{\frac{ad}{\lambda}} - 1}\frac{t^n}{n!}.
\end{aligned}$$

□

3. Symmetric properties using generalized falling factorial sum

In this section, we find some symmetric identities for the degenerate generalized (p,q)-poly-Bernoulli polynomials. Furthermore, we investigate another symmetric results that are related with the generalized falling factorial sum.

Theorem 3.1. *Let n, k be integers with $n \geq 0$ and $m_1, m_2 (m_1 \neq m_2) > 0$. The symmetric identity is obtained:*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r} m_1^{n-r} m_2^r B^{(k)}_{r,\lambda,p,q}(m_1 y;a) B^{(k)}_{n-r,\lambda,p,q}(m_2 x;a) \\
&\qquad = \sum_{r=0}^{n}\binom{n}{r} m_1^{r} m_2^{n-r} B^{(k)}_{r,\lambda,p,q}(m_2 y;a) B^{(k)}_{n-r,\lambda,p,q}(m_1 x;a).
\end{aligned}$$

Proof. For $n(n \geq 0), k \in \mathbb{Z}$ and $m_1, m_2 > 0$ with $m_1 \neq m_2$, we construct special function as below:

$$\begin{aligned}
F(t) &= \frac{L_{i_{k,p,q}}(1-e^{-m_1 t}) L_{i_{k,p,q}}(1-e^{-m_2 t})}{\left((1+\lambda t)^{\frac{am_1}{\lambda}} - 1\right)\left((1+\lambda t)^{\frac{am_2}{\lambda}} - 1\right)}(1+\lambda t)^{\frac{m_1 m_2 (x+y)}{\lambda}} \\
&= \sum_{n=0}^{\infty} B^{(k)}_{n,\lambda,p,q}(m_2 x;a)\frac{(m_1 t)^n}{n!} \sum_{r=0}^{\infty} B^{(k)}_{r,\lambda,p,q}(m_1 y;a)\frac{(m_2 t)^r}{r!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} m_1^{n-r} m_2^r B_{r,\lambda,p,q}^{(k)}(m_1 y; a) B_{n-r,\lambda,p,q}^{(k)}(m_2 x; a) \frac{t^n}{n!}.$$

In similarly, we obtain next result.

$$F(t) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} m_1^r m_2^{n-r} B_{r,\lambda,p,q}^{(k)}(m_2 y; a) B_{n-r,\lambda,p,q}^{(k)}(m_1 x; a) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ in above results, we can verify that the theorem is correct. □

Put $p = 1$ in the generating function of Theorem 3.1, then we get the next Corollary.

Corollary 3.2. *Let n, k be integers with $n \geq 0$ and $m_1, m_2 > 0$ with $m_1 \neq m_2$. Then we obtain the result of the degenerate generalized q-poly-Bernoulli polynomials :*

$$\sum_{r=0}^{n} \binom{n}{r} m_1^{n-r} m_2^r B_{r,\lambda,q}^{(k)}(m_1 y; a) B_{n-r,\lambda,q}^{(k)}(m_2 x; a) = \sum_{r=0}^{n} \binom{n}{r} m_1^r m_2^{n-r} B_{r,\lambda,q}^{(k)}(m_2 y; a) B_{n-r,\lambda,q}^{(k)}(m_1 x; a).$$

If $q \to 1$ and $\lambda \to 0$, then the symmetric identities in Corollary 3.2 is reduced that of the generalized poly-Bernoulli polynomials.

Corollary 3.3. *Let n, k be integers with $n \geq 0$ and $m_1, m_2 > 0$ with $m_1 \neq m_2$. Then we obtain the result of the degenerate generalized poly-Bernoulli polynomials :*

$$\sum_{r=0}^{n} \binom{n}{r} m_1^{n-r} m_2^r B_r^{(k)}(m_1 y; a) B_{n-r}^{(k)}(m_2 x; a) = \sum_{r=0}^{n} \binom{n}{r} m_1^r m_2^{n-r} B_r^{(k)}(m_2 y; a) B_{n-r}^{(k)}(m_1 x; a).$$

In [10,11], the generalized falling factorial sum is defined by

$$\sum_{n=0}^{\infty} \sigma_n(m-1; \lambda) \frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{m}{\lambda}} - 1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}.$$

Note that $\lim_{\lambda \to 0} \sigma_n(m-1; \lambda) = \sigma_n(m)$ where $\sigma_n(m) = \sum_{k=0}^{m} k^n$ is the power sums of the first integers(see [4,5,9]). Using the generalized falling factorial sum and (2.2), we obtain a symmetric identities of the degenerate generalized (p, q)-poly-Bernoulli numbers and polynomials.

Theorem 3.4. *Let $m_1, m_2 (m_1 \neq m_2) \in \mathbb{N}$ and $n, k \in \mathbb{Z}$ with $n \geq 0$. Then we have*

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i}\frac{(-1)^{i+l}m_2^i}{[l]_{p,q}^k}l!S_2(i,l)\sum_{r=0}^{n-i}\binom{n-i}{r}a^{n-i-r}m_1^r m_2^{n-i-r}\\ \times B_{r,\lambda,p,q}^{(k)}(am_2x;a)\sigma_{n-i-r}(\frac{m_1-1}{a};\lambda)\\ =\sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i}\frac{(-1)^{i+l}m_1^i}{[l]_{p,q}^k}l!S_2(i,l)\sum_{r=0}^{n-i}\binom{n-i}{r}a^{n-i-r}m_1^{n-i-r}m_2^r\\ \times B_{r,\lambda,p,q}^{(k)}(am_1x;a)\sigma_{n-i-r}(\frac{m_2-1}{a};\lambda)\end{aligned}$$

Proof. If we start the next function

$$\widetilde{F}(t)=\frac{Li_{k,p,q}(1-e^{-m_1t})Li_{k,p,q}(1-e^{-m_2t})\left((1+\lambda t)^{\frac{m_1m_2}{\lambda}}-1\right)(1+\lambda t)^{\frac{am_1m_2x}{\lambda}}}{\left((1+\lambda t)^{\frac{am_1}{\lambda}}-1\right)\left((1+\lambda t)^{\frac{am_2}{\lambda}}-1\right)}, \tag{3.1}$$

then it holds

$$\begin{aligned}\widetilde{F}(t)&=\sum_{n=0}^{\infty}\sum_{l=1}^{n}\frac{(-1)^{n+l}m_2^n}{[l]_{p,q}^k}l!S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty}B_{n,\lambda,p,q}^{(k)}(am_2x;a)\frac{(m_1t)^n}{n!}\\ &\quad\times\sum_{l=0}^{\infty}\sigma_l(\frac{m_1-1}{a};\lambda)\frac{(am_2t)^l}{l!}\\ &=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i}\frac{(-1)^{i+l}m_2^i}{[l]_{p,q}^k}l!S_2(i,l)\sum_{r=0}^{n-i}\binom{n-i}{r}a^{n-i-r}m_1^r m_2^{n-i-r}\\ &\quad\times B_{r,\lambda,p,q}^{(k)}(am_2x;a)\sigma_{n-i-r}(\frac{m_1-1}{a};\lambda)\frac{t^n}{n!}.\end{aligned}\tag{3.2}$$

In a similar way, (3.1) follows as below

$$\begin{aligned}\widetilde{F}(t)&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i}\frac{(-1)^{i+l}m_1^i}{[l]_{p,q}^k}l!S_2(i,l)\sum_{r=0}^{n-i}\binom{n-i}{r}a^{n-i-r}m_1^{n-i-r}m_2^r\\ &\quad\times B_{r,\lambda,p,q}^{(k)}(am_1x;a)\sigma_{n-i-r}(\frac{m_2-1}{a};\lambda)\frac{t^n}{n!}.\end{aligned}\tag{3.3}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.2) and (3.3), then it gives a symmetric identity of the theorem. □

From the multinomial coefficient, the result of Theorem 3.4 is represented as follows

Corollary 3.5. *Let $m_1, m_2 (m_1 \neq m_2) \in \mathbb{N}$ and $n, k \in \mathbb{Z}$ with $n \geq 0$. Then we obtain*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \geq 0}}^{n} \binom{n}{n_1,n_2,n_3} \sum_{l=1}^{n_1} \frac{(-1)^{n_1+l} m_2^{n_1} l!}{[l]_{p,q}^k} S_2(n_1,l) \\
&\qquad \times a^{n_3} m_1^{n_2} m_2^{n_3} B_{n_2,\lambda,p,q}^{(k)}(am_2x;a)\ \sigma_{n_3}(\frac{m_1-1}{a};\lambda) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3 \geq 0}}^{n} \binom{n}{n_1,n_2,n_3} \sum_{l=1}^{n_1} \frac{(-1)^{n_1+l} m_1^{n_1} l!}{[l]_{p,q}^k} S_2(n_1,l) \\
&\qquad \times a^{n_3} m_1^{n_3} m_2^{n_2} B_{n_2,\lambda,p,q}^{(k)}(am_1x;a)\ \sigma_{n_3}(\frac{m_2-1}{a};\lambda) \frac{t^n}{n!}
\end{aligned}$$

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