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GENERALIZED EULER POWER SERIES†

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Abstract. This work is a continuation of our investigations for p-adic analogue of the alternating form Dirichlet L-functions

$$L_E(s,\chi)=\sum_{n=1}^{\infty}\frac{(-1)^n\chi(n)}{n^s},\quad \mathrm{Re}(s)>0.$$

Let $L_{p,E}(s,t;\chi)$ be the p-adic Euler L-function of two variables. In this paper, for any $\alpha\in\mathbb{C}_p, |\alpha|_p\leq 1$, we give a power series expansion of $L_{p,E}(s,t;\chi)$ in terms of the variable t. From this, we derive a power series expansion of the generalized Euler polynomials with negative index, that is, we prove that

$$E_{-n,\chi}(t)=\sum_{m=0}^{\infty}\binom{-n}{m}E_{-(m+n),\chi}t^m,\quad n\in\mathbb{N},$$

where $t\in\mathbb{C}_p$ with $|t|_p<1$. Some further properties for $L_{p,E}(s,t;\chi)$ has also been shown.

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1. Introduction

For a fixed primitive Dirichlet character χ with odd conductor f_χ, the generalized Euler polynomials $E_{n,\chi}(t)\in\mathbb{Q}(\chi(1),\chi(2),\ldots,t)$ are defined by the generating function

$$\sum_{a=1}^{f_\chi}\frac{2(-1)^a\chi(a)e^{(a+t)x}}{e^{f_\chi x}+1}=\sum_{n=0}^{\infty}E_{n,\chi}(t)\frac{x^n}{n!}\tag{1}$$

(see [3, 5, 9, 10]). The corresponding generalized Euler numbers can be defined by $E_{n,\chi}=E_{n,\chi}(0)$. With this definition, the generalized Euler polynomials can also be expressed in terms of the expansion $E_{n,\chi}(t)=\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}t^{n-k}$. This

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can also be derived from (1). Another property this kind of polynomial satisfying is that for $n \geq 0$,

$$(-1)^{m-1}E_{n,\chi}(t+mf_\chi)+E_{n,\chi}(t)=2\sum_{a=1}^{mf_\chi}(-1)^a\chi(a)(t+a)^n, \tag{2}$$

where χ is the fixed primitive Dirichlet character with odd conductor f_χ and $m \geq 1$ (see [6, p. 376, (10)]). This can be derived from (1). Note that letting $\chi = 1$, the trivial character, and letting $t = 0$, (2) becomes to

$$\frac{1}{2}((-1)^{m-1}E_{n,1}(m)+E_{n,1}(0))=\sum_{a=1}^{m}(-1)^a a^n. \tag{3}$$

The ordinary Euler polynomials $E_n(t) \in \mathbb{Q}(t)$ is defined by the generating function

$$\frac{2e^{tx}}{e^x+1}=\sum_{n=0}^{\infty}E_n(t)\frac{x^n}{n!}. \tag{4}$$

Here are some important properties of Euler polynomials

$$\begin{aligned} E_n(t+1)+E_n(t)&=2t^n,\\ E_n(1-t)&=(-1)^nE_n(t), \end{aligned} \tag{5}$$

where $n \geq 0$. Each of these results can be derived from the generating function (4) above. Similar to (2) for the generalized Euler polynomials, whenever $m \geq 1$ and $n \geq 0$,

$$\frac{1}{2}((-1)^{m-1}E_n(m)+E_n(0))=\sum_{a=0}^{m-1}(-1)^a a^n, \tag{6}$$

where we take 0^0 to be 1 in the case of $a = 0$ and $n = 0$. Note that this can be derived from the first identity of (5) since

$$(-1)^{m-1}E_n(m)+E_n(0)=\sum_{a=0}^{m-1}(-1)^a(E_n(a+1)+E_n(a)). \tag{7}$$

From (1), we may conclude that the numbers $E_n(0)$ are related to the generalized Euler polynomials, that is letting $\chi = 1$ we have

$$-\frac{2e^x}{e^x+1}=\sum_{n=0}^{\infty}E_{n,1}(0)\frac{x^n}{n!} \tag{8}$$

and since

$$-\frac{2e^x}{e^x+1}=-2+\frac{2}{e^x+1}, \tag{9}$$

we see that

$$E_{n,1}(0)=E_n(0) \text{ for all } n\neq 0 \quad\text{and}\quad E_{0,1}(0)=-E_0(0)=-1 \tag{10}$$

and this can be written as $E_{n,1}(0) = (-1)^{n-1}E_n(0)$ for $n \geq 0$, and for the polynomials, $E_{n,1}(t) = (-1)^{n-1}E_n(-t)$ for $n \geq 0$.

The main interest of these numbers is that they give the values at negative integers of Euler L-functions: An alternating form of Dirichlet L-function

$$L_E(s,\chi)=\sum_{n=1}^{\infty}\frac{(-1)^n\chi(n)}{n^s},\quad \mathrm{Re}(s)>0, \tag{11}$$

is called Euler L-function (see [6, 7]). We see that $L_E(s,\chi)$ is indeed the following Dirichlet eta function with a character

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \tag{12}$$

where $\mathrm{Re}(s)>0$. The Dirichlet eta function $\eta(s)$ is a particular case of Witten's zeta functions in mathematical physics and it has been used by Euler to obtain a functional equation of Riemann zeta function $\zeta(s)$ (see [4]). In particular, Kim and Hu [7] derived the p-adic Euler L-function $L_{p,E}(s,\chi)$ by using the p-adic Huriwitz-type Euler zeta functions as building blocks. The p-adic function $L_{p,E}(s,\chi)$ may be served as a p-adic counterpart of $L_E(s,\chi)$ (11), the alternating form of Dirichlet L-functions.

The two variable p-adic L-functions have been studied by Fox [1], Simsek [10] and Young [11]. These functions interpolate the generalized Bernoulli polynomials at nonpositive integers. By using these functions, Kummer's congruences for generalized Bernoulli polynomials are established. In [6] Kim proved the existence of p-adic Euler L-function of two variables $L_{p,E}(s,t;\chi)$ (see (17) below), considered several properties of $L_{p,E}(s,t;\chi)$.

In this paper, we give a power series expansion of $L_{p,E}(s,t;\chi)$ in the variable t about any $\alpha\in\mathbb{C}_p, |\alpha|_p\le 1$ (see Theorem 2.6 below). Furthermore, we prove that

$$E_{-n,\chi}(t)=\sum_{m=0}^{\infty}\binom{-n}{m}E_{-(m+n),\chi}t^m,\quad n\in\mathbb{N},$$

where $t\in\mathbb{C}_p$ with $|t|_p<1$. We also obtain some properties of these functions.

2. Properties of p-adic Euler L-function with two variables

Let p be an odd prime number. Let $\mathbb{Q}_p$ be the topological completion of $\mathbb{Q}$ with respect to the metric topology induced by $|\cdot|_p$. Let $\mathbb{C}_p$ be the field of p-adic completion of algebraic closure of $\mathbb{Q}_p$. Let v_p denote the p-adic exponential valuation on $\mathbb{C}_p$, normalized so that $v_p(p)=1$.

Note that there exist $\phi(p)$ distinct solutions, modulo p, to the equation $x^{\phi(p)}-1=0$, and each solution must be congruent to one of the values $a\in\mathbb{Z}$, where $1\le a\le p, (a,p)=1$. Thus, by Hensel's Lemma, given $a\in\mathbb{Z}$ with $(a,p)=1$, there exists a unique $\omega(a)\in\mathbb{Z}_p$, where $\omega(a)^{\phi(p)}=1$, such that $\omega(a)\equiv a \pmod{p\mathbb{Z}_p}$. Letting $\omega(a)=0$ for $a\in\mathbb{Z}$ such that $(a,p)\neq 1$, it can be seen that ω is actually a Dirichlet character having conductor $f_\omega=p$, called the Teichmüller character. Let

$$\langle a\rangle=\omega^{-1}(a)a. \tag{13}$$

Then $\langle a\rangle \equiv 1 \pmod{p\mathbb{Z}_p}$. If $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbb{Z}, a+pt \equiv a \pmod{p\mathbb{Z}_p[t]}$. Thus, we define $\omega(a+pt) = \omega(a)$ for these values of t. We also define

$$\langle a+pt\rangle = \omega^{-1}(a)(a+pt) \tag{14}$$

for such t. Therefore, $\langle a+pt\rangle = \langle a\rangle + p\omega^{-1}(a)t$, so that $\langle a+pt\rangle \equiv 1 \pmod{p\mathbb{Z}_p[t]}$.

We also define a particular subring of $\mathbb{C}_p$ by

$$D = \left\{ s \in \mathbb{C}_p \,:\, v_p(s) > -1 + \frac{1}{p-1} \right\}. \tag{15}$$

Since $1 \in D$ and any point of a p-adic disc is its center, D is the same as the set $D = \{s \in \mathbb{C}_p \,:\, v_p(1-s) > -1 + \frac{1}{p-1}\}$.

Let $\mathbb{Q}_p(\chi)$ denote the field generated over $\mathbb{Q}_p$ by $\chi(a), a \in \mathbb{Z}$ in an algebraic closure of $\mathbb{Q}_p$. $\mathbb{Q}_p(\chi)$ is a locally compact topological field containing $\mathbb{Q}(\chi)$ as a dense subfield. Let $t \in \mathbb{C}_p, |t|_p \leq 1$, and let $\mathbb{Q}_p(\chi, t)$, the field generated over $\mathbb{Q}_p$ by adjoining t and the values $\chi(a), a \in \mathbb{Z}$. For $n \in \mathbb{N}$, we define χ_n to be the primitive character associated with the character $\chi_n : (\mathbb{Z}/\text{l.c.m.}(f_\chi, p)\mathbb{Z})^\times \to \mathbb{C}^\times$ defined by $\chi_n(a) = \chi(a)\omega^{-n}(a)$. We define a sequence of elements $\epsilon_{n,\chi}(t), n \geq 0$, in $\mathbb{Q}_p(\chi, t)$ by

$$\epsilon_{n,\chi}(t) = E_{n,\chi_n}(pt) - \chi_n(p)p^n E_{n,\chi_n}(t), \tag{16}$$

where $E_{n,\chi_n}(t)$ is the generalized Euler polynomial and $n \geq 0$. Note that $\chi_n(a)$ is in $\mathbb{Q}_p(\chi)$ for any $n \geq 0$ and $a \in \mathbb{Z}$.

Now we consider a p-adic Euler L-function of two variables and a power series expansion has been given in [6].

Define

$$L_{p,E}(s,t;\chi) = \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{f_\chi p^N} (-1)^a \chi(a) \langle a+pt\rangle^{1-s}, \tag{17}$$

which is analytic for $s \in D$ and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$ (see [5, 8, 10, 11]).

In the more generalized form, the p-adic Euler L-functions of two variables $L_{p,E}(s,t;\chi)$ must satisfy $L_{p,E}(s,0;\chi) = L_{p,E}(s,\chi)$, and so $L_{p,E}(s,0;\chi)$ vanishes for all $s \in D$ when $\chi(-1) = 1$, but this property does not hold for all t for any given χ (see [6, p. 376, Theorem 3.3]).

We have the following theorem.

Theorem 2.1 ([6, p. 375])**.** *Let χ be a Dirichlet character with odd conductor f_χ. For each $t \in \mathbb{C}_p$, with $|t|_p \leq 1$, there exists a unique p-adic analytic function with following properties:*

(1) *$L_{p,E}(s,t;\chi)$ has a series expansion*

$$L_{p,E}(s,t;\chi) = \sum_{n=0}^{\infty} (-1)^n a_n(t)(s-1)^n, \quad a_n(t) \in \mathbb{Q}_p(\chi, t),$$

where the power series converges in the domain D.

(2) *For all positive integer* n,

$$L_{p,E}(1-n,t;\chi)=\epsilon_{n,\chi}(t),$$

where $\epsilon_{n,\chi}(t)$ *is defined in (16).*

Remark 2.1. Putting $t=0$ in Theorem 2.1(2), we find that

$$L_{p,E}(1-n,\chi)=(1-\chi_n(p)p^n)E_{n,\chi_n},\quad n\in\mathbb{N}.$$

From this, we conclude that the definition of $L_{p,E}(1-n,\chi)$ is equivalent to the definition in [7] following Kubota-Leopoldt's approach (cf. [7, p. 3007, Proposition 5.9(2)]).

In the case $\chi=\omega^n$, Theorem 2.1(2) gives the following.

Corollary 2.2. *For all positive integer* n, *we obtain*

$$L_{p,E}(1-n,t;\omega^n)=E_{n,1}(pt)-p^nE_{n,1}(t).$$

In particular, we have

$$L_{p,E}(1-n,\omega^n)=(1-p^n)E_{n,1}(0)=(1-p^n)E_n(0)$$

for all $n\geq 1$.

Theorem 2.3. *Let* $t\in\mathbb{C}_p, |t|_p\leq 1$, *and* $s\in D$. *Then*

$$L_{p,E}(s,t;\chi)=\sum_{n=0}^{\infty}\binom{1-s}{n}p^nt^nL_{p,E}(s+n,\chi_n).$$

Proof. From (13) and (14), it is easy to see that for $t\in\mathbb{C}_p, |t|_p\leq 1$, and $a\in\mathbb{Z}_p^{\times}$,

$$\begin{aligned}\langle a+pt\rangle^{1-s}&=\langle a\rangle^{1-s}\left(\frac{a+pt}{a}\right)^{1-s}\\&=\langle a\rangle^{1-s}\sum_{n=0}^{\infty}\binom{1-s}{n}\left(\frac{pt}{a}\right)^n\\&=\sum_{n=0}^{\infty}\binom{1-s}{n}p^nt^n\omega^{-n}(a)\langle a\rangle^{1-s-n}.\end{aligned}\tag{18}$$

Combining (17) with (18), we obtain

$$
\begin{aligned}
L_{p,E}(s,t;\chi) &= \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{f_\chi p^N} (-1)^a \chi(a)\langle a+pt\rangle^{1-s} \\
&= \sum_{n=0}^{\infty} \binom{1-s}{n} p^n t^n \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{f_\chi p^N} (-1)^a \chi(a)\omega^{-n}(a)\langle a\rangle^{1-s-n} \\
&= \sum_{n=0}^{\infty} \binom{1-s}{n} p^n t^n \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{f_\chi p^N} (-1)^a \chi_n(a)\langle a\rangle^{1-(s+n)} \\
&= \sum_{n=0}^{\infty} \binom{1-s}{n} p^n t^n L_{p,E}(s+n,\chi_n),
\end{aligned} \tag{19}
$$

which completes the proof. □

Since we can now express $L_{p,E}(s,t;\chi)$ in terms of a power series in t, we can take a derivative of this function with respect to t.

Lemma 2.4. *Let $t \in \mathbb{C}_p, |t|_p \leq 1,$ and $s \in D$. Then*

$$\frac{\partial^n}{\partial t^n} L_{p,E}(s,t;\chi) = n! p^n \binom{1-s}{n} L_{p,E}(s+n,t;\chi_n),$$

where $n \in \mathbb{Z}, n \geq 0$.

Proof. The proof proceeds by induction. The case $n=0$ is clear. First we consider $n=1$. By Theorem 2.3 and

$$m\binom{1-s}{m} = (1-s)\binom{-s}{m-1}, \tag{20}$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial t} L_{p,E}(s,t;\chi) &= \sum_{m=1}^{\infty} m\binom{1-s}{m} p^m t^{m-1} L_{p,E}(s+m,\chi_m) \\
&= \sum_{m=1}^{\infty} (1-s)\binom{-s}{m-1} p^m t^{m-1} L_{p,E}(s+m,\chi_m) \\
&= p(1-s)\sum_{m=0}^{\infty} \binom{-s}{m} p^m t^m L_{p,E}(s+m+1,\chi_{m+1}) \\
&= p(1-s) L_{p,E}(s+1,t;\chi_1).
\end{aligned} \tag{21}
$$

Suppose that

$$\frac{\partial^n}{\partial t^n} L_{p,E}(s,t;\chi) = n! p^n \binom{1-s}{n} L_{p,E}(s+n,t;\chi_n)$$

for $n \in \mathbb{N}$. Then, by (21),

$$\begin{aligned}\frac{\partial^{n+1}}{\partial t^{n+1}} L_{p,E}(s,t;\chi) &= \frac{\partial}{\partial t}\left(\frac{\partial^n}{\partial t^n} L_{p,E}(s,t;\chi)\right) \\ &= n! p^n \binom{1-s}{n} \frac{\partial}{\partial t} L_{p,E}(s+n,t;\chi_n) \\ &= n! p^n \binom{1-s}{n} p(-s-n+1) L_{p,E}(s+n+1,t;\chi_{n+1}) \\ &= (n+1)! p^{n+1} \binom{1-s}{n+1} L_{p,E}(s+n+1,t;\chi_{n+1}),\end{aligned} \tag{22}$$

which completes the proof. □

Lemma 2.5 ([1, Proposition 2.6] and [2, p. 107]). *Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$ be a power series, and suppose $f(x)$ converges. If $f(x) = \sum_{n=0}^{\infty} a_n (x-\alpha)^n$ converges on some closed ball B in $\mathbb{C}_p$. Then for each $x \in B$, the k-th derivative $f^{(k)}(x)$ exists, and is given by*

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x-\alpha)^{n-k},$$

in particular, we have

$$a_k = \frac{f^{(k)}(\alpha)}{k!}.$$

From Lemma 2.4 and Lemma 2.5, we can derive a more general power series expansion of $L_{p,E}(s,t;\chi)$ in the variable t about any $\alpha \in \mathbb{C}_p, |\alpha|_p \leq 1$.

Theorem 2.6. *Let $t \in \mathbb{C}_p, |t|_p \leq 1$, and $s \in D$. Then*

$$L_{p,E}(s,t;\chi) = \sum_{n=0}^{\infty} \binom{1-s}{n} p^n (t-\alpha)^n L_{p,E}(s+n,\alpha;\chi_n),$$

where $\alpha \in \mathbb{C}_p, |\alpha|_p \leq 1$,

Remark 2.2. We remark that Theorem 2.6 is equivalent to Theorem 2.3 when $\alpha = 0$.

Proof of Theorem 2.6. Using Lemma 2.5, we can write $L_{p,E}(s,t;\chi)$ in the form

$$L_{p,E}(s,t;\chi) = \sum_{n=0}^{\infty} a_n (t-\alpha)^n,$$

where

$$a_n = \frac{1}{n!} \frac{\partial^n}{\partial t^n} L_{p,E}(s,t;\chi)\Big|_{t=\alpha}.$$

By Lemma 2.4, we obtain

$$\frac{1}{n!} \frac{\partial^n}{\partial t^n} L_{p,E}(s,t;\chi) = p^n \binom{1-s}{n} L_{p,E}(s+n,t;\chi_n),$$

and so

$$a_n = p^n \binom{1-s}{n} L_{p,E}(s+n, \alpha; \chi_n),$$

completing the proof. □

3. Generalized Euler polynomials of negative index

For $n \in \mathbb{N}$, and for $t \in \mathbb{C}_p, |t|_p \leq |p|_p$, we define the generalized Euler polynomials of negative index by (cf. [7, p. 3012, Definition 5.14])

$$E_{-n,\chi}(t) = \lim_{k\to\infty} E_{\phi(p^k)-n,\chi}(t), \tag{23}$$

where ϕ is the Euler-phi function and the limit here is taken p-adically. Since $E_{\phi(p^k)-n,1}(0) = E_{\phi(p^k)-n}(0)$ for $n, k \in \mathbb{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $E_{-n,1}(0) = E_{-n}(0)$ for all such n.

Denote $\chi_n = \chi\omega^{-n}$. Using (23), we can show that, since

$$\omega^{\phi(p^k)} = \omega^{p^{k-1}(p-1)} = 1 \quad \text{and} \quad \chi_{\phi(p^k)-n} = \chi\omega^{n-\phi(p^k)} = \chi\omega^n \tag{24}$$

for all characters χ and for all $n \in \mathbb{N}$,

$$\begin{aligned} E_{-n,\chi}(pt) &= \lim_{k\to\infty} \left(E_{\phi(p^k)-n,\chi_{\phi(p^k)}}(pt) - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n} E_{\phi(p^k)-n,\chi_{\phi(p^k)}}(t) \right) \\ &= \lim_{k\to\infty} L_{p,E}\left(1-(\phi(p^k)-n), t; \chi_n\right) \\ &= L_{p,E}\left(n+1, t; \chi_n\right). \end{aligned} \tag{25}$$

Since $L_{p,E}(n+1, t; \chi_n)$ exists for each $n \in \mathbb{N}$ and $t \in \mathbb{C}_p, |t|_p \leq 1$, we see that $E_{-n,\chi}(pt)$ must also exist for such t. Thus $E_{-n,\chi}(t)$ exists for $t \in \mathbb{C}_p, |t|_p \leq |p|_p$.

Theorem 3.1. *Let χ be a primitive character modulo f_χ and ϕ be the Euler-phi function. Then for all $n \in \mathbb{N}$, we obtain*

$$L_{p,E}(n+1, t; \chi) = \lim_{k\to\infty} E_{\phi(p^k)-n,\chi\omega^n}(pt).$$

In particular, we have $\lim_{k\to\infty} E_{\phi(p^k)-n,1}(0) = L_{p,E}(n+1, \omega^{-n})$.

Proof. Since $L_{p,E}(s, t; \chi)$ is a continuous function of s, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_{p,E}(n+1, t; \chi) &= \lim_{k\to\infty} L_{p,E}(n+1-\phi(p^k), t; \chi) \\ &= \lim_{k\to\infty} L_{p,E}(1-(\phi(p^k)-n), t; \chi) \\ &= \lim_{k\to\infty} \Big(E_{\phi(p^k)-n,\chi_{\phi(p^k)-n}}(pt) \\ &\qquad - \chi_{\phi(p^k)-n}(p) p^{\phi(p^k)-n} E_{\phi(p^k)-n,\chi_{\phi(p^k)-n}}(t) \Big) \end{aligned}$$

using Theorem 2.1(2). From (24), we obtain

$$L_{p,E}(n+1, t; \chi) = \lim_{k\to\infty} E_{\phi(p^k)-n,\chi\omega^n}(pt).$$

This completes the proof. □

Theorem 3.2. (1) *For all $n \in \mathbb{N}$ and $t \in \mathbb{C}_p, |t|_p < 1,$ we have*

$$E_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} E_{-(m+n),\chi} t^m.$$

(2) *For all $m, n \in \mathbb{N}$ with $p \mid mf_\chi$, we have*

$$\frac{1}{2}\left((-1)^{m-1}E_{-n,\chi}(mf_\chi) + E_{-n,\chi}\right) = \sum_{\substack{a=1\\(a,p)=1}}^{mf_\chi} (-1)^a \chi(a) a^{-n}.$$

Proof. (1) Put $t = 0$ in (25). Then

$$E_{-n,\chi} = E_{-n,\chi}(0) = L_{p,E}\left(n+1, \chi_n\right), \quad n \in \mathbb{N}.$$

Thus using (25) and Theorem 2.3, we have

$$\begin{aligned} E_{-n,\chi}(pt) &= L_{p,E}\left(n+1, t; \chi_n\right) \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} p^m t^m L_{p,E}(m+n+1, \chi_{m+n}) \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} E_{-(m+n),\chi} (pt)^m, \end{aligned}$$

this converges for $|pt|_p < 1$, since $|E_{-(m+n),\chi}|_p \leq \max\left\{|p|_p^{-1}, |f_\chi|_p^{-1}\right\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m}.$$

(2) If we put $t = 0$ in (2) and use definition (23), since $|mf_\chi|_p \leq |p|_p$, we get

$$\begin{aligned} &(-1)^{m-1}E_{-n,\chi}(mf_\chi) + E_{-n,\chi} \\ &\quad = \lim_{k\to\infty} \left((-1)^{m-1}E_{\phi(p^k)-n,\chi}(mf_\chi) + E_{\phi(p^k)-n,\chi}\right) \\ &\quad = 2\lim_{k\to\infty} \sum_{a=1}^{mf_\chi} (-1)^a \chi(a) a^{\phi(p^k)-n} \\ &\quad = 2 \sum_{\substack{a=1\\(a,p)=1}}^{mf_\chi} (-1)^a \chi(a) a^{-n}. \end{aligned}$$

We therefore obtain the theorem. □

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