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# WHICH WEIGHTED SHIFTS ARE FLAT ?

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Abstract. The flatness property of a unilateral weighted shifts is important to study the gaps between subnormality and hyponormality. In this paper, we first summerize the results on the flatness for some special kinds of a weighted shifts. And then, we consider the flatness property for a local-cubically hyponormal weighted shifts, which was introduced in [2]. Let $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=2}^{\infty}$ and let $W_\alpha$ be the associated weighted shift. We prove that $W_\alpha$ is a local-cubically hyponormal weighted shift $W_\alpha$ of order $\theta = \frac{\pi}{4}$ by numerical calculation.

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## 1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if there exists a normal operator $N$ on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $T = N|_{\mathcal{H}}$. To discuss gaps between hyponormality and subnormality, several classes of operators have been introduced, for example, $k$-hyponormal and weakly $k$-hyponormal operators (cf. [4]), whose definitions will be given below. An $n$-tuple $(T_1, \cdots, T_n)$ of operators on $B(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of $n$ copies of $\mathcal{H}$, where $[X, Y] = XY - YX$ for $X, Y \in B(\mathcal{H})$. An $n$-tuple $(T_1, \cdots, T_n)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n$, where $\mathbb{C}$ is the set of complex numbers. An operator $T \in B(\mathcal{H})$ is said to be *polynomially hyponormal* if $p\,(T)$ is hyponormal for all complex polynomials $p$. For a positive integer $k \geq 1$ and $T \in B(\mathcal{H})$, $T$ is *$k$-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. An operator $T \in B(\mathcal{H})$ is *weakly $k$-hyponormal* if $(T, T^2, \cdots, T^k)$ is weakly hyponormal. It is well known that subnormal $\Rightarrow$ $k$-hyponormal $\Rightarrow$

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weakly $k$-hyponormal, for every $k \geq 1$. In particular, weak 2- and weak 3-hyponormality are often referred to as quadratic- and cubic-hyponormality ([6], [7], [10], [11], [12]). In [9], the classes of semi-weakly $k$-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality. An operator $T \in B(\mathcal{H})$ is called *semi-weakly $k$-hyponormal* if $T + sT^k$ is hyponormal for all $s \in \mathbb{C}$. It is trivial that semi-weak 2-hyponormality is equivalent to weak 2-hyponormality. In particular, $T$ is said to be *completely semi-weakly hyponormal* if is semi-weakly $k$-hyponormal for all $k \geq 2$. The followings are the relations of all above operators.

| | | | | |
|---|---|---|---|---|
| subnormal | $\Rightarrow$ $\not\Leftarrow$ | poly. hypo | $\Rightarrow$ | completely semi-weakly hypo |
| $\Downarrow$ | | $\Downarrow$ | | |
| $\vdots$ | | $\vdots$ | | |
| $\Downarrow$ | | $\Downarrow$ | | |
| $n$-hypo | $\Rightarrow$ | weakly $n$-hypo | $\Rightarrow$ | semi-weakly $n$-hypo |
| $\Downarrow$ | | $\Downarrow$ | | |
| $\vdots$ | | $\vdots$ | | |
| $\Downarrow$ | | $\Downarrow$ | | |
| 3-hypo | $\Rightarrow$ | cub. hypo | $\Rightarrow$ | semi-weakly 3-hypo |
| $\Downarrow$ | | $\Downarrow$ | | $\nparallel$ |
| 2-hypo | $\Rightarrow$ | quad. hypo | $\Leftrightarrow$ | semi-weakly 2-hypo |
| $\Downarrow$ | | $\Downarrow$ | | $\Downarrow$ |
| hypo | | hypo | | hypo |

In [8], Curto-Putinar proved that there exists an operator that is polynomially hyponormal but not 2-hyponormal. This solved a long standing open problem (cf. [8]): "if $T \in B(\mathcal{H})$ is polynomially hyponormal, must $T$ be subnormal ?" Although the existence of a weighted shift which is polynomially hyponormal but not subnormal was established in [8], concrete examples of such weighted shifts have not yet been found.

Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a bounded weight sequence in the set $\mathbb{R}_+$ of the positive real numbers. The *weighted shift* $W_\alpha$ acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_j = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The weighted shifts have played a fundamental role in studying properties of weak subnormality. Indeed the *flatness* of weighted shift operators makes an important role to detect the structure of $k$-hyponormality and weak $k$-hyponormality of weighted shifts ([1], [3], [5], [6], [7], etc.).

We say that $W_\alpha$ is *flat*, if $\alpha_0 \leq \alpha_1 = \alpha_2 = \cdots$. In particular, if $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \cdots$, we say $W_\alpha$ is *completely flat*. Obviously, if $W_\alpha$ is flat, then $W_\alpha$ is subnormal. Thus, we must avoid the flatness of the weighted shift $W_\alpha$ to solve the above problem.

Suppose:

(I) $\alpha_0 = \alpha_1$;
(II) $\alpha_n = \alpha_{n+1}$, for some $n \in \mathbb{N}$.
Then we have the following well-known results on the flatness.

- Under condition (I), if $W_\alpha$ is subnormal, then $W_\alpha$ is completely flat ([15]).
- Under condition (II), if $W_\alpha$ is subnormal, then $W_\alpha$ is flat ([15]).
- Under condition (I), if $W_\alpha$ is 2-hyponormal, then $W_\alpha$ is completely flat ([5]).
- Under condition (II), if $W_\alpha$ is 2-hyponormal, then $W_\alpha$ is flat ([5]).
- Under condition (I), if $W_\alpha$ is quadratically hyponormal, then $W_\alpha$ is not flat. It is well known that the associated weighted shift $W_\alpha$ with a weight sequence $\alpha : \sqrt{2/3}, \sqrt{2/3}, \sqrt{(k+1)/(k+2)}$ $(k \geq 2)$ is quadratically hyponormal ([5]).
- Under condition (II), if $W_\alpha$ is quadratically hyponormal, then $W_\alpha$ is flat ([3]).
- Under condition (I), if $W_\alpha$ is semi-weakly 3-hyponormal, then $W_\alpha$ is not flat. It is well known that the associated weighted shift $W_\alpha$ with a weight sequence $\alpha : \sqrt{2/3}, \sqrt{2/3}, \sqrt{(k+1)/(k+2)}$ $(k \geq 2)$ is semi-weakly 3-hyponormal ([9]).
- Under condition (II), if $W_\alpha$ is semi-weakly 3-hyponormal, then $W_\alpha$ is flat ([9]).
- Under condition (I), if $W_\alpha$ is cubically hyponormal, then $W_\alpha$ is completely flat ([9]).
- Under condition (II), if $W_\alpha$ is cubically hyponormal, then $W_\alpha$ is flat (Trivial).

In [2], the authors introduced a local-cubically hyponormal weighted shift of order $\theta$ with $0 \leq \theta \leq \frac{\pi}{2}$, which is a new notion of operators between cubic hyponormality and quadratic hyponormality and showed that a local-cubically hyponormal weighted shift $W_\alpha$ with first *three* equal weights of order $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfies the flatness property ([2, Th. 4.2]). In [13, Th. 3.3], the author improved this result that a local-cubically hyponormal weighted shift $W_\alpha$ with *two* (except *first two*) equal weights of order $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfies the flatness property. *Under condition (I), if $W_\alpha$ is local-cubically hyponormal, is $W_\alpha$ completely flat or not ?* In this article, we consider a problem that suggested in [2, Prob. 3.4] as following:

**Problem 1.1.** *Let $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=2}^{\infty}$ and let $W_\alpha$ be the associated weighted shift. For arbitrary $\theta \in (0, \frac{\pi}{2})$, is it true that $W_\alpha$ is not a local-cubically hyponormal weighted shift of order $\theta$?*

The authors in [2, Prob. 3.4] showed that there exists a subinterval $J$ of $\left(0, \frac{\pi}{2}\right)$ such that, for any $\theta \in J$, $W_\alpha$ can not be local-cubically hyponormal of order $\theta$. In particular, they found $\theta = \frac{9\pi}{200} \in J$. In this paper, we prove that $W_\alpha$

is a local-cubically hyponormal weighted shift $W_\alpha$ of order $\theta=\frac{\pi}{4}$ by numerical calculation.

This paper consists of four sections. In Section 2, we introduce the local cubic hyponormal weighted shifts of order $\theta$. In Section 3 we answer Problem 1.1. In Section 4, we show the detail calculations of some pentadiagonal matrices that support our results.

## 2. The local-cubically hyponormal weighted shifts of order $\theta$

**Definition 2.1**. ([2]) Let $\alpha=\{\alpha_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers and let $W_\alpha$ be the associated weighted shift with a sequence $\alpha$. For $\theta\in[0,\frac{\pi}{2}]$, a weighted shift $W_\alpha$ is called a *local-cubically hyponormal of order* $\theta$ if $W_\alpha+s(\cos\theta)W_\alpha^2+s(\sin\theta)W_\alpha^3$ is hyponormal for all $s\in\mathbb{C}$, i.e.,

$$\left[\left(W_\alpha+s\cos\theta W_\alpha^2+s\sin\theta W_\alpha^3\right)^*,W_\alpha+s\cos\theta W_\alpha^2+s\sin\theta W_\alpha^3\right]\geq 0,\quad s\in\mathbb{C}.$$

It is easy to know that $W_\alpha$ is local-cubically hyponormal of order 0 if and only if it is quadratically hyponormal; and $W_\alpha$ is local-cubically hyponormal of order $\frac{\pi}{2}$ if and only if it is semi-cubically hyponormal.

Let $P_n$ denote the orthogonal projection onto $\vee_{i=0}^{n}\{e_i\}$. For $n\geq 0$ and $s\in\mathbb{C}$ and $\theta\in\left(0,\frac{\pi}{2}\right)$, define

$$D_n:=D_n(s,\theta)\equiv P_n\left[\left(W_\alpha+sW_\alpha^2+s\tan\theta W_\alpha^3\right)^*,W_\alpha+sW_\alpha^2+s\tan\theta W_\alpha^3\right]P_n$$

$$=\begin{pmatrix} q_0 & r_0 & z_0 & 0 & 0 & 0 & & \\ \overline{r}_0 & q_1 & r_1 & z_1 & 0 & 0 & & \\ \overline{z}_0 & \overline{r}_1 & q_2 & r_2 & z_2 & 0 & & \\ 0 & \overline{z}_1 & \overline{r}_2 & q_3 & r_3 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & \ddots & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & 0 & \overline{z}_{n-2} & \overline{r}_{n-1} & q_n \end{pmatrix}$$

where

$$\begin{cases} q_n:=\alpha_n^2-\alpha_{n-1}^2+(\alpha_n^2\alpha_{n+1}^2-\alpha_{n-2}^2\alpha_{n-1}^2)|s|^2 \\ \qquad +(\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2-\alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)|s|^2\Delta^2, \\ r_n:=\alpha_n(\alpha_{n+1}^2-\alpha_{n-1}^2)\overline{s}+\alpha_n(\alpha_{n+1}^2\alpha_{n+2}^2-\alpha_{n-1}^2\alpha_{n-2}^2)|s|^2\Delta, \\ z_n:=\alpha_n\alpha_{n+1}(\alpha_{n+2}^2-\alpha_{n-1}^2)\overline{s}\Delta, \end{cases}$$

with $\Delta=\tan\theta$ and $\alpha_{-3}=\alpha_{-2}=\alpha_{-1}=0$. It is obvious that if $W_\alpha$ is local-cubically hyponormal of order $\theta\in\left(0,\frac{\pi}{2}\right)$ if and only if $D_n(s,\Delta)\geq 0$ for every $s\in\mathbb{C}$, $\Delta\in(0,+\infty)$ and $n\geq 0$. We consider the determinant for the pentadiagonal matrix $D_n(s,\Delta)$, $d_n\equiv d_n(s,\Delta):=\det D_n(s,\Delta)$.

## 3. The local-cubically hyponormal weighted shifts of order $\theta = \frac{\pi}{4}$

For convenience, we change some notations. Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers and let $W_\alpha$ be the associated weighted shift with a sequence $\alpha$. First, we know that $W_\alpha$ is *local-cubically hyponormal of order* $\theta = \frac{\pi}{4}$ if $W_\alpha^3 + W_\alpha^2 + tW_\alpha$ is hyponormal for all $t \in \mathbb{C}$, i.e.,

$$\left[\left(W_\alpha^3 + W_\alpha^2 + tW_\alpha\right)^*, W_\alpha^3 + W_\alpha^2 + tW_\alpha\right] \geq 0, \quad t \in \mathbb{C}.$$

Let $P_n$ denote the orthogonal projection onto $\vee_{i=0}^{n}\{e_i\}$. For $n \geq 0$ and $t \in \mathbb{C}$, we define

$$M_n(t) := P_n \left[\left(W_\alpha^3 + W_\alpha^2 + tW_\alpha\right)^*, W_\alpha^3 + W_\alpha^2 + tW_\alpha\right] P_n$$
$$= \begin{pmatrix} q_0 & r_0 & z_0 & 0 & 0 & 0 & & \\ \overline{r}_0 & q_1 & r_1 & z_1 & 0 & 0 & & \\ \overline{z}_0 & \overline{r}_1 & q_2 & r_2 & z_2 & 0 & & \\ 0 & \overline{z}_1 & \overline{r}_2 & q_3 & r_3 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & \ddots & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & 0 & \overline{z}_{n-2} & \overline{r}_{n-1} & q_n \end{pmatrix}$$

where

$$\begin{cases} q_n := \left(\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2\right) \\ \qquad + \left(\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2\right) + |t|^2\left(\alpha_n^2 - \alpha_{n-1}^2\right), \\ r_n := \alpha_n\left(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2\right) + t\alpha_n\left(\alpha_{n+1}^2 - \alpha_{n-1}^2\right), \\ z_n := t\alpha_n\alpha_{n+1}\left(\alpha_{n+2}^2 - \alpha_{n-1}^2\right), \end{cases} \tag{3.1}$$

with $\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0$. It is obvious that if $W_\alpha$ is local-cubically hyponormal of order $\theta = \frac{\pi}{4}$ if and only if $M_n(t) \geq 0$ for every $t \in \mathbb{C}$ and $n \geq 0$. We consider the determinant for the pentadiagonal matrix $M_n(t)$, $d_n := \det M_n(t)$. The followings are the main results of this article.

**Theorem 3.1.** *Let* $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=2}^{\infty}$ *and let* $W_\alpha$ *be the associated weighted shift. Then* $W_\alpha$ *is local-cubically hyponormal weighted shift of order* $\theta = \frac{\pi}{4}$.

**Corollary 3.2.** *Let* $W_\alpha$ *be a weighted shift with* $\alpha_0 = \alpha_1$. *Then* $W_\alpha$ *is local-cubically hyponormal for some* $\theta$, *and not for some other* $\theta$. *That is, the flatness of local-cubic hyponormality with first two equal weights is not satisfied.*

Let $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=2}^{\infty}$. We have known that $W_\alpha$ is not local-cubic hyponormal for $\theta = \frac{9\pi}{200}$. Theorem 3.1 means the flatness is not satisfied for local-cubic hyponormality of order $\theta = \frac{\pi}{4}$. Naturally, we can consider the following problems but we leave it to interested readers.

**Problem 3.3.** *Let* $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=2}^{\infty}$ *and let* $W_\alpha$ *be the associated weighted shift. Find the interval of* $\theta$ *such that* $W_\alpha$ *is local-cubically hyponormal weighted shift of order* $\theta$.

## 4. Proof of Theorem 3.1

All of the calculations in this paper, we use the software tool *Scientific WorkPlace* [16]. Let

$$\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots .$$

By (3.1), we obtain ($s = \bar{t}$)

$$\begin{cases} q_0 = \frac{2}{3}st + \frac{7}{9}, & q_1 = \frac{9}{10}, & q_2 = \frac{1}{12}st + \frac{59}{90}, \\ q_3 = \frac{1}{20}st + \frac{17}{42}, & q_4 = \frac{1}{30}st + \frac{19}{56}, & q_5 = \frac{1}{42}st + \frac{1}{4}, \\ q_6 = \frac{1}{56}st + \frac{121}{630}, & q_7 = \frac{1}{72}st + \frac{67}{440}, & q_8 = \frac{1}{90}st + \frac{49}{396}, \end{cases}$$

$$\begin{cases} r_0 = \frac{1}{6}\sqrt{6} + \frac{2}{9}\sqrt{6}t, & r_4 = \frac{1}{40}\sqrt{30} + \frac{1}{105}\sqrt{30}t, \\ r_1 = \frac{1}{5}\sqrt{6} + \frac{1}{36}\sqrt{6}t, & r_5 = \frac{1}{63}\sqrt{42} + \frac{1}{168}\sqrt{42}t, \\ r_2 = \frac{1}{9}\sqrt{3} + \frac{1}{15}\sqrt{3}t, & r_6 = \frac{3}{140}\sqrt{14} + \frac{1}{126}\sqrt{14}t, \\ r_3 = \frac{3}{35}\sqrt{5} + \frac{1}{30}\sqrt{5}t, & r_7 = \frac{1}{22}\sqrt{2} + \frac{1}{60}\sqrt{2}t, \end{cases}$$

and

$$\begin{cases} z_0 = \frac{1}{2}t, & \\ z_1 = \frac{1}{15}\sqrt{2}t, & z_4 = \frac{3}{280}\sqrt{35}t, \\ z_2 = \frac{1}{28}\sqrt{6}t, & z_5 = \frac{1}{72}\sqrt{12}t, \\ z_3 = \frac{1}{28}\sqrt{6}t, & z_6 = \frac{1}{70}\sqrt{7}t. \end{cases}$$

Let $t = x + yi, s = x - yi$. We know that $d_n$ is as follows

$$d_n = \sum_{k=0}^{n} p_{n,k}(x)\, y^{2k}.$$

In particular, by direct calculation, we obtain

$$\begin{aligned} d_1 &= \left(\frac{41}{135}x^2 - \frac{4}{9}x + \frac{8}{15}\right) + \frac{41}{135}y^2, \\ d_2 &= p_{2,0}(x) + p_{2,1}(x)\, y^2 + + \frac{1}{45}y^4, \quad \text{with} \\ &\quad p_{2,0}(x) = \tfrac{1}{45}x^4 - \tfrac{2}{45}x^3 + \tfrac{121}{810}x^2 - \tfrac{58}{405}x + \tfrac{22}{135}, \\ &\quad p_{2,1}(x) = \tfrac{2}{45}x^2 - \tfrac{2}{45}x + \tfrac{121}{810}, \end{aligned}$$

and

$$d_3 = p_{3,0}(x) + p_{3,1}(x)\,y^2 + p_{3,2}(x)\,y^4 + \tfrac{1}{1620}y^6, \quad \text{with}$$

$$p_{3,0}(x) = \tfrac{1}{1620}x^6 - \tfrac{1}{810}x^5 + \tfrac{59}{4536}x^4 - \tfrac{193}{5670}x^3 + \tfrac{2323}{34020}x^2 - \tfrac{437}{8505}x + \tfrac{131}{2835},$$

$$p_{3,1}(x) = \tfrac{1}{540}x^4 - \tfrac{1}{405}x^3 + \tfrac{59}{2268}x^2 - \tfrac{193}{5670}x + \tfrac{1651}{34020},$$

$$p_{3,2}(x) = \tfrac{1}{540}x^2 - \tfrac{1}{810}x + \tfrac{59}{4536},$$

and

$$d_4 = p_{4,0}(x) + p_{4,1}(x)\,y^2 + p_{4,2}(x)\,y^4 + p_{4,3}(x)\,y^6 + \tfrac{1}{48\,600}y^8, \quad \text{with}$$

$$p_{4,0}(x) = \tfrac{1}{48600}x^8 - \tfrac{1}{24300}x^7 + \tfrac{11}{38880}x^6 - \tfrac{821}{680400}x^5 + \tfrac{5287}{907200}x^4 - \tfrac{8299}{680400}x^3 + \tfrac{24\,251}{1360800}x^2 - \tfrac{571}{48600}x + \tfrac{157}{16200},$$

$$p_{4,1}(x) = \tfrac{1}{12150}x^6 - \tfrac{1}{8100}x^5 + \tfrac{11}{12960}x^4 - \tfrac{821}{340200}x^3 + \tfrac{2083}{194400}x^2 - \tfrac{8299}{680400}x + \tfrac{2189}{194400},$$

$$p_{4,2}(x) = \tfrac{1}{8100}x^4 - \tfrac{1}{8100}x^3 + \tfrac{11}{12960}x^2 - \tfrac{821}{680400}x + \tfrac{13\,301}{2721600},$$

$$p_{4,3}(x) = \tfrac{1}{12150}x^2 - \tfrac{1}{24300}x + \tfrac{11}{38880},$$

$$d_5 = p_{5,0}(x) + p_{5,1}(x)\,y^2 + p_{5,2}(x)\,y^4 + p_{5,3}(x)\,y^6 + p_{5,4}(x)\,y^8 + \tfrac{1}{2041200}y^{10},$$

with

$$p_{5,0}(x) = \tfrac{1}{2041200}x^{10} - \tfrac{1}{1020600}x^9 + \tfrac{37}{8164800}x^8 - \tfrac{53}{4082400}x^7 + \tfrac{271}{1814400}x^6 - \tfrac{521}{816480}x^5 + \tfrac{4919}{2721600}x^4 - \tfrac{127}{45360}x^3 + \tfrac{1241}{388800}x^2 - \tfrac{1283}{680400}x + \tfrac{353}{226800},$$

$$p_{5,1}(x) = \tfrac{1}{408240}x^8 - \tfrac{1}{255150}x^7 + \tfrac{37}{2041200}x^6 - \tfrac{53}{1360800}x^5 + \tfrac{7157}{16329600}x^4 - \tfrac{521}{408240}x^3 + \tfrac{257}{90720}x^2 - \tfrac{571}{226800}x + \tfrac{5023}{2721600},$$

$$p_{5,2}(x) = \tfrac{1}{204120}x^6 - \tfrac{1}{170100}x^5 + \tfrac{37}{1360800}x^4 - \tfrac{53}{1360800}x^3 + \tfrac{6997}{16329600}x^2 - \tfrac{521}{816480}x + \tfrac{2791}{2721600},$$

$$p_{5,3}(x) = \tfrac{1}{204120}x^4 - \tfrac{1}{255150}x^3 + \tfrac{37}{2041200}x^2 - \tfrac{53}{4082400}x + \tfrac{2279}{16329600},$$

$$p_{5,4}(x) = \tfrac{1}{408240}x^2 - \tfrac{1}{1020600}x + \tfrac{37}{8164800},$$

$$d_6 = p_{6,0}(x) + p_{6,1}(x)y^2 + p_{6,2}(x)y^4 + p_{6,3}(x)y^6 + p_{6,4}(x)y^8 + p_{6,5}(x)y^{10}$$

$$+\frac{1}{114307200}y^{12},$$

with

$$p_{6,0}(x) = \frac{1}{114307200}x^{12} - \frac{1}{57153600}x^{11} + \frac{439}{5143824000}x^{10} - \frac{317}{1285956000}x^9 + \frac{1387}{857304000}x^8$$
$$- \frac{23\,893}{2571912000}x^7 + \frac{554\,809}{10287648000}x^6 - \frac{87\,091}{514382400}x^5 + \frac{84\,737}{244944000}x^4$$
$$- \frac{110\,879}{257191200}x^3 + \frac{2158\,621}{5143824000}x^2 - \frac{298\,649}{1285956000}x + \frac{84\,179}{428652000},$$

$$p_{6,1}(x) = \frac{1}{19051200}x^{10} - \frac{1}{11430720}x^9 + \frac{439}{1028764800}x^8 - \frac{317}{321489000}x^7 + \frac{449}{71442000}x^6$$
$$- \frac{23\,893}{857304000}x^5 + \frac{1501\,547}{10287648000}x^4 - \frac{85\,171}{257191200}x^3$$
$$+ \frac{51\,061}{102876480}x^2 - \frac{9451}{26244000}x + \frac{1178\,029}{5143824000},$$

$$p_{6,2}(x) = \frac{1}{7620480}x^8 - \frac{1}{5715360}x^7 + \frac{439}{514382400}x^6 - \frac{317}{214326000}x^5 + \frac{1307}{142884000}x^4$$
$$- \frac{23\,893}{857304000}x^3 + \frac{1338\,667}{10287648000}x^2 - \frac{1699}{10497600}x + \frac{773\,573}{5143824000},$$

$$p_{6,3}(x) = \frac{1}{5715360}x^6 - \frac{1}{5715360}x^5 + \frac{439}{514382400}x^4 - \frac{317}{321489000}x^3$$
$$+ \frac{181}{30618000}x^2 - \frac{23\,893}{2571912000}x + \frac{130\,643}{3429216000},$$

$$p_{6,4}(x) = \frac{1}{7620480}x^4 - \frac{1}{11430720}x^3 + \frac{439}{1028764800}x^2 - \frac{317}{1285956000}x + \frac{409}{285768000},$$

$$p_{6,5}(x) = \frac{1}{19051200}x^2 - \frac{1}{57153600}x + \frac{439}{5143824000},$$

$$d_7 = p_{7,0}(x) + p_{7,1}(x)y^2 + p_{7,2}(x)y^4 + p_{7,3}(x)y^6 + p_{7,4}(x)y^8 + p_{7,5}(x)y^{10}$$

$$+p_{7,6}(x)y^{12} + \frac{1}{8230118400}y^{14},$$ with

$$p_{7,0}(x) = \frac{1}{8230118400}x^{14} - \frac{1}{4115059200}x^{13} + \frac{559}{452656512000}x^{12} - \frac{29}{808152000}x^{11}$$
$$+ \frac{557}{28291032000}x^{10} - \frac{5653}{75442752000}x^9 + \frac{180\,601}{301771008000}x^8 - \frac{753\,971}{226328256000}x^7$$
$$+ \frac{1420\,369}{113164128000}x^6 - \frac{821\,123}{28291032000}x^5 + \frac{6929\,801}{150885504000}x^4 - \frac{5485\,391}{113164128000}x^3$$
$$+ \frac{19\,338\,533}{452656512000}x^2 - \frac{2598\,433}{113164128000}x + \frac{750\,571}{37721376000},$$

$$p_{7,1}(x) = \frac{1}{1175731200}x^{12} - \frac{1}{685843200}x^{11} + \frac{559}{75442752000}x^{10} - \frac{29}{1616630400}x^9$$
$$+ \frac{677}{7072758000}x^8 - \frac{5653}{18860688000}x^7 + \frac{172\,561}{75442752000}x^6 - \frac{750\,451}{75442752000}x^5$$

$$+\frac{3512\,819}{113164128000}x^{4}-\frac{748\,679}{14145516000}x^{3}+\frac{13\,781\,147}{226328256000}x^{2}-\frac{4331\,087}{113164128000}x$$
$$+\frac{1467\,491}{64665216000},$$

$$p_{7,2}(x)=\frac{1}{391910400}x^{10}-\frac{1}{274337280}x^{9}+\frac{559}{30177100800}x^{8}-\frac{29}{808315200}x^{7}+\frac{877}{4715172000}x^{6}$$
$$-\frac{5653}{12573792000}x^{5}+\frac{23\,503}{7185024000}x^{4}-\frac{248\,977}{25147584000}x^{3}$$
$$+\frac{5129}{209952000}x^{2}-\frac{19\,321}{808315200}x+\frac{7381\,147}{452656512000},$$

$$p_{7,3}(x)=\frac{1}{235146240}x^{8}-\frac{1}{205752960}x^{7}+\frac{559}{22632825600}x^{6}-\frac{29}{808315200}x^{5}+\frac{1277}{7072758000}x^{4}$$
$$-\frac{5653}{18860688000}x^{3}+\frac{156\,481}{75442752000}x^{2}-\frac{743\,411}{226328256000}x+\frac{224\,027}{37721376000},$$

$$p_{7,4}(x)=\frac{1}{235146240}x^{6}-\frac{1}{274337280}x^{5}+\frac{559}{30177100800}x^{4}-\frac{29}{1616630400}x^{3}$$
$$+\frac{2477}{28291032000}x^{2}-\frac{5653}{75442752000}x+\frac{148\,441}{301771008000},$$

$$p_{7,5}(x)=\frac{1}{391910400}x^{4}-\frac{1}{685843200}x^{3}+\frac{559}{75442752000}x^{2}-\frac{29}{8083152000}x+\frac{1}{58939650},$$

$$p_{7,6}(x)=\frac{1}{1175731200}x^{2}-\frac{1}{4115059200}x+\frac{559}{452656512000}.$$

Let $m_{n,k}=\min p_{n,k}(x)$. Then

$$d_n=\sum_{k=0}^{n}p_{n,k}(x)\,y^{2k}\geq\sum_{k=0}^{n}m_{n,k}y^{2k}.$$

Numerically, we can obtain some of the values $m_{n,k}$ as following

| $k\rightarrow$ $n\downarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | $\frac{76}{205}$ | $\frac{41}{135}$ | $\diagdown$ | $\diagdown$ | $\diagdown$ | $\diagdown$ | $\diagdown$ |
| 2 | 0.12398 | $\frac{56}{405}$ | $\frac{1}{45}$ | $\diagdown$ | $\diagdown$ | $\diagdown$ | $\diagdown$ |
| 3 | $3.4077\times10^{-2}$ | $3.7032\times10^{-2}$ | $\frac{871}{68\,040}$ | $\frac{1}{1620}$ | $\diagdown$ | $\diagdown$ | $\diagdown$ |
| 4 | $7.0696\times10^{-3}$ | $7.3336\times10^{-3}$ | $4.4455\times10^{-3}$ | $\frac{1}{3600}$ | $\frac{1}{48600}$ | $\diagdown$ | $\diagdown$ |
| 5 | $1.1559\times10^{-3}$ | $1.1305\times10^{-3}$ | $7.7941\times10^{-4}$ | $1.3713\times10^{-4}$ | $\frac{181}{40824000}$ | $\frac{1}{2041200}$ | $\diagdown$ |
| 6 | $1.483\times10^{-4}$ | $1.3692\times10^{-4}$ | $9.3125\times10^{-5}$ | $3.4282\times10^{-5}$ | $1.3944\times10^{-6}$ | $\frac{863}{10287648000}$ | $\frac{1}{114307200}$ |

Since $m_{n,k} > 0$ for all $n, k$, we know that $d_n > 0$ for all $n \in \mathbb{N}$. Hence, $W_\alpha$ is local-cubically hyponormal weighted shift of order $\theta = \frac{\pi}{4}$.

*Remark.* We have known that

$$d_n = \sum_{k=0}^{n} p_{n,k}(x)\, y^{2k}, \tag{4.1}$$

where

$$p_{n,k}(x) = \sum_{i=0}^{2n-2k} (-1)^i a_i x^{2n-2k-i}, \quad \text{with } a_i > 0, \text{ for } 0 \le i \le 2(n-k). \tag{4.2}$$

Without loss of generality, we consider the polynomial as following

$$q_n(x) = a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-2} - \cdots - a_{2n-1} x + a_{2n}, \tag{4.3}$$

with $a_i > 0$, for all $0 \le i \le 2n$.

1. For $n = 1$, since

$$q_1(x) = a_0 x^2 - a_1 x + a_2 = a_0 \left( x - \frac{a_1}{2a_0} \right)^2 + a_2 - \frac{a_1^2}{4a_0},$$

we have $q_1(x) > 0$ for all $x \in \mathbb{R}$ if and only if $\Delta_1 := a_2 - \frac{a_1^2}{4a_0} > 0$.

2. For $n = 2$, since

$$\begin{aligned} q_2(x) &= a_0 x^4 - a_1 x^3 + a_2 x^2 - a_3 x + a_4 \\ &= x^2 \left( a_0 x^2 - a_1 x + a_2 \right) - a_3 x + a_4 \\ &= x^2 \left( a_0 \left( x - \frac{a_1}{2a_0} \right)^2 \right) + \Delta_1 x^2 - a_3 x + a_4 \\ &= x^2 \left( a_0 \left( x - \frac{a_1}{2a_0} \right)^2 \right) + \Delta_1 \left( x - \frac{a_3}{2\Delta_1} \right)^2 + a_4 - \frac{a_3^2}{4\Delta_1}, \end{aligned}$$

we know that if $\Delta_2 := a_4 - \frac{a_3^2}{4\Delta_1} > 0$, then $q_2(x) > 0$ for all $x \in \mathbb{R}$.

Thus, in general, we have obtain a sufficient condition of positivity for $q_n(x)$ as in (4.3).

**Theorem 4.1.** *If* $\Delta_k := a_{2k} - \frac{a_{2k-1}^2}{4\Delta_{k-1}} > 0$ *for* $k = 1, 2, ..., n$, *then* $q_n(x) > 0$ *for all* $x \in \mathbb{R}$.

By Theorem 4.1, we can prove the positivity of $p_{n,k}(x)$ as shown in (4.2). For example,

$$d_3 = p_{3,0}(x) + p_{3,1}(x)\,y^2 + p_{3,2}(x)\,y^4 + \tfrac{1}{1620}y^6, \quad \text{with}$$

$$p_{3,0}(x) = \tfrac{1}{1620}x^6 - \tfrac{1}{810}x^5 + \tfrac{59}{4536}x^4 - \tfrac{193}{5670}x^3 + \tfrac{2323}{34020}x^2 - \tfrac{437}{8505}x + \tfrac{131}{2835},$$

$$p_{3,1}(x) = \tfrac{1}{540}x^4 - \tfrac{1}{405}x^3 + \tfrac{59}{2268}x^2 - \tfrac{193}{5670}x + \tfrac{1651}{34020},$$

$$p_{3,2}(x) = \tfrac{1}{540}x^2 - \tfrac{1}{810}x + \tfrac{59}{4536}.$$

(i) Define $a_0 = \frac{1}{540}, a_1 = \frac{1}{810}, a_2 = \frac{59}{4536}$. Then $\Delta_1 = a_2 - \frac{a_1^2}{4a_0} = \frac{871}{68040} > 0$. Thus

$$p_{3,2}(x) = \frac{1}{540}x^2 - \frac{1}{810}x + \frac{59}{4536} > 0.$$

(ii) Define $a_0 = \frac{1}{540}, a_1 = \frac{1}{405}, a_2 = \frac{59}{2268}$, then $\Delta_1 = \frac{871}{68040} > 0$. And define $\Delta_1 = \frac{871}{68040}, a_3 = \frac{193}{5670}, a_4 = \frac{1651}{34020}$, then $\Delta_2 = \frac{767539}{29631420} > 0$. Thus,

$$p_{3,1}(x) = \frac{1}{540}x^4 - \frac{1}{405}x^3 + \frac{59}{2268}x^2 - \frac{193}{5670}x + \frac{1651}{34020} > 0.$$

(iii) Define $a_0 = \frac{1}{1620}, a_1 = \frac{1}{810}, a_2 = \frac{59}{4536}$, then $\Delta_1 = \frac{281}{22680} > 0$. And define $\Delta_1 = \frac{281}{22680}, a_3 = \frac{193}{5670}, a_4 = \frac{2323}{34020}$, then $\Delta_2 = \frac{429269}{9559620} > 0$. And define $\Delta_2 = \frac{429269}{9559620}, a_5 = \frac{437}{8505}, a_6 = \frac{131}{2835}$, then $\Delta_3 = \frac{115040428}{3650932845} > 0$. Thus,

$$p_{3,0}(x) = \frac{1}{1620}x^6 - \frac{1}{810}x^5 + \frac{59}{4536}x^4 - \frac{193}{5670}x^3 + \frac{2323}{34020}x^2 - \frac{437}{8505}x + \frac{131}{2835} > 0.$$

Finally, we know that $d_3 > 0$. Similarly, we can show that $d_n > 0$, for any $n \in \mathbb{N}$.

**Conclusion.** This paper summerized the flatness of $k$-hyponormal or weakly $k$-hyponormal weighted shifts, discussed the local-cubic hyponormality for Bergman shift operator, and showed that Bergman shift with first two equal weights is local-cubic hyponormal for $\theta = \frac{\pi}{4}$, which means the flatness is not always satisfied for local-cubic hyponormal weighted shifts.

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