

ITERATED LEFT ABSTRACT FRACTIONAL LANDAU INEQUALITIES

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ABSTRACT. We present uniform and L_p left Caputo-Bochner abstract iterated fractional Landau inequalities over \mathbb{R}_+ . These estimate the size of second and third iterated left abstract fractional derivatives of a Banach space valued function over \mathbb{R}_+ . We give an application when the basic fractional order is $\frac{1}{2}$.

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1. Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f , such that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \quad (1)$$

where $\|\cdot\|_{p,I}$ is the p -norm on the interval I , see [1], [5].

The research on these inequalities started by E. Landau [10] in 1913. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \quad \text{and} \quad C_\infty(\mathbb{R}) = \sqrt{2}, \quad (2)$$

are the best constants in (1).

In 1932, G. H. Hardy and J. E. Littlewood [7] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \quad \text{and} \quad C_2(\mathbb{R}) = 1. \quad (3)$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [8] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \quad \text{for } p \in [1, \infty), \quad (4)$$

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which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$.

In fact, in [6] and [9] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

We need the following concept from abstract fractional calculus.

Our integral next is of Bochner type [11].

We need

Definition 1.1. ([4], p. 150) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a, b] \rightarrow X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo-Bochner left fractional derivative of order α :

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m - \alpha)} \int_a^x (x - t)^{m - \alpha - 1} f^{(m)}(t) dt, \quad \forall x \in [a, b], \quad (5)$$

where Γ is the gamma function.

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary X -valued derivative (defined similar to numerical one, see [12], p. 83), and also set $D_{*a}^0 f := f$.

By ([4], p. 2), $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^\alpha f \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_\infty([a, b], X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by ([4], p. 3), $D_{*a}^\alpha f \in C([a, b], X)$, hence $\|D_{*a}^\alpha f\| \in C([a, b])$.

We mention the important:

Corollary 1.2. ([4], p. 157) Let $f \in C^m([a, b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a, b]$. Then $D_{*x_0}^\alpha f(x)$ is jointly continuous functions in (x, x_0) from $[a, b]^2$ into X , X is a Banach space.

By convention we suppose that

$$D_{*x_0}^\alpha f(x) = 0, \quad \text{for } x < x_0,$$

for all $x, x_0 \in [a, b]$.

The author has already done an extensive body of work on fractional Landau inequalities, see [3], and on abstract fractional Landau inequalities, see [4]. However there the proving methods came out of applications of fractional Ostrowski inequalities ([2], [4]). Usually there the domains where $[A, +\infty)$ or $(-\infty, B]$, with $A, B \in \mathbb{R}$ and in one mixed case the domain was all of \mathbb{R} .

In this work with less assumptions we establish uniform and L_p type left Caputo-Bochner abstract iterated fractional Landau inequalities over \mathbb{R}_+ . The method of proving is based on left Caputo-Bochner iterated fractional Taylor's formula with integral remainder, see [4], pp. 129.

We give also an application for $\alpha = \frac{1}{2}$. Regardless to say that we are also inspired by [3], [4].

2. Main results

We consider ($\alpha > 0$) the composition $D_{*a}^{n\alpha} := D_{*a}^\alpha D_{*a}^\alpha \dots D_{*a}^\alpha$ (n -times), $n \in \mathbb{N}$.

We mention the following left modified X -valued Taylor's formula, $(X, \|\cdot\|)$ is a Banach space.

Theorem 2.1. ([4], p. 129) Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a, b], X)$. For $k = 1, \dots, n$, we assume that $D_{*a}^{k\alpha} f \in C^1([a, b], X)$ and $D_{*a}^{(n+1)\alpha} f \in C([a, b], X)$. Then

$$f(x) = \sum_{i=0}^n \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{*a}^{i\alpha} f)(a) + \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} (D_{*a}^{(n+1)\alpha} f)(t) dt, \tag{6}$$

$\forall x \in [a, b]$.

When $0 < \alpha < 1$ and $f \in C^1([a, b], X)$, by [4], p. 130, we get that $(D_{*a}^\alpha f)(a) = 0$.

We present the following abstract left fractional Landau inequalities over \mathbb{R}_+ .

Theorem 2.2. Let $0 < \alpha < 1$, $f \in C^1(\mathbb{R}_+, X)$ with $\|f\|_{\infty, \mathbb{R}_+}, \|f'\|_{\infty, \mathbb{R}_+} < \infty$. For $k = 1, 2, 3$, we assume that $D_{*a}^{k\alpha} f \in C^1([a, \infty), X)$ and $D_{*a}^{4\alpha} f \in C([a, \infty), X)$, $\forall a \in \mathbb{R}_+$. We further assume that

$$K := \left\| \|D_{*a}^{4\alpha} f(t)\| \right\|_{\infty, \mathbb{R}_+^2} < \infty, \tag{7}$$

where $(a, t) \in \mathbb{R}_+^2$.

Then

$$\sup_{a \in \mathbb{R}_+} \|(D_{*a}^{2\alpha} f)(a)\| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}(2^\alpha-1)} \sqrt{\frac{2^{3\alpha+1}(2^{3\alpha}+1)(2^\alpha+1)}{\Gamma(4\alpha+1)}} \|f\|_{\infty, \mathbb{R}_+} K, \tag{8}$$

and

$$\sup_{a \in \mathbb{R}_+} \|(D_{*a}^{3\alpha} f)(a)\| \leq \frac{4\sqrt[4]{2}\Gamma(3\alpha+1)(\Gamma(4\alpha+1))^{-\frac{3}{4}}(2^{2\alpha}+1)}{(\sqrt[4]{3})^3(\sqrt{2})^\alpha(2^\alpha-1)} \|f\|_{\infty, \mathbb{R}_+}^{\frac{1}{4}} K^{\frac{3}{4}}. \tag{9}$$

That is $\sup_{a \in \mathbb{R}_+} \|(D_{*a}^{2\alpha} f)(a)\|, \sup_{a \in \mathbb{R}_+} \|(D_{*a}^{3\alpha} f)(a)\| < \infty$.

Proof. We notice again here that $(D_{*a}^\alpha f)(a) = 0, \forall a \in \mathbb{R}_+$.

We make use of Theorem 2.1 for $0 < \alpha < 1$ and $n = 3$, applied for any $a \in \mathbb{R}_+$ and $b = \infty$.

Momentarily we fix $a \in \mathbb{R}_+$. Let $x_2 > x_1 > a$, then

$$f(x_1) - f(a) = \frac{(x_1-a)^{2\alpha}}{\Gamma(2\alpha+1)} (D_{*a}^{2\alpha} f)(a) + \frac{(x_1-a)^{3\alpha}}{\Gamma(3\alpha+1)} (D_{*a}^{3\alpha} f)(a) + \frac{1}{\Gamma(4\alpha)} \int_a^{x_1} (x_1-t)^{4\alpha-1} (D_{*a}^{4\alpha} f)(t) dt, \tag{10}$$

and

$$f(x_2) - f(a) = \frac{(x_2-a)^{2\alpha}}{\Gamma(2\alpha+1)} (D_{*a}^{2\alpha} f)(a) + \frac{(x_2-a)^{3\alpha}}{\Gamma(3\alpha+1)} (D_{*a}^{3\alpha} f)(a) + \tag{11}$$

$$\frac{1}{\Gamma(4\alpha)} \int_a^{x_2} (x_2 - t)^{4\alpha-1} (D_{*a}^{4\alpha} f)(t) dt.$$

That is

$$\frac{(x_1 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} (D_{*a}^{2\alpha} f)(a) + \frac{(x_1 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} (D_{*a}^{3\alpha} f)(a) = \quad (12)$$

$$f(x_1) - f(a) - \frac{1}{\Gamma(4\alpha)} \int_a^{x_1} (x_1 - t)^{4\alpha-1} (D_{*a}^{4\alpha} f)(t) dt =: A,$$

and

$$\frac{(x_2 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} (D_{*a}^{2\alpha} f)(a) + \frac{(x_2 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} (D_{*a}^{3\alpha} f)(a) = \quad (13)$$

$$f(x_2) - f(a) - \frac{1}{\Gamma(4\alpha)} \int_a^{x_2} (x_2 - t)^{4\alpha-1} (D_{*a}^{4\alpha} f)(t) dt =: B.$$

We are solving the above system of two equations with two unknowns $(D_{*a}^{2\alpha} f)(a)$, $(D_{*a}^{3\alpha} f)(a)$.

The main determinant of system is

$$D := \begin{vmatrix} \frac{(x_1 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} & \frac{(x_1 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} \\ \frac{(x_2 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} & \frac{(x_2 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} \end{vmatrix} =$$

$$\frac{1}{\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} \left[(x_1 - a)^{2\alpha} (x_2 - a)^{3\alpha} - (x_1 - a)^{3\alpha} (x_2 - a)^{2\alpha} \right] =$$

$$\frac{(x_1 - a)^{2\alpha} (x_2 - a)^{2\alpha}}{\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} [(x_2 - a)^\alpha - (x_1 - a)^\alpha] > 0.$$

I.e.

$$D = \frac{(x_1 - a)^{2\alpha} (x_2 - a)^{2\alpha}}{\Gamma(2\alpha + 1)\Gamma(3\alpha + 1)} [(x_2 - a)^\alpha - (x_1 - a)^\alpha] > 0. \quad (14)$$

We obtain the unique solution

$$(D_{*a}^{2\alpha} f)(a) = \frac{\begin{vmatrix} A & \frac{(x_1 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} \\ B & \frac{(x_2 - a)^{3\alpha}}{\Gamma(3\alpha + 1)} \end{vmatrix}}{D}, \quad (15)$$

$$(D_{*a}^{3\alpha} f)(a) = \frac{\begin{vmatrix} \frac{(x_1 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} & A \\ \frac{(x_2 - a)^{2\alpha}}{\Gamma(2\alpha + 1)} & B \end{vmatrix}}{D}.$$

Therefore we have

$$\begin{aligned} (D_{*a}^{2\alpha} f)(a) &= \frac{(x_2-a)^{3\alpha} A - (x_1-a)^{3\alpha} B}{D}, \\ \text{and} \\ (D_{*a}^{3\alpha} f)(a) &= \frac{(x_1-a)^{2\alpha} B - (x_2-a)^{2\alpha} A}{D}. \end{aligned} \tag{16}$$

We have the following

$$\begin{aligned} \|A\| &= \left\| f(x_1) - f(a) - \frac{1}{\Gamma(4\alpha)} \int_a^{x_1} (x_1-t)^{4\alpha-1} (D_{*a}^{4\alpha} f)(t) dt \right\| \leq \\ & 2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{\| \|D_{*a}^{4\alpha} f\| \|_{\infty, \mathbb{R}_+^2}}{\Gamma(4\alpha+1)} (x_1-a)^{4\alpha}, \end{aligned} \tag{17}$$

under the assumption $\| \|f\| \|_{\infty, \mathbb{R}_+} < \infty$.

That is

$$\|A\| \leq 2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_1-a)^{4\alpha}, \tag{18}$$

and similarly,

$$\|B\| \leq 2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_2-a)^{4\alpha}, \tag{19}$$

where by assumption

$$K := \| \|D_{*a}^{4\alpha} f\| \|_{\infty, \mathbb{R}_+^2} < \infty, \tag{20}$$

with $(a, t) \in \mathbb{R}_+^2$.

Consequently we have

$$\begin{aligned} & \| (D_{*a}^{2\alpha} f)(a) \| \\ & \leq \frac{1}{\Gamma(3\alpha+1)D} \left[(x_2-a)^{3\alpha} \left(2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_1-a)^{4\alpha} \right) \right. \\ & \quad \left. + (x_1-a)^{3\alpha} \left(2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_2-a)^{4\alpha} \right) \right], \end{aligned} \tag{21}$$

and

$$\begin{aligned} & \| (D_{*a}^{3\alpha} f)(a) \| \\ & \leq \frac{1}{\Gamma(2\alpha+1)D} \left[(x_1-a)^{2\alpha} \left(2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_2-a)^{4\alpha} \right) \right. \\ & \quad \left. + (x_2-a)^{2\alpha} \left(2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} (x_1-a)^{4\alpha} \right) \right]. \end{aligned} \tag{22}$$

Set now $x_1 := a + h, x_2 := a + 2h$, where $h > 0$, so that $x_1 - a = h, x_2 - a = 2h$.

Hence we get

$$D = \frac{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} > 0. \tag{23}$$

Therefore we derive (from (16))

$$\| (D_{*a}^{2\alpha} f)(a) \| \leq \frac{\Gamma(2\alpha+1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} \left[2^{3\alpha} h^{3\alpha} \left(2 \| \|f\| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} h^{4\alpha} \right) \right]$$

$$+h^{3\alpha} \left(2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} 2^{4\alpha} h^{4\alpha} \right) \Big] = \quad (24)$$

$$\begin{aligned} & \frac{\Gamma(2\alpha+1)}{2^{2\alpha} h^{5\alpha} (2\alpha-1)} \left[2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} (2^{3\alpha}+1) h^{3\alpha} + \frac{K}{\Gamma(4\alpha+1)} (2^{3\alpha}+2^{4\alpha}) h^{7\alpha} \right] \\ &= \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2\alpha-1)} \right) \left[\frac{2 (2^{3\alpha}+1) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{2^{3\alpha} (2\alpha+1)}{\Gamma(4\alpha+1)} K h^{2\alpha} \right]. \end{aligned} \quad (25)$$

That is

$$\begin{aligned} \| (D_{*a}^{2\alpha} f)(a) \| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2\alpha-1)} \right) \\ &\left[\frac{2 (2^{3\alpha}+1) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{2^{3\alpha} (2\alpha+1)}{\Gamma(4\alpha+1)} K h^{2\alpha} \right], \end{aligned} \quad (26)$$

$\forall a \in \mathbb{R}_+, \forall h > 0$.

I.e. it holds

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \| (D_{*a}^{2\alpha} f)(a) \| &\leq \left(\frac{\Gamma(2\alpha+1)}{2^{2\alpha} (2\alpha-1)} \right) \\ &\left[\frac{2 (2^{3\alpha}+1) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{2^{3\alpha} (2\alpha+1)}{\Gamma(4\alpha+1)} K h^{2\alpha} \right] < \infty, \end{aligned} \quad (27)$$

$\forall h > 0, 0 < \alpha < 1$.

By (16) we derive

$$\begin{aligned} \| (D_{*a}^{3\alpha} f)(a) \| &\leq \frac{\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2\alpha-1)} \left[h^{2\alpha} \left(2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} 2^{4\alpha} h^{4\alpha} \right) \right. \\ &\quad \left. + 2^{2\alpha} h^{2\alpha} \left(2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{K}{\Gamma(4\alpha+1)} h^{4\alpha} \right) \right] = \\ &\frac{\Gamma(3\alpha+1)}{2^{2\alpha} h^{5\alpha} (2\alpha-1)} \left[2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} (2^{2\alpha}+1) h^{2\alpha} + \frac{K}{\Gamma(4\alpha+1)} (2^{4\alpha}+2^{2\alpha}) h^{6\alpha} \right] \\ &= \left(\frac{\Gamma(3\alpha+1)}{2^{2\alpha} (2\alpha-1)} \right) \left[\frac{2 (2^{2\alpha}+1) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{2^{2\alpha} (2^{2\alpha}+1)}{\Gamma(4\alpha+1)} K h^\alpha \right] \\ &= \frac{\Gamma(3\alpha+1) (2^{2\alpha}+1)}{2^{2\alpha} (2\alpha-1)} \left[\frac{2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{2^{2\alpha} K}{\Gamma(4\alpha+1)} h^\alpha \right]. \end{aligned} \quad (28)$$

That is

$$\begin{aligned} \| (D_{*a}^{3\alpha} f)(a) \| &\leq \frac{\Gamma(3\alpha+1) (2^{2\alpha}+1)}{2^{2\alpha} (2\alpha-1)} \\ &\left[\frac{2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{2^{2\alpha} K}{\Gamma(4\alpha+1)} h^\alpha \right], \end{aligned} \quad (29)$$

$\forall a \in \mathbb{R}_+, \forall h > 0$.

I.e. it holds

$$\sup_{a \in \mathbb{R}_+} \|(D_{*a}^{3\alpha} f)(a)\| \leq \frac{\Gamma(3\alpha + 1)(2^{2\alpha} + 1)}{2^{2\alpha}(2^\alpha - 1)} \left[\frac{2 \| \|f\| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{2^{2\alpha} K}{\Gamma(4\alpha + 1) h^\alpha} \right] < \infty, \tag{30}$$

$\forall h > 0, 0 < \alpha < 1.$

Call

$$\mu := 2(2^{3\alpha} + 1) \| \|f\| \|_{\infty, \mathbb{R}_+}, \tag{31}$$

$$\theta = \frac{2^{3\alpha}(2^\alpha + 1)K}{\Gamma(4\alpha + 1)}, \tag{32}$$

both are greater than zero.

Set also $\rho := 2\alpha; 0 < \rho < 2.$ We consider the function

$$y(h) := \mu h^{-\rho} + \theta h^\rho, \quad \forall h > 0. \tag{33}$$

We have

$$y'(h) = -\rho \mu h^{-\rho-1} + \rho \theta h^{\rho-1} = 0, \tag{34}$$

then

$$\theta h^{\rho-1} = \mu h^{-\rho-1}$$

and

$$\theta h^{2\rho} = \mu,$$

with a unique solution

$$h_0 := h_{crit.no.} = \left(\frac{\mu}{\theta} \right)^{\frac{1}{2\rho}}. \tag{35}$$

We have that

$$y''(h) = \rho(\rho + 1) \mu h^{-\rho-2} + \rho(\rho - 1) \theta h^{\rho-2}. \tag{36}$$

We see that

$$\begin{aligned} y''(h_0) &= y'' \left(\left(\frac{\mu}{\theta} \right)^{\frac{1}{2\rho}} \right) = \rho(\rho + 1) \mu \left(\frac{\mu}{\theta} \right)^{-\frac{\rho-2}{2\rho}} + \rho(\rho - 1) \theta \left(\frac{\mu}{\theta} \right)^{\frac{\rho-2}{2\rho}} = \\ &= \rho \left(\frac{\mu}{\theta} \right)^{-\frac{1}{\rho}} \left[(\rho + 1) \mu \left(\frac{\mu}{\theta} \right)^{-\frac{1}{2}} + (\rho - 1) \theta \left(\frac{\mu}{\theta} \right)^{\frac{1}{2}} \right] = \\ &= \rho \left(\frac{\theta}{\mu} \right)^{\frac{1}{\rho}} \left[(\rho + 1) \sqrt{\mu\theta} + (\rho - 1) \sqrt{\mu\theta} \right] = \\ &= \rho \left(\frac{\theta}{\mu} \right)^{\frac{1}{\rho}} (2\rho \sqrt{\mu\theta}) = 2\rho^2 \sqrt{\mu\theta} \left(\frac{\theta}{\mu} \right)^{\frac{1}{\rho}} > 0. \end{aligned}$$

Therefore y has a global minimum at $h_0 = \left(\frac{\mu}{\theta} \right)^{\frac{1}{2\rho}}$, which is

$$y(h_0) = \mu \left(\frac{\mu}{\theta} \right)^{-\frac{1}{2}} + \theta \left(\frac{\mu}{\theta} \right)^{\frac{1}{2}} = \mu \left(\frac{\theta}{\mu} \right)^{\frac{1}{2}} + \sqrt{\theta\mu} = 2\sqrt{\theta\mu}.$$

We have proved that (see (27))

$$\sup_{a \in \mathbb{R}_+} \|(D_{*a}^{2\alpha} f)(a)\| \leq \frac{\Gamma(2\alpha + 1)}{2^{2\alpha-1}(2^\alpha - 1)} \sqrt{\frac{2^{3\alpha+1}(2^{3\alpha} + 1)(2^\alpha + 1)}{\Gamma(4\alpha + 1)}} \|f\|_{\infty, \mathbb{R}_+} K. \quad (37)$$

Call

$$\begin{aligned} \xi &:= 2 \|f\|_{\infty, \mathbb{R}_+}, \\ \psi &:= \frac{2^{2\alpha} K}{\Gamma(4\alpha + 1)}, \end{aligned} \quad (38)$$

both are greater than zero.

We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^\alpha, \quad \forall h > 0. \quad (39)$$

We have

$$\gamma'(h) = -3\alpha \xi h^{-3\alpha-1} + \alpha \psi h^{\alpha-1} = 0,$$

then

$$\psi h^{\alpha-1} = 3\xi h^{-3\alpha-1}$$

and

$$\psi h^{4\alpha} = 3\xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}. \quad (40)$$

We have that

$$\gamma''(h) = 3\alpha(3\alpha + 1)\xi h^{-3\alpha-2} + \alpha(\alpha - 1)\psi h^{\alpha-2}. \quad (41)$$

We see

$$\begin{aligned} \gamma''(h_0) &= 3\alpha(3\alpha + 1)\xi \left(\frac{3\xi}{\psi}\right)^{\frac{-3\alpha-2}{4\alpha}} + \alpha(\alpha - 1)\psi \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} = \\ &\alpha \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} \left[3(3\alpha + 1)\xi \left(\frac{3\xi}{\psi}\right)^{-1} + (\alpha - 1)\psi \right] = \\ &\alpha \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} \left[3(3\alpha + 1)\xi \frac{\psi}{3\xi} + (\alpha - 1)\psi \right] = \\ &\alpha \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} [(3\alpha + 1)\psi + (\alpha - 1)\psi] = \\ &\alpha \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} (4\alpha\psi) = 4\alpha^2\psi \left(\frac{3\xi}{\psi}\right)^{\frac{\alpha-2}{4\alpha}} > 0. \end{aligned} \quad (42)$$

Therefore γ has a global minimum at $h_0 = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4\alpha}}$, which is

$$\begin{aligned} \gamma(h_0) &= \xi \left(\frac{3\xi}{\psi}\right)^{-\frac{3}{4}} + \psi \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} = \\ & \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} \left[\xi \left(\frac{3\xi}{\psi}\right)^{-1} + \psi \right] = \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} \left(\xi \frac{\psi}{3\xi} + \psi\right) = \\ & \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} \left(\frac{4}{3}\psi\right) = \frac{4}{3}\psi \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}}. \end{aligned} \tag{43}$$

Consequently,

$$\gamma(h_0) = \frac{4}{3}\psi \left(\frac{3\xi}{\psi}\right)^{\frac{1}{4}} = \frac{4}{(\sqrt[4]{3})^3} \psi^{\frac{3}{4}} \xi^{\frac{1}{4}}. \tag{44}$$

We have proved that (see (30))

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \|(D_{*a}^{3\alpha} f)(a)\| &\leq \frac{4\Gamma(3\alpha + 1)(2^{2\alpha} + 1)}{(\sqrt[4]{3})^3 2^{2\alpha}(2\alpha - 1)} \\ & \left(2\|f\|_{\infty, \mathbb{R}_+}\right)^{\frac{1}{4}} \left(\frac{2^{2\alpha}K}{\Gamma(4\alpha + 1)}\right)^{\frac{3}{4}} = \\ & \frac{4\sqrt[4]{2}\Gamma(3\alpha + 1)\Gamma(4\alpha + 1)^{-\frac{3}{4}}(2^{2\alpha} + 1)}{(\sqrt[4]{3})^3 2^{\frac{\alpha}{2}}(2\alpha - 1)} \|f\|_{\infty, \mathbb{R}_+}^{\frac{1}{4}} K^{\frac{3}{4}}. \end{aligned} \tag{45}$$

The theorem is proved. □

We continue with abstract L_p left fractional Landau inequalities over \mathbb{R}_+ .

Theorem 2.3. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, 0 < \alpha < 1$. Let $f \in C^1(\mathbb{R}_+, X)$ with $\|f\|_{\infty, \mathbb{R}_+}, \|f'\|_{\infty, \mathbb{R}_+} < \infty$. For $k = 1, 2, 3$, we assume that $D_{*a}^{k\alpha} f \in C^1([a, \infty), X)$ and $D_{*a}^{4\alpha} f \in C([a, \infty), X), \forall a \in \mathbb{R}_+$. We further assume that*

$$\left(\sup_{a \in \mathbb{R}_+} \|D_{*a}^{4\alpha} f\|_{p, \mathbb{R}_+}\right) < \infty. \tag{46}$$

Then

1) under $\frac{1}{2p} < \alpha < 1$, we get

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \|(D_{*a}^{2\alpha} f)(a)\| &\leq \left[\left(\frac{2^\alpha \Gamma(2\alpha) \left(4\alpha - \frac{1}{p}\right)}{2^\alpha - 1}\right) \left(\frac{4\alpha(1 + 2^{-3\alpha})}{2\alpha - \frac{1}{p}}\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \right. \\ & \left. \left(\frac{1 + 2^{\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \right] \|f\|_{\infty, \mathbb{R}_+}^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \end{aligned}$$

$$\left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right)^{\left(\frac{2\alpha}{4\alpha - \frac{1}{p}} \right)} < \infty. \quad (47)$$

2) under $\frac{1}{p} < \alpha < 1$, we get

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \| (D_{*a}^{3\alpha} f)(a) \| &\leq \left[\left(\frac{\Gamma(3\alpha) \left(4\alpha - \frac{1}{p} \right)}{2\alpha - 1} \right) \left(\frac{6\alpha (1 + 2^{-2\alpha})}{\alpha - \frac{1}{p}} \right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}} \right)} \right. \\ &\quad \left. \left(\frac{1 + 2^{2\alpha - \frac{1}{p}}}{\Gamma(4\alpha) (q(4\alpha - 1) + 1)^{\frac{1}{q}}} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}} \right)} \right] \| \| f \| \|_{\infty, \mathbb{R}_+}^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}} \right)} \\ &\quad \left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}} \right)} < \infty. \end{aligned} \quad (48)$$

That is $\sup_{a \in \mathbb{R}_+} \| (D_{*a}^{2\alpha} f)(a) \|, \sup_{a \in \mathbb{R}_+} \| (D_{*a}^{3\alpha} f)(a) \| < \infty$.

Proof. As in the proof of Theorem 2.2 we have that

$$\begin{aligned} \|A\| &\stackrel{(12)}{=} \left\| f(x_1) - f(a) - \frac{1}{\Gamma(4\alpha)} \int_a^{x_1} (x_1 - t)^{4\alpha - 1} (D_{*a}^{4\alpha} f)(t) dt \right\| \leq \\ &2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{1}{\Gamma(4\alpha)} \int_a^{x_1} (x_1 - t)^{4\alpha - 1} \| (D_{*a}^{4\alpha} f)(t) \| dt \leq \\ &2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{1}{\Gamma(4\alpha)} \frac{(x_1 - a)^{\frac{(q(4\alpha - 1) + 1)}{q}}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right)^{(x_1 - a =: h > 0)} \\ &2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{1}{\Gamma(4\alpha)} \frac{h^{(4\alpha - \frac{1}{p})}}{(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right), \end{aligned} \quad (49)$$

with $\frac{1}{4p} < \alpha < 1$.

That is

$$\|A\| \leq 2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha) (q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right), \quad (50)$$

where $\frac{1}{4p} < \alpha < 1$.

We also have

$$\begin{aligned} \|B\| &\stackrel{(13)}{=} \left\| f(x_2) - f(a) - \frac{1}{\Gamma(4\alpha)} \int_a^{x_2} (x_2 - t)^{4\alpha - 1} (D_{*a}^{4\alpha} f)(t) dt \right\| \leq \\ &2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{(x_2 - a)^{\frac{(q(4\alpha - 1) + 1)}{q}}}{\Gamma(4\alpha) (q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \left\| \| D_{*a}^{4\alpha} f \| \right\|_{p, \mathbb{R}_+} \right)^{(x_2 - a =: 2h)} \end{aligned} \quad (51)$$

$$2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \| \| \| D_{*a}^{4\alpha} f \| \| \|_{p, \mathbb{R}_+} \right).$$

That is

$$\| B \| \leq 2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}} \left(\sup_{a \in \mathbb{R}_+} \| \| \| D_{*a}^{4\alpha} f \| \| \|_{p, \mathbb{R}_+} \right), \tag{52}$$

where $\frac{1}{4p} < \alpha < 1$.

We have assumed that

$$M := \left(\sup_{a \in \mathbb{R}_+} \| \| \| D_{*a}^{4\alpha} f \| \| \|_{p, \mathbb{R}_+} \right) < \infty. \tag{53}$$

For convenience we call

$$c := \Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}} > 0. \tag{54}$$

So we have

$$\begin{aligned} \| A \| &\leq 2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{h^{4\alpha - \frac{1}{p}}}{c} M, \\ \text{and} \\ \| B \| &\leq 2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} M, \end{aligned} \tag{55}$$

where $\frac{1}{4p} < \alpha < 1$.

Next we estimate the (16)-quantities and we have:

$$\begin{aligned} \| (D_{*a}^{2\alpha} f)(a) \| &\leq \frac{1}{D\Gamma(3\alpha + 1)} [2^{3\alpha} h^{3\alpha} \| A \| + h^{3\alpha} \| B \|] \stackrel{(23)}{=} \\ &\frac{h^{3\alpha} \Gamma(2\alpha + 1)}{2^{2\alpha} h^{5\alpha} (2\alpha - 1)} [2^{3\alpha} \| A \| + \| B \|] \stackrel{(55)}{\leq} \\ &\frac{\Gamma(2\alpha + 1)}{2^{2\alpha} (2\alpha - 1) h^{2\alpha}} \left[2^{3\alpha + 1} \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{2^{3\alpha} h^{4\alpha - \frac{1}{p}}}{c} M + \right. \\ &\quad \left. 2 \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} M \right] = \\ &\frac{\Gamma(2\alpha + 1)}{2^{2\alpha} (2\alpha - 1) h^{2\alpha}} \left[(2^{3\alpha + 1} + 2) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+} + \frac{(2^{3\alpha} + 2^{4\alpha - \frac{1}{p}})}{c} M h^{4\alpha - \frac{1}{p}} \right] = \\ &\frac{\Gamma(2\alpha + 1)}{2^{2\alpha} (2\alpha - 1)} \left[\frac{(2^{3\alpha + 1} + 2) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{(2^{3\alpha} + 2^{4\alpha - \frac{1}{p}})}{c} M h^{2\alpha - \frac{1}{p}} \right] = \\ &\frac{2^\alpha \Gamma(2\alpha + 1)}{(2\alpha - 1)} \left[\frac{2(1 + 2^{-3\alpha}) \| \| \| f \| \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{(1 + 2^{\alpha - \frac{1}{p}}) M}{c} h^{2\alpha - \frac{1}{p}} \right]. \end{aligned} \tag{57}$$

That is

$$\| (D_{*a}^{2\alpha} f)(a) \| \leq$$

$$\left(\frac{2^\alpha \Gamma(2\alpha + 1)}{2^\alpha - 1} \right) \left[\frac{2(1 + 2^{-3\alpha}) \| \| f \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{(1 + 2^{\alpha - \frac{1}{p}}) M}{c} h^{2\alpha - \frac{1}{p}} \right], \quad (58)$$

$\forall a \in \mathbb{R}_+, \forall h > 0.$

I.e. it holds

$$\sup_{a \in \mathbb{R}_+} \| (D_{*a}^{2\alpha} f)(a) \| \leq \left(\frac{2^\alpha \Gamma(2\alpha + 1)}{2^\alpha - 1} \right) \left[\frac{2(1 + 2^{-3\alpha}) \| \| f \| \|_{\infty, \mathbb{R}_+}}{h^{2\alpha}} + \frac{(1 + 2^{\alpha - \frac{1}{p}}) M}{c} h^{2\alpha - \frac{1}{p}} \right], \quad (59)$$

$\forall h > 0$, under $\frac{1}{4p} < \alpha < 1.$

Again from (16) we get

$$\begin{aligned} \| (D_{*a}^{3\alpha} f)(a) \| &\leq \frac{1}{\Gamma(2\alpha + 1) D} [h^{2\alpha} \| B \| + 2^{2\alpha} h^{2\alpha} \| A \|] = \\ &\frac{h^{2\alpha} \Gamma(3\alpha + 1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} [\| B \| + 2^{2\alpha} \| A \|] \leq \\ &\left(\frac{h^{2\alpha} \Gamma(3\alpha + 1)}{2^{2\alpha} h^{5\alpha} (2^\alpha - 1)} \right) \left[2 \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{2^{4\alpha - \frac{1}{p}} h^{4\alpha - \frac{1}{p}}}{c} M + \right. \\ &\left. 2^{2\alpha + 1} \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{2^{2\alpha} h^{4\alpha - \frac{1}{p}}}{c} M \right] = \\ &\left(\frac{\Gamma(3\alpha + 1)}{2^{2\alpha} (2^\alpha - 1) h^{3\alpha}} \right) \left[(2 + 2^{2\alpha + 1}) \| \| f \| \|_{\infty, \mathbb{R}_+} + \frac{(2^{2\alpha} + 2^{4\alpha - \frac{1}{p}})}{c} M h^{4\alpha - \frac{1}{p}} \right] = \\ &\frac{\Gamma(3\alpha + 1)}{2^{2\alpha} (2^\alpha - 1)} \left[\frac{(2 + 2^{2\alpha + 1}) \| \| f \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{(2^{2\alpha} + 2^{4\alpha - \frac{1}{p}})}{c} M h^{\alpha - \frac{1}{p}} \right] = \\ &\frac{\Gamma(3\alpha + 1)}{(2^\alpha - 1)} \left[\frac{2(1 + 2^{-2\alpha}) \| \| f \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{(1 + 2^{2\alpha - \frac{1}{p}})}{c} M h^{\alpha - \frac{1}{p}} \right]. \end{aligned} \quad (61)$$

That is

$$\begin{aligned} &\| (D_{*a}^{3\alpha} f)(a) \| \\ &\leq \left(\frac{\Gamma(3\alpha + 1)}{2^\alpha - 1} \right) \left[\frac{2(1 + 2^{-2\alpha}) \| \| f \| \|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{(1 + 2^{2\alpha - \frac{1}{p}})}{c} M h^{\alpha - \frac{1}{p}} \right], \end{aligned} \quad (62)$$

$\forall a \in \mathbb{R}_+, \forall h > 0.$

I.e. it holds

$$\sup_{a \in \mathbb{R}_+} \| (D_{*a}^{3\alpha} f)(a) \| \leq \left(\frac{\Gamma(3\alpha + 1)}{2^\alpha - 1} \right)$$

$$\left[\frac{2(1+2^{-2\alpha})\|f\|_{\infty, \mathbb{R}_+}}{h^{3\alpha}} + \frac{(1+2^{2\alpha-\frac{1}{p}})M}{c} h^{\alpha-\frac{1}{p}} \right], \tag{63}$$

$\forall h > 0, \frac{1}{4p} < \alpha < 1.$

Call

$$\begin{aligned} \mu &:= 2(1+2^{-3\alpha})\|f\|_{\infty, \mathbb{R}_+}, \\ \theta &:= \frac{(1+2^{\alpha-\frac{1}{p}})M}{c}, \end{aligned} \tag{64}$$

both are greater than zero.

We consider the function

$$y(h) = \mu h^{-2\alpha} + \theta h^{2\alpha-\frac{1}{p}}, \quad \forall h > 0. \tag{65}$$

We have

$$y'(h) = -2\alpha\mu h^{-2\alpha-1} + \left(2\alpha - \frac{1}{p}\right)\theta h^{2\alpha-\frac{1}{p}-1} = 0, \tag{66}$$

then

$$\left(2\alpha - \frac{1}{p}\right)\theta h^{2\alpha-\frac{1}{p}-1} = 2\alpha\mu h^{-2\alpha-1}$$

and

$$\left(2\alpha - \frac{1}{p}\right)\theta h^{2\alpha-\frac{1}{p}-1} h^{2\alpha+1} = 2\alpha\mu,$$

i.e.

$$\left(2\alpha - \frac{1}{p}\right)\theta h^{4\alpha-\frac{1}{p}} = 2\alpha\mu,$$

with a unique solution

$$h_0 := h_{crit.no.} = \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{1}{4\alpha-\frac{1}{p}}} \tag{67}$$

(assuming $\frac{1}{2p} < \alpha < 1$).

We have that

$$y''(h) = 2\alpha(2\alpha+1)\mu h^{-2\alpha-2} + \left(2\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p} - 1\right)\theta h^{2\alpha-\frac{1}{p}-2}. \tag{68}$$

We see that

$$\begin{aligned} y''(h_0) &= 2\alpha(2\alpha+1)\mu \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}} + \\ &\left(2\alpha - \frac{1}{p}\right)\left(2\alpha - \frac{1}{p} - 1\right)\theta \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{2\alpha-\frac{1}{p}-2}{4\alpha-\frac{1}{p}}} = \end{aligned}$$

$$\begin{aligned}
& \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}} [2\alpha(2\alpha+1)\mu + \\
& \left(2\alpha - \frac{1}{p}\right) \left(2\alpha - \frac{1}{p} - 1\right) \theta \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)] = \quad (69) \\
& \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{-2\alpha-2}{4\alpha-\frac{1}{p}}} \left[2\alpha(2\alpha+1)\mu + 2\alpha\mu \left(2\alpha - \frac{1}{p} - 1\right) \right] = \\
& 2\alpha\mu \left(\frac{\left(2\alpha - \frac{1}{p}\right)\theta}{2\alpha\mu} \right)^{\left(\frac{2(\alpha+1)}{4\alpha-\frac{1}{p}}\right)} \left(4\alpha - \frac{1}{p}\right) > 0.
\end{aligned}$$

Therefore y has a global minimum at

$$h_0 = \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{1}{4\alpha-\frac{1}{p}}},$$

which is

$$\begin{aligned}
y(h_0) &= \mu \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}} + \theta \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{2\alpha-\frac{1}{p}}{4\alpha-\frac{1}{p}}} = \quad (70) \\
& \left(\frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right)^{\frac{-2\alpha}{4\alpha-\frac{1}{p}}} \left[\mu + \theta \frac{2\alpha\mu}{\left(2\alpha - \frac{1}{p}\right)\theta} \right] \\
&= \mu \left(\frac{\left(2\alpha - \frac{1}{p}\right)\theta}{2\alpha\mu} \right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)} \left(1 + \frac{2\alpha}{2\alpha - \frac{1}{p}} \right) \\
&= \mu \left(\frac{\left(2\alpha - \frac{1}{p}\right)\theta}{2\alpha\mu} \right)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)} \left(\frac{4\alpha - \frac{1}{p}}{2\alpha - \frac{1}{p}} \right) = \\
& \left(\frac{4\alpha - \frac{1}{p}}{2\alpha - \frac{1}{p}} \right) \left(\frac{2\alpha - \frac{1}{p}}{2\alpha} \right)^{\left(\frac{-2\alpha}{4\alpha-\frac{1}{p}}\right)} \mu \left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}} \right) \theta \left(\frac{-2\alpha}{4\alpha-\frac{1}{p}} \right) = \\
& \frac{\left(4\alpha - \frac{1}{p}\right)}{(2\alpha)^{\left(\frac{2\alpha}{4\alpha-\frac{1}{p}}\right)}} \left(2\alpha - \frac{1}{p}\right)^{\frac{-2\alpha+\frac{1}{p}}{4\alpha-\frac{1}{p}}} \mu \left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}} \right) \theta \left(\frac{-2\alpha}{4\alpha-\frac{1}{p}} \right). \quad (71)
\end{aligned}$$

That is

$$y(h_0) = \frac{\left(4\alpha - \frac{1}{p}\right) \left(2\alpha - \frac{1}{p}\right)^{\left(\frac{-2\alpha + \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}}{\left(2\alpha\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}} \mu^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \theta^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)}. \tag{72}$$

Therefore we derive (see (59))

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \|(D_{*a}^{2\alpha} f)(a)\| &\leq \left(\frac{2^\alpha \Gamma(2\alpha)}{2^\alpha - 1}\right) \left(\frac{2\alpha}{2\alpha - \frac{1}{p}}\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(4\alpha - \frac{1}{p}\right) \\ &\left(2(1 + 2^{-3\alpha})\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{1 + 2^{\alpha - \frac{1}{p}}}{\Gamma(4\alpha)(q(4\alpha - 1) + 1)^{\frac{1}{q}}}\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \|\|f\|\|_{\infty, \mathbb{R}_+}^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \\ &\left(\sup_{a \in \mathbb{R}_+} \|\|D_{*a}^{4\alpha} f\|\|_{p, \mathbb{R}_+}\right)^{\left(\frac{2\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} < \infty, \end{aligned} \tag{73}$$

where $\frac{1}{2p} < \alpha < 1$.

Call

$$\begin{aligned} \xi &:= 2(1 + 2^{-2\alpha}) \|\|f\|\|_{\infty, \mathbb{R}_+}, \\ \psi &:= \frac{(1 + 2^{2\alpha - \frac{1}{p}})}{c} M, \end{aligned} \tag{74}$$

both are greater than zero.

We consider the function

$$\gamma(h) := \xi h^{-3\alpha} + \psi h^{\alpha - \frac{1}{p}}, \quad \forall h > 0. \tag{75}$$

We have

$$\gamma'(h) = -3\alpha \xi h^{-3\alpha - 1} + \left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 0,$$

then

$$\left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} = 3\alpha \xi h^{-3\alpha - 1}$$

and

$$\left(\alpha - \frac{1}{p}\right) \psi h^{\alpha - \frac{1}{p} - 1} h^{3\alpha + 1} = 3\alpha \xi,$$

and

$$\left(\alpha - \frac{1}{p}\right) \psi h^{4\alpha - \frac{1}{p}} = 3\alpha \xi,$$

with unique solution

$$h_0 := h_{crit.no.} = \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\frac{1}{4\alpha - \frac{1}{p}}} \quad (76)$$

(assuming $\frac{1}{p} < \alpha < 1$).

We have that

$$\gamma''(h) = 3\alpha(3\alpha + 1)\xi h^{-3\alpha-2} + \left(\alpha - \frac{1}{p}\right)\left(\alpha - \frac{1}{p} - 1\right)\psi h^{\alpha - \frac{1}{p} - 2}. \quad (77)$$

We observe that

$$\begin{aligned} \gamma''(h_0) &= 3\alpha(3\alpha + 1)\xi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha-2}{4\alpha - \frac{1}{p}}\right)} + \\ &\quad \left(\alpha - \frac{1}{p}\right)\left(\alpha - \frac{1}{p} - 1\right)\psi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{\alpha - \frac{1}{p} - 2}{4\alpha - \frac{1}{p}}\right)} = \\ &\quad \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha-2}{4\alpha - \frac{1}{p}}\right)} \left[3\alpha(3\alpha + 1)\xi + \left(\alpha - \frac{1}{p}\right)\left(\alpha - \frac{1}{p} - 1\right)\psi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right) \right] \\ &= \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha-2}{4\alpha - \frac{1}{p}}\right)} \left[3\alpha(3\alpha + 1)\xi + \left(\alpha - \frac{1}{p} - 1\right)3\alpha\xi \right] = \\ &\quad 3\alpha\xi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha-2}{4\alpha - \frac{1}{p}}\right)} \left[(3\alpha + 1) + \left(\alpha - \frac{1}{p} - 1\right) \right] = \\ &\quad 3\alpha\xi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha-2}{4\alpha - \frac{1}{p}}\right)} \left(4\alpha - \frac{1}{p} \right) > 0. \end{aligned} \quad (78)$$

Therefore y has a global minimum at

$$h_0 = \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\frac{1}{4\alpha - \frac{1}{p}}},$$

which is

$$\gamma(h_0) = \xi h_0^{-3\alpha} + \psi h_0^{\alpha - \frac{1}{p}} = h_0^{-3\alpha} \left(\xi + \psi h_0^{4\alpha - \frac{1}{p}} \right) = \quad (79)$$

$$\begin{aligned} & \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha}{4\alpha - \frac{1}{p}}\right)} \left(\xi + \psi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right) \right) = \\ & \xi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha}{4\alpha - \frac{1}{p}}\right)} \left(1 + \frac{3\alpha}{\alpha - \frac{1}{p}} \right) = \\ & \xi \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha}{4\alpha - \frac{1}{p}}\right)} \left(\frac{4\alpha - \frac{1}{p}}{\alpha - \frac{1}{p}} \right). \end{aligned}$$

That is

$$\begin{aligned} \gamma(h_0) &= \xi \left(\frac{4\alpha - \frac{1}{p}}{\alpha - \frac{1}{p}} \right) \left(\frac{3\alpha\xi}{\left(\alpha - \frac{1}{p}\right)\psi} \right)^{\left(\frac{-3\alpha}{4\alpha - \frac{1}{p}}\right)} = \tag{80} \\ & \left(\frac{4\alpha - \frac{1}{p}}{\alpha - \frac{1}{p}} \right) \xi \left(\frac{\left(\alpha - \frac{1}{p}\right)\psi}{3\alpha\xi} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} = \\ & \left(\frac{4\alpha - \frac{1}{p}}{\alpha - \frac{1}{p}} \right) \xi^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{\left(\alpha - \frac{1}{p}\right)}{3\alpha} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} \psi^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)}. \end{aligned}$$

I.e. we have found

$$\gamma(h_0) = \frac{\left(4\alpha - \frac{1}{p}\right) \xi^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \psi^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)}}{\left(\alpha - \frac{1}{p}\right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} (3\alpha)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)}}. \tag{81}$$

We have proved that (see (63))

$$\begin{aligned} \sup_{a \in \mathbb{R}_+} \|(D_{*a}^{3\alpha} f)(a)\| &\leq \left(\frac{\left(4\alpha - \frac{1}{p}\right) \Gamma(3\alpha)}{2^\alpha - 1} \right) \left(\frac{3\alpha}{\alpha - \frac{1}{p}} \right)^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \tag{82} \\ & (2(1 + 2^{-2\alpha}))^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \left(\frac{\left(1 + 2^{2\alpha - \frac{1}{p}}\right)}{\Gamma(4\alpha) (q(4\alpha - 1) + 1)^{\frac{1}{q}}} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} \|\|f\|\|_{\infty, \mathbb{R}_+}^{\left(\frac{\alpha - \frac{1}{p}}{4\alpha - \frac{1}{p}}\right)} \\ & \left(\sup_{a \in \mathbb{R}_+} \|\|D_{*a}^{4\alpha} f\|\|_{p, \mathbb{R}_+} \right)^{\left(\frac{3\alpha}{4\alpha - \frac{1}{p}}\right)} < \infty, \end{aligned}$$

where $\frac{1}{p} < \alpha < 1$.

The theorem is proved. \square

We give an application when $\alpha = \frac{1}{2}$.

Corollary 2.4. *Let $f \in C^1(\mathbb{R}_+, X)$ with $\|f\|_{\infty, \mathbb{R}_+}, \|f'\|_{\infty, \mathbb{R}_+} < \infty$, where $(X, \|\cdot\|)$ is a Banach space. For $k = 1, 2, 3$, we assume that $D_{*a}^{k\frac{1}{2}} f \in C^1([a, \infty), X)$ and $D_{*a}^{4\frac{1}{2}} f \in C([a, \infty), X), \forall a \in \mathbb{R}_+$. We further assume that*

$$K := \left\| \left\| D_{*a}^{4\frac{1}{2}} f(t) \right\| \right\|_{\infty, \mathbb{R}_+^2} < \infty, \quad (83)$$

where $(a, t) \in \mathbb{R}_+^2$.

Then

$$\sup_{a \in \mathbb{R}_+} \left\| \left(D_{*a}^{2\frac{1}{2}} f \right) (a) \right\| \leq \left(\frac{\sqrt{12 + 6\sqrt{2}}}{\sqrt{2} - 1} \right) \|f\|_{\infty, \mathbb{R}_+}^{\frac{1}{2}} \left(\left\| \left\| D_{*a}^{4\frac{1}{2}} f(t) \right\| \right\|_{\infty, \mathbb{R}_+^2} \right)^{\frac{1}{2}} < \infty, \quad (84)$$

and

$$\begin{aligned} & \sup_{a \in \mathbb{R}_+} \left\| \left(D_{*a}^{3\frac{1}{2}} f \right) (a) \right\| \leq \\ & \left(\frac{9\sqrt{\pi}}{(2 - \sqrt{2}) \sqrt[4]{2} (\sqrt[4]{3})^3} \right) \|f\|_{\infty, \mathbb{R}_+}^{\frac{1}{4}} \left(\left\| \left\| D_{*a}^{4\frac{1}{2}} f(t) \right\| \right\|_{\infty, \mathbb{R}_+^2} \right)^{\frac{3}{4}} < \infty. \end{aligned} \quad (85)$$

That is $\sup_{a \in \mathbb{R}_+} \left\| \left(D_{*a}^{2\frac{1}{2}} f \right) (a) \right\|, \sup_{a \in \mathbb{R}_+} \left\| \left(D_{*a}^{3\frac{1}{2}} f \right) (a) \right\| < \infty$.

Proof. By Theorem 2.2. \square

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