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RESIDUATED CONNECTIONS INDUCED BY RESIDUATED FRAMES†

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Abstract. In this paper, we introduce the notions of (dual) residuated frames for a fuzzy logic as an extension of residuated frames for classical relational semantics. We investigate the relations between residuated connections and residuated frames on Alexandrov topologies based on $[0, \infty]$. Moreover, we study their properties and give their examples.

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1. Introduction

Blyth and Janovitz [2] introduced the residuated connection as a pair of maps on partially ordered sets. Many researchers [4,5,10,11] studied residuated frames for relational semantics for a logic. Orłowska and Rewitzky [10,11] investigated various residuated connections from the viewpoint of (dual) residuated frames from many valued logics.

Pawlak [12,13] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Ward et al. [10] introduced a complete residuated lattice L as an important algebraic structure for many valued logics [1,6-9,14,15].

For an extension of Pawlak's rough sets, many researchers developed L-lower and L-upper approximation operators in complete residuated lattices [6-9,14,15,18].

Discrete and stone dualities are dualities between algebras and logical relational systems such as Boolean algebras and classical propositional logic; MV-algebra and Lukasiewicz logic; BL-algebra and basic fuzzy logics [3-5,17-19].

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In this paper, as a duality between algebras and logical relational systems, we introduce the notion of (dual) residuated connections and (dual) residuated frames in fuzzy logics. We investigate the relations between residuated connections and residuated frames on Alexandrov topologies based on $[0,\infty]$. Moreover, we study their properties and give their examples.

2. Preliminaries

Let $([0,\infty],\leq,\vee,+,\wedge,\nearrow,\infty,0)$ be a nonnegative extended real number defined as

$$x\nearrow y=\bigwedge\{z\in[0,\infty]\mid z+x\geq y\}=(y-x)\vee 0,$$
$$\infty+a=a+\infty=\infty,\forall a\in[0,\infty],\infty\nearrow\infty=0.$$

Definition 2.1. Let X be a set. A function $d_X: X\times X\to[0,\infty]$ is called a *non-symmetric pseudo-metric* if it satisfies the following conditions:

(M1) $d_X(x,x)=0$ for all $x\in X$,

(M2) $d_X(x,y)+d_X(y,z)\geq d_X(x,z)$, for all $x,y,z\in X$.

(M3) If $d_X(x,y)=d_X(x,y)=0$, $x=y$.

The pair (X,d_X) is called a *non-symmetric pseudo-metric space.*

Remark 2.1. (1) We define a function $d_{[0,\infty]^X}:[0,\infty]^X\times[0,\infty]^X\to[0,\infty]$ as $d_{[0,\infty]^X}(A,B)=\bigvee_{x\in X}(A(x)\to B(x))=\bigvee_{x\in X}((B(x)-A(x))\vee 0)$. Then $([0,\infty]^X,d_{[0,\infty]^X})$ is a non-symmetric pseudo-metric space.

(2) If (X,d_X) is a non-symmetric pseudo-metric space and we define a function $d_X^{-1}(x,y)=d_X(y,x)$, then (X,d_X^{-1}) is a non-symmetric pseudo-metric space.

(3) Let (X,d_X) be a non-symmetric pseudo-metric space and define $(d_X\oplus d_X)(x,z)=\bigwedge_{y\in X}(d_X(x,y)+d_X(y,z))$ for each $x,z\in X$. By (M2), $(d_X\oplus d_X)(x,z)\geq d_X(x,z)$ and $(d_X\oplus d_X)(x,z)\leq d_X(x,x)+d_X(x,z)=d(x,z)$. Hence $(d_X\oplus d_X)=d_X$.

(4) If d_X is a non-symmetric pseudo-metric and $d_X(x,y)=d_X(y,x)$ for each $x,y\in X$, then d_X is a pseudo-metric

(5) For $x,y,z\in[0,\infty]$, $x+y\geq z$ iff $x\geq y\nearrow z$.

Lemma 2.2. For each $x,y,z,x_i,y_i\in[0,\infty]$, we have the following properties.

(1) If $y\leq z$, $x\nearrow y\leq x\nearrow z$ and $z\nearrow x\leq y\nearrow x$.

(2) $x\nearrow(\bigvee_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\nearrow y_i)$ and $(\bigwedge_{i\in\Gamma}x_i)\nearrow y=\bigvee_{i\in\Gamma}(x_i\nearrow y)$.

(3) $x+(x\nearrow y)\geq y$, $(x\nearrow y)\nearrow y\leq y$ and $(x\nearrow y)+(y\nearrow z)\geq x\nearrow z$.

(4) $(x+y)\nearrow z=x\nearrow(y\nearrow z)=y\nearrow(x\nearrow z)$.

(5) $x\nearrow y\geq(x+z)\nearrow(y+z)$ and $x\nearrow y\geq(y\nearrow z)\nearrow(x\nearrow z)$.

(6) $x\nearrow x=0$ and $0\nearrow x=x$.

(7) $\bigvee_{\alpha\in[0,\infty]}((x\nearrow\alpha)\nearrow(y\nearrow\alpha))=y\nearrow x$ for each $x,y\in[0,\infty]$,

Proof. (1) Since $y\leq z\leq x+(x\nearrow z)$, $x\nearrow y\leq x\nearrow z$. Since $x\leq y+(y\nearrow x)\leq z+(y\nearrow x)$, $z\nearrow x\leq y\nearrow x$.

(2) By (1), $x\nearrow(\bigvee_{i\in\Gamma}y_i)\geq\bigvee_{i\in\Gamma}(x\nearrow y_i)$. Since $x+\bigvee_{i\in\Gamma}(x\nearrow y_i)\geq\bigvee_{i\in\Gamma}(x+(x\nearrow y_i))\geq\bigvee_{i\in\Gamma}y_i$, $x\nearrow(\bigvee_{i\in\Gamma}y_i)\leq\bigvee_{i\in\Gamma}(x\nearrow y_i)$.

By (1), $(\bigwedge_{i\in\Gamma} x_i) \nearrow y \geq \bigvee_{i\in\Gamma}(x_i \nearrow y)$. Since $(\bigwedge_{i\in\Gamma} x_i) + \bigvee_{i\in\Gamma}(x_i \nearrow y) \geq \bigwedge_{i\in\Gamma}(x_i + (x_i \nearrow y)) \geq y$, $(\bigwedge_{i\in\Gamma} x_i) \nearrow y \leq \bigvee_{i\in\Gamma}(x_i \nearrow y)$.

(3) Since $x \nearrow y \geq x \nearrow y$, $x + (x \nearrow y) \geq y$. Moreover, $x \geq (x \nearrow y) \nearrow y$. Since $x + (x \nearrow y) + (y \nearrow z) \geq y + (y \nearrow z) \geq z$, $(x \nearrow y) + (y \nearrow z) \geq x \nearrow z$.

(4) Since $x + y + ((x + y) \nearrow z) \geq z$ iff $x + ((x + y) \nearrow z) \geq y \nearrow z$, $(x + y) \nearrow z \geq x \nearrow (y \nearrow z)$.

Since $x + y + (x \nearrow (y \nearrow z)) \geq y + (y \nearrow z) \geq z$, $x \nearrow (y \nearrow z) \geq (x + y) \nearrow z$.

Similarly, $(x + y) \nearrow z = y \nearrow (x \nearrow z)$.

(5) Since $(x + z) + (x \nearrow y) \geq y + z$, $x \nearrow y \geq (x + z) \nearrow (y + z)$. Since $x + (x \nearrow y) + (y \nearrow z) \geq z$, $x \nearrow y \geq (y \nearrow z) \nearrow (x \nearrow z)$.

(6) For $x \in X$, $x \nearrow x = \bigwedge\{z \in [0,\infty] \mid x + z \geq x\} = 0$ and $0 \nearrow x = \bigwedge\{z \in [0,\infty] \mid 0 + z \geq x\} = x$.

(7) $I = \bigvee_{\alpha\in[0,\infty]}((x \nearrow \alpha) \nearrow (y \nearrow \alpha)) = \bigvee_{\alpha\in[0,\infty]}((y \nearrow \alpha) - (x \nearrow \alpha)) \vee 0 = \bigvee_{\alpha\in[0,\infty]}((\alpha - y) \vee 0 - (\alpha - x) \vee 0) \vee 0$. If $\alpha \geq x \geq y$, then $I = x - y$. If $x \geq \alpha \geq y$, then $I = \alpha - y \leq x - y$. Other cases, $I = 0$. Hence $I = (x - y) \vee 0 = y \nearrow x$.

□

3. Fuzzy join and meet preserving maps on Alexandrov L-pretopologies

Definition 3.1. Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (d_X, f, g, d_Y) is called a *residuated connection* if for all $x \in X, y \in Y$, $d_Y(f(x), y) = d_X(x, g(y))$.

(2) (d_X, f, g, d_Y) is called a *dual residuated connection* if for all $x \in X, y \in Y$, $d_Y(y, f(x)) = d_X(g(y), x)$.

We redefine the following definition as a sense in [6-8].

Definition 3.2. A subset $\tau_X \subset [0,\infty]^X$ is called an *Alexandrov topology* on X iff it satisfies the following conditions:

(AT1) $\alpha_X \in \tau_X$ where $\alpha_X(x) = \alpha$ for each $x \in X$ and $\alpha \in [0,\infty]$.

(AT2) If $A_i \in \tau_X$ for all $i \in I$, then $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau_X$.

(AT3) If $A \in \tau_X$ and $\alpha \in [0,\infty]$, then $\alpha + A, \alpha \to A \in \tau_X$ where $(\alpha \to A)(x) = (A(x) - \alpha) \vee 0$.

The pair (X, τ_X) is called an *Alexandrov topological space*.

For $R_1 \in [0,\infty]^{X\times Y}, R_2 \in [0,\infty]^{Y\times Z}$, We define

$$R_1 \oplus R_2(x, z) = \bigwedge_y (R_1(x, y) + R_2(y, z)), \quad R^{-1}(y, x) = R(x, y).$$

Lemma 3.3. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces. For $R \in [0,\infty]^{X\times Y}$, we have the following properties:*

(1) $(d_X \oplus R)^{-1} = R^{-1} \oplus d_X^{-1}$ *and* $(R \oplus d_X)^{-1} = d_X^{-1} \oplus R^{-1}$.

(2) $d_X \oplus R \oplus d_Y \geq R$ *iff* $d_X \oplus R \geq R$ *and* $R \oplus d_Y \geq R$.

(3) $d_X^{-1} \oplus R \oplus d_Y^{-1} \geq R$ *iff* $d_X^{-1} \oplus R \geq R$ *and* $R \oplus d_Y^{-1} \geq R$.

Proof. (1) It is easily proved.

(2) It follows $(d_X \oplus R \oplus d_Y)(x,y) = \bigwedge_{y_1 \in Y}((d_X \oplus R)(x,y_1) + d_Y(y_1,y)) \leq (d_X \oplus R)(x,y) + d_Y(y,y) = (d_X \oplus R)(x,y)$. Similarly, $R \oplus d_Y \geq R$. Conversely, it easily proved.

(3) is similarly proved as (2). □

Definition 3.4. Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces. For $R \in [0,\infty]^{X\times Y}, S \in [0,\infty]^{Y\times X}$, a structure (d_X, R, S, d_Y) is called:

(1) a residuated frame if $S = R^{-1}$ and $d_X \oplus R \oplus d_Y \geq R$.

(2) a dual residuated frame if $S = R^{-1}$ and $d_X^{-1} \oplus R \oplus d_Y^{-1} \geq R$.

Lemma 3.5. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $f : X \to Y$ and $g : Y \to X$ maps. Then the followings hold.*

(1) If (d_X, f, g, d_Y) is a residuated connection and define maps $R : X \times Y \to [0,\infty]$ and $S : Y \times X \to [0,\infty]$ as

$$R(x,y) = d_X(x, g(y)) = d_Y(f(x), y), \quad S(y,x) = R(x,y).$$

Then (d_X, R, S, d_Y) is a residuated frame.

(2) If (d_X, f, g, d_Y) is a dual residuated connection and define maps $R : X \times Y \to [0,\infty]$ and $S : Y \times X \to [0,\infty]$ as

$$R(x,y) = d_X(g(y), x) = d_Y(y, f(x)), \quad S(y,x) = R(x,y).$$

Then (d_X, R, S, d_Y) is a dual residuated frame.

(3) If $d_X(g(y_1), g(y_2)) \leq d_Y(y_1, y_2)$ for each $y_1, y_2 \in Y$ and $R_1(x,y) = d_X(x, g(y))$ (resp. $R_2(x,y) = d_X(g(y), x)$), then $d_X \oplus R_1 \oplus d_Y \geq R_1$ (resp. $d_X^{-1} \oplus R_2 \oplus d_Y^{-1} \geq R_2$).

(4) If $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ for each $x_1, x_2 \in X$ and $R_1(x,y) = d_Y(y, f(x))$ (resp. $R_2(x,y) = d_Y(f(x), y)$), then $d_X^{-1} \oplus R_1 \oplus d_Y^{-1} \geq R_1$ (resp. $d_X \oplus R_2 \oplus d_Y \geq R_2$).

Proof. (1) For each $x, x_1 \in X, y, y_1 \in Y$,

$$\begin{aligned} & d_X(x,x_1) + R(x_1,y_1) + d_Y(y_1,y) \\ & = d_X(x,x_1) + d_X(x_1, g(y_1)) + d_Y(y_1,y) \\ & \geq d_X(x, g(y_1)) + d_X(y_1,y) \\ & = d_Y(f(x), y_1) + d_Y(y_1,y) \\ & \geq d_Y(f(x), y) = R(x,y) \end{aligned}$$

Hence $d_X \oplus R \oplus d_Y \geq R$.

(3) For each $x, x_1 \in X, y, y_1 \in Y$,

$$\begin{aligned} & d_X(x,x_1) + R_1(x_1,y_1) + d_Y(y_1,y) \\ & = d_X(x,x_1) + d_X(x_1, g(y_1)) + d_Y(y_1,y) \\ & \geq d_X(x,x_1) + d_X(x_1, g(y_1)) + d_X(g(y_1), g(y)) \\ & \geq d_X(x,x_1) + d_X(x_1, g(y)) \\ & \geq d_X(x, g(y)) = R(x,y) \end{aligned}$$

Hence $d_X \oplus R_1 \oplus d_Y \geq R_1$.

Other case, (2) and (4) are similarly proved.

□

Theorem 3.6. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces and $R \in [0,\infty]^{X\times Y}$. Define*

$$\tau_{d_X} = \{A \in [0,\infty]^X \mid A(x) + d_X(x,z) \geq A(z)\},$$

$$\tau_{d_Y} = \{B \in [0,\infty]^Y \mid B(y) + d_Y(y,w) \geq B(w)\}.$$

Then the followings hold.

(1) τ_{d_X} and τ_{d_Y} are Alexandrov topologies on X.

(2)If $(d_X)_x = d_X(x,-) \in [0,\infty]^X$ and $((d_X)_x^{-1} \nearrow \alpha)(z) = (d_X)_x^{-1}(z) \nearrow \alpha = d_X(z,x) \nearrow \alpha$, then $(d_X)_x \in \tau_{d_X}$ and $(d_X)_x^{-1} \nearrow \alpha \in \tau_{d_X}$.

(3) If $d_X \oplus R \geq R$ for $R \in [0,\infty]^{X\times Y}$ and put $R_y^{-1}(x) = R(x,y)$ for each $x \in X$, then $(R_y^{-1} \nearrow \alpha) \in \tau_{d_X}$ and $R_y^{-1} \in \tau_{d_X^{-1}}$. Moreover, $\bigvee_{y\in Y}(R(-,y) \nearrow B(y)) \in \tau_{d_X}$ and $\bigwedge_{y\in Y}(B(y) + R(-,y)) \in \tau_{d_X^{-1}}$.

(4) If $d_X^{-1} \oplus R \geq R$ for $R \in [0,\infty]^{X\times Y}$, then $R_y^{-1} \in \tau_{d_X}$ and $\alpha \nearrow R_y^{-1} \in \tau_{d_X}$. Moreover, $\bigwedge_{y\in Y}(R(-,y) + B(y)) \in \tau_{d_X}$ and $\bigvee_{y\in Y}(B(y) \nearrow R(-,y)) \in \tau_{d_X}$.

(5) If $R \oplus d_Y \geq R$ for $R \in [0,\infty]^{X\times Y}$ and put $R_x(z) = R(x,z)$, then $R_x \in \tau_{d_X}$ and $\alpha \nearrow R_x \in \tau_{d_Y}$. Moreover, $\bigwedge_{x\in X}(R(x,-) + A(x)) \in \tau_{d_Y}$ and $\bigvee_{x\in X}(A(x) \nearrow R(x,-)) \in \tau_{d_Y}$.

(6) If $R \oplus d_Y^{-1} \geq R$ for $R \in [0,\infty]^{X\times Y}$, then $(R_x \nearrow \alpha) \in \tau_{d_Y}$ and $R_x \in \tau_{d_Y^{-1}}$. Moreover, $\bigvee_{x\in X}(R(x,-) \nearrow A(x)) \in \tau_{d_Y}$ and $\bigwedge_{x\in X}(R(x,-) + A(x)) \in \tau_{d_Y^{-1}}$.

(7) Define $d_{\tau_{d_X}} : \tau_{d_X} \times \tau_{d_X} \to [0,\infty]$ as $d_{\tau_{d_X}}(A,B) = \bigvee_{x\in X}(A(x) \nearrow B(x))$. Then $(\tau_{d_X}, d_{\tau_{d_X}})$ is a non-symmetric pseudo-metric space.

Proof. (1) Since $\alpha_X(x) + d_X(x,y) \geq \alpha_X(y)$, we have $\alpha_X \in \tau_{d_X}$.

If $A_i \in \tau_{d_X}$ for all $i \in I$, then

$$\begin{array}{l}(\bigwedge_{i\in I} A_i) + d_X(x,y) = \bigwedge_{i\in I}(A_i + d_X(x,y)) \geq \bigwedge_{i\in I} A_i,\\ (\bigvee_{i\in I} A_i) + d_X(x,y) = \bigvee_{i\in I}(A_i + d_X(x,y)) \geq \bigvee_{i\in I} A_i,\end{array}$$

then $\bigwedge_{i\in I} A_i, \bigvee_{i\in I} A_i \in \tau_{d_X}$.

If $A \in \tau_{d_X}$ and $\alpha \in L$, then $\alpha + (\alpha \nearrow A(x)) + d_X(x,y) \geq A(x) + d_X(x,y) \geq A(y)$ implies $(\alpha \nearrow A(x)) + d_X(x,y) \geq (\alpha \nearrow A(y))$. So, $\alpha \nearrow A \in \tau_{d_X}$. Easily, $\alpha + A \in \tau_{d_X}$. Hence τ_{d_X} is an Alexandrov topology on X.

Similarly, τ_{d_Y} is an Alexandrov topology on Y.

(2) Since $(d_X)_x(y) + d_X(y,z) \geq (d_X)_x(z)$, $(d_X)_x \in \tau_{d_X}$. Moreover, $(d_X)_x^{-1} \nearrow \alpha) \in \tau_{d_X}$ from

$$\begin{array}{l}(d_X(z,x) \nearrow \alpha) + d_X(z,w) + d_X(w,x) \geq (\alpha - d_X(z,x)) \vee 0 + d_X(z,x) \geq \alpha,\\ (\Rightarrow)(d_X(z,x) \nearrow \alpha) + d_X(z,w) \geq (\alpha - d_X(w,x)) \vee 0\\ (\Rightarrow)(d_x^{-1}(z) \nearrow \alpha) + d_X(z,w) \geq d_x^{-1}(w) \nearrow \alpha.\end{array}$$

(3) Since $d_X(x,y)+R(y,z) \geq (d_X \oplus R)(x,z) \geq R(x,z)$ iff $d_X^{-1}(y,x)+R_z^{-1}(y) \geq R_z^{-1}(x)$, $R_z^{-1} \in \tau_{d_X^{-1}}$. We have $(R_y^{-1} \nearrow \alpha) \in \tau_{d_X}$ from:

$$\begin{array}{l}(R_y^{-1}(x) \nearrow \alpha) + d_X(x,z) + R(z,y) \geq (R_y^{-1}(x) \nearrow \alpha) + R(x,y) \geq \alpha \\ (\Rightarrow)\ (R_y^{-1}(x) \nearrow \alpha) + d_X(x,z) \geq (R_y^{-1}(z) \nearrow \alpha).\end{array}$$

Moreover, by (1),$\bigvee_{y\in Y}(R(-,y) \nearrow B(y)) \in \tau_{d_X}$ and $\bigwedge_{y\in Y}(B(y)+R(-,y)) \in \tau_{d_X^{-1}}$.

(4) Since $d_X^{-1}(x,y)+R(y,z) \geq R(x,z)$, $R_z^{-1} \in \tau_{d_X}$. By (1), $\alpha \nearrow R_y^{-1} \in \tau_{d_X}$. Moreover, $\bigwedge_{y\in Y}(R(-,y)+B(y)) \in \tau_{d_X}$ and $\bigvee_{y\in Y}(B(y) \nearrow R(-,y)) \in \tau_{d_X}$.

(5) Since $R(x,y)+d_Y(y,z) \geq R(x,z)$, $R_x \in \tau_{d_Y}$. By (1), $\alpha \nearrow R_x \in \tau_{d_Y}$, $\bigwedge_{x\in X}(R(x,-)+A(x)) \in \tau_{d_Y}$ and $\bigvee_{x\in X}(A(x) \nearrow R(x,-)) \in \tau_{d_Y}$.

(6) Since $R(x,w)+d_Y^{-1}(w,y) \geq R(x,y)$, $R_x \in \tau_{d_Y^{-1}}$. $(R_x \nearrow \alpha) \in \tau_{d_Y}$ from

$$\begin{array}{l}(R_x(y) \nearrow \alpha) + d_Y(y,w) + R(x,w) \geq (R_x(y) \nearrow \alpha) + R(x,y) \geq \alpha \\ (\Rightarrow)(R_x(y) \nearrow \alpha) + d_Y(y,w) \geq R(x,w) \nearrow \alpha.\end{array}$$

Moreover, $\bigvee_{x\in X}(R(x,-) \nearrow A(x)) \in \tau_{d_Y}$ and $\bigwedge_{x\in X}(R(x,-)+A(x)) \in \tau_{d_Y^{-1}}$.

(7) (M1) $d_{\tau_{d_X}}(A,A) = \bigvee_{x\in X}(A(x) \to A(x)) = 0$ for all $A \in \tau_{d_X}$,

(M2) Since $d_{\tau_{d_X}}(A,B)+d_{\tau_{d_X}}(B,C) = \bigvee_{x\in X}(A(x) \to B(x))+\bigvee_{x\in X}(B(x) \to C(x)) \geq \bigvee_{x\in X}((B(x)-A(x))\vee 0)+(C(x)-B(x))\vee 0 \geq \bigvee_{x\in X}((C(x)-A(x))\vee 0) = d_{\tau_{d_X}}(A,C)$, for all $A,B,C \in \tau_X$,

(M3) If $d_{\tau_{d_X}}(A,B) = d_{\tau_{d_X}}(B,A) = 0$, $A = B$.

□

If (X,d_X) is a non-symmetric pseudo-metric space, then $d_X \oplus d_X = d_X$ and $d_X^{-1} \oplus d_X^{-1} = d_X^{-1}$. From Theorem 3.6, the following corollary hold.

Corollary 3.7. *If (X,d_X) is a non-symmetric pseudo-metric space, then* $(d_X)_x, ((d_X)_z^{-1} \nearrow \alpha), \bigwedge_{x\in X}(d_X(x,-)+A(x)), \bigvee_{x\in X}(A(x) \nearrow d_X(x,-)), \bigvee_{y\in X}(d_X(-,y) \nearrow B(y)) \in \tau_{d_X}$.

Theorem 3.8. *Let (X,d_X) and (Y,d_Y) be non-symmetric pseudo-metric spaces. Let τ_{d_X} and τ_{d_Y} be Alexandrov topologies. Then the followings hold.*

(1) (d_X,f,g,d_Y) is a residuated connection, that is, $d_Y(f(x),y) = d_X(x,g(y))$ for all $x,y \in X$ iff there exist relations $R : \tau_{d_X} \times \tau_{d_Y} \to [0,\infty]$ and $S : \tau_{d_Y} \times \tau_{d_X} \to [0,\infty]$ as

$$R(A,B) = \bigvee_{x\in X}(A(x) \nearrow B(f(x))),$$

$$S(B,A) = \bigvee_{y\in Y}(A(g(y)) \nearrow B(y))$$

with maps $f : X \to Y$, $g : Y \to X$ and $d_X(x_1,x_2) \geq d_Y(f(x_1),f(x_2))$, $d_Y(y_1,y_2) \geq d_X(g(y_1),g(y_2))$, for each $x_1,x_2 \in X$, $y_1,y_2 \in Y$ such that $(d_{\tau_{d_X}},R,S,d_{\tau_{d_Y}})$ is a residuated frame.

(2) In (1),

$$R(A,B) = d_{\tau_{d_X}}(A,f^{\leftarrow}(B)) = d_{\tau_{d_Y}}(F(A),B) = d_{\tau_{d_X}}(A,G(B))$$

where $F(A)(y) = \bigwedge_{z\in X}(d_Y(f(z),y) + A(z))$ *and* $G(B) = \bigvee_{y\in Y}(d_Y(f(z),y) \nearrow B(y)))$.

$$S(B,A) = d_{\tau_{d_Y}}(g^{\leftarrow}(A),B) = d_{\tau_{d_Y}}(F_1(A),B) = d_{\tau_{d_X}}(A,G_1(B))$$

where $F_1(A)(w) = \bigwedge_{z\in X}(d_Y(z,g(w))+A(z))$ *and* $G_1(B)(z) = \bigvee_{w\in Y}(d_Y(z,g(w)) \nearrow B(w))$.

Proof. (1) ($\Rightarrow$) For $x \in X$ and $y \in Y$, $d_X(x,g(f(x))) = d_Y(f(x),f(x)) = 0$ and $d_Y(f(g(y)),y) = d_X(g(y),g(y)) = 0$. Moreover, $d_Y(f(x_1),f(x_2)) = d_X(x_1,g(f(x_2)) \le d_X(x_1,x_2) + d_X(x_2,g(f(x_2)) = d_X(x_1,x_2)$ for each $x_1,x_2 \in X$ and $d_X(g(y_1),g(y_2)) = d_Y(f(g(y_1),y_2) \le d_X(f(g(y_1),y_1) + d_Y(y_1,y_2) = d_Y(y_1,y_2)$ for each $y_1,y_2 \in Y$.

For $A \in \tau_{d_X}$ and $B \in \tau_{d_Y}$, since $B(f(g(y))) + d_Y(f(g(y)),y) \ge B(y)$, $d_Y(f(g(y)),y) = 0$, $A(x)+d_X(x,g(f(x))) \ge A(g(f(x)))$ and $d_X(x,g(f(x))) = 0$, we have $R(A,B) = S(B,A)$ from:

$$\begin{aligned} R(A,B) &= \bigvee_{x\in X}(A(x) \nearrow B(f(x))) \\ &\ge \bigvee_{y\in Y}(A(g(y)) \nearrow B(f(g(y))) + d_Y(f(g(y)),y)) \\ &\ge \bigvee_{y\in Y}(A(g(y)) \nearrow B(y)) = S(B,A), \end{aligned}$$

$$\begin{aligned} S(B,A) &= \bigvee_{y\in Y}(A(g(y)) \nearrow B(y)) \\ &\ge \bigvee_{x\in X}(A(g(f(x))) \nearrow B(f(x))) \\ &\ge \bigvee_{x\in X}(A(x) + d_X(x,g(f(x))) \nearrow B(f(x))) = R(A,B). \end{aligned}$$

For each $A, A_1 \in \tau_{d_X}, B, B_1 \in \tau_{d_Y}$, we have $d_{\tau_{d_X}} \oplus R \oplus d_{\tau_{d_Y}} \ge R$ from:

$$\begin{aligned} & d_{\tau_{d_X}}(A,A_1) + R(A_1,B_1) + d_{\tau_{d_Y}}(B_1,B) \\ &= d_{\tau_{d_X}}(A,A_1) + \bigvee_{x\in X}(A_1(x) \nearrow B_1(f(x))) + \bigvee_{x\in X}(B_1(f(x)) \nearrow B(f(x))) \\ &\ge \bigvee_{x\in X}(A(x) \nearrow B(f(x))) = R(A,B). \end{aligned}$$

($\Leftarrow$) For $(d_X)_x \in \tau_{d_X}$ and $((d_Y)_y^{-1} \nearrow \alpha) \in \tau_{d_Y}$ from Theorem 3.6(2),

$$\begin{aligned} & R((d_X)_x,((d_Y)_y^{-1} \nearrow \alpha)) = \bigvee_{z\in X}((d_X)_x(z) \nearrow ((d_Y)_y^{-1} \nearrow \alpha)(f(z))) \\ &\ge (d_X)_x(x) \nearrow ((d_Y)_y^{-1} \nearrow \alpha)(f(x)) = d_Y(f(x),y) \nearrow \alpha \end{aligned}$$

Since $d_X(x,z)+d_Y(f(z),y) \ge d_Y(f(x),f(z))+d_Y(f(z),y) \ge d_Y(f(x),y)$, $d_X(x,z) \nearrow (d_Y(f(z),y) \nearrow \alpha) \le d_Y(f(x),y) \nearrow \alpha$. Hence $R((d_X)_x,((d_Y)_y^{-1} \nearrow \alpha)) = d_Y(f(x),y) \nearrow \alpha$.

$$\begin{aligned} S(((d_Y)_y^{-1} \nearrow \alpha),(d_X)_x) &= \bigvee_{z\in X}((d_X)_x(g(z)) \nearrow ((d_Y)_y^{-1} \nearrow \alpha)(z)) \\ &\le (d_X)_x(g(y)) \nearrow ((d_Y)_y^{-1} \nearrow \alpha)(y) = d_X(x,g(y)) \nearrow \alpha \end{aligned}$$

Since $d_X(x,g(z)) + d_Y(z,y) \le d_X(x,g(z)) + d_X(g(z),\ g(y)) \ge d_X(x,g(y))$, $d_X(x,g(z)) \nearrow (d_Y(z,y) \nearrow \alpha) \le d_X(x,g(y)) \nearrow \alpha$. Hence $S(((d_Y)_y^{-1} \nearrow \alpha),(d_X)_x) = d_X(x,g(y)) \nearrow \alpha$. Thus $d_Y(f(x),y) \nearrow \alpha = d_X(x,g(y)) \nearrow \alpha$ for all $x,y \in X$ from:

$$\begin{aligned} & R((d_X)_x,((d_Y)_y^{-1} \nearrow \alpha)) = d_Y(f(x),y) \nearrow \alpha \\ &= S(((d_Y)_y^{-1} \nearrow \alpha),(d_X)_x) = d_X(x,g(y)) \nearrow \alpha. \end{aligned}$$

Put $\alpha = d_X(x, g(y))$. Since $d_Y(f(x), y) \nearrow d_X(x, g(y)) = d_X(x, g(y)) \nearrow d_X(x, g(y)) = 0$, $d_Y(f(x), y) \geq d_X(x, g(y))$. Put $\alpha = d_Y(f(x), y)$. Similarly, $d_Y(f(x), y) \leq d_X(x, g(y))$. Hence $d_Y(f(x), y) = d_X(x, g(y))$.

(2) For $A \in \tau_{d_X}$ and $B \in \tau_{d_Y}$, since $A = \bigwedge_{z \in X}(A(z) + d_X(z, -))$ and $B = \bigvee_{\alpha \in [0,\infty]} \bigvee_{y \in Y}((B(y) \nearrow \alpha) \nearrow (d_Y(-, y) \nearrow \alpha)) = \bigvee_{y \in Y}(d_Y(-, y) \nearrow B(y))$,

$$\begin{aligned}
R(A, B) &= \bigvee_{x \in X}(A(x) \nearrow B(f(x))) \\
&= \bigvee_{x \in X}(\bigwedge_{z \in X}(A(z) + d_X(z, x)) \nearrow \bigvee_{y \in Y}(d_Y(f(x), y) \nearrow B(y))) \\
&= \bigvee_{x,y,z \in X}((B(y) - d_Y(f(x), y)) \vee 0 - (A(z) + d_X(z, x))) \vee 0 \\
&= \bigvee_{y,z \in X}((B(y) - A(z)) - \bigwedge_{x \in X}(d_Y(f(x), y) + d_X(z, x))) \vee 0 \\
&= \bigvee_{y,z \in X}((B(y) - A(z)) - \bigwedge_{x \in X}(d_X(x, g(y)) + d_X(z, x))) \vee 0 \\
&= \bigvee_{y,z \in X}((B(y) - A(z)) - d_X(z, g(y))) \vee 0 \\
&= \bigvee_{y \in X}(B(y) - \bigwedge_{z \in X}(A(z) + d_X(z, g(y))) \vee 0 = d_{\tau_{d_Y}}(F(A), B) \\
&= \bigvee_{z \in X}(\bigvee_{y \in X}(B(y) - d_X(z, g(y))) - A(z)) \vee 0 \\
&= \bigvee_{z \in X}(A(z) \nearrow \bigvee_{y \in Y}(d_Y(f(z), y) \nearrow B(y))) = d_{\tau_{d_X}}(A, G(B)).
\end{aligned}$$

$$\begin{aligned}
S(B, A) &= \bigvee_{y \in Y}(A(g(y)) \nearrow B(y)) \\
&= \bigvee_{y \in Y}(\bigwedge_{z \in X}(A(z) + d_X(z, g(y))) \nearrow \bigvee_{w \in Y}(d_Y(y, w) \nearrow B(w))) \\
&= \bigvee_{y,z,w \in Y}((B(w) - d_Y(y, w)) \vee 0 - (A(z) + d_X(z, g(y)))) \vee 0 \\
&= \bigvee_{y,z,w \in Y}((B(w) - A(z) - \bigwedge_{y \in Y}(d_Y(y, w) + d_X(z, g(y)))) \vee 0) \\
&= \bigvee_{y,z,w \in Y}((B(w) - A(z) - \bigwedge_{y \in Y}(d_Y(y, w) + d_Y(f(z), y))) \vee 0) \\
&= \bigvee_{y,z,w \in Y}((B(w) - A(z) - d_Y(f(z), w)) \vee 0) \\
&= \bigvee_{w \in X}(B(w) - \bigwedge_{z \in X}(A(z) + d_Y(f(z), w))) \vee 0 = d_{\tau_{d_Y}}(F_1(A), B) \\
&= \bigvee_{z \in X}(\bigvee_{w \in X}(B(w) - d_X(z, g(w))) - A(z)) \vee 0 \\
&= d_{\tau_{d_X}}(A, G_1(B)).
\end{aligned}$$

□

Theorem 3.9. *Let (X, d_X) and (Y, d_Y) be non-symmetric pseudo-metric spaces. Then the followings hold.*

(1) *If $R : X \times Y \to [0, \infty]$ with $d_X \oplus R \geq R$ and $R \oplus d_Y \geq R$ such that*

$$F(A)(y) = \bigwedge_{x \in X}(A(x) + R(x, y)), \quad G(B)(x) = \bigvee_{y \in Y}(R(x, y) \nearrow B(y)),$$

then $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection.

(2) *If $R : X \times Y \to [0, \infty]$ with $d_X^{-1} \oplus R \geq R$, $R \oplus d_Y^{-1} \geq R$ such that*

$$F(A)(y) = \bigvee_{x \in X}(R(x, y) \nearrow A(x)), \quad G(B)(x) = \bigwedge_{y \in Y}(R(x, y) + B(y)),$$

then $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a dual residuated connection.

Proof. (1) Since $R \oplus d_Y \geq R$, by Theorem 3.6(5), $F(A) = \bigwedge_{x\in Y}(R(x,-) + A(x)) \in \tau_{d_Y}$. Since $d_X \oplus R \geq R$, by Theorem 3.6(3), $G(B) = \bigvee_{x\in X}(R(-,y) \nearrow B(y)) \in \tau_{d_X}$. Moreover,

$$
\begin{aligned}
d_{\tau_{d_Y}}(F(A),B) &= \bigvee_{y\in Y}(F(A)(y) \nearrow B(y)) \\
&= \bigvee_{y\in Y}\Big(\bigwedge_{x\in X}(R(x,y)+A(x)) \nearrow B(y)\Big) \\
&= \bigvee_{y\in Y}\Big((B(y) - \bigwedge_{x\in X}(R(x,y)+A(x))) \vee 0\Big) \\
&= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((B(y) - R(x,y) - A(x))) \vee 0\Big) \\
&= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((((B(y) - R(x,y)) \vee 0) - A(x)) \vee 0\Big) \\
&= \bigvee_{x\in X}\bigvee_{y\in Y}\Big(A(x) \nearrow (R(x,y) \nearrow B(y))\Big) \\
&= \bigvee_{x\in X}\Big(A(x) \nearrow \bigvee_{x\in X}(R(x,y) \nearrow B(y))\Big) \\
&= \bigvee_{x\in X}\Big(A(x) \nearrow G(B)(x)\Big) = d_{\tau_{d_X}}(A,G(B)).
\end{aligned}
$$

(2) Since $R \oplus d_Y^{-1} \geq R$, by Theorem 3.6(6), $F(A) = \bigvee_{x\in X}(R(x,-) \nearrow A(x)) \in \tau_{d_Y}$. Since $d_X^{-1} \oplus R \geq R$, by Theorem 3.6(4), $G(B) = \bigwedge_{x,y\in Y}(B(y) + R(-,y)) \in \tau_{d_X}$.

$$
\begin{aligned}
d_{\tau_{d_Y}}(B,F(A)) &= \bigvee_{y\in Y}(B(y) \nearrow \bigvee_{x\in X}(R(x,y) \nearrow A(x))) \\
&= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((((A(x) - R(x,y)) \vee 0) - B(x)) \vee 0\Big) \\
&= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((A(x) - \bigvee_{y\in Y}(R(x,y) + B(y))) \vee 0\Big) \\
&= \bigvee_{x\in X}(G(B)(x) \nearrow A(x)) = d_{\tau_{d_X}}(G(B),A).
\end{aligned}
$$

□

Example 3.10. Let $(X = \{a,b,c\}, d_i)$ be non-symmetric pseudo-metric spaces and $R_i \in [0,\infty]^X$ $i = 1,2$ as follows:

$$
d_1 = \begin{pmatrix} 0 & 6 & 5 \\ 6 & 0 & 7 \\ 5 & 7 & 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 0 & 7 & 5 \\ 4 & 0 & 3 \\ 3 & 5 & 0 \end{pmatrix}
$$

$$
R_1 = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 5 \end{pmatrix} \quad R_2 = \begin{pmatrix} 7 & 4 & 3 \\ 6 & 8 & 5 \\ 3 & 5 & 8 \end{pmatrix}.
$$

(1) Since $R_1 \oplus d_1 = R_1 = d_1 \oplus R_1$, by Theorem 3.9(1), $(d_{\tau_{d_1}}, F, G, d_{\tau_{d_1}})$ is a residuated connection where $F(A)(y) = \bigwedge_{x\in X}(A(x) + R_2(x,y))$, $G(B)(x) = \bigvee_{y\in Y}(R_1(x,y) \nearrow B(y))$.

Since $R_1 \oplus d_1^{-1} = R_1 = d_1^{-1} \oplus R_1$, by Theorem 3.9(2), $(d_{\tau_{d_1}}, F, G, d_{\tau_{d_1}})$ is a dual residuated connection where $F(A)(y) = \bigvee_{x\in X}(R_1(x,y) \nearrow A(x))$, $G(B)(x) = \bigwedge_{y\in Y}(R_1(x,y) + B(y))$.

(2) Since

$$R_2 \oplus d_2 = d_2^{-1} \oplus R_2 = \begin{pmatrix} 6 & 4 & 3 \\ 6 & 8 & 5 \\ 3 & 5 & 8 \end{pmatrix} d_2 \oplus R_2 = R_2 \oplus d_2^{-1} = \begin{pmatrix} 7 & 4 & 3 \\ 6 & 8 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

By Theorem 3.9(1), let $F(A)(y) = \bigwedge_{x\in X}(A(x) + R_2(x, y))$, $G(B)(x) = \bigvee_{y\in Y}$ $(R_2(x, y) \nearrow B(y))$. Since

$$\begin{aligned} 6 &= F(d_2(a, -))(c) + d_2(c, a) = R_2(a, c) + d_2(c, a) \\ &\not\geq R_2(a, a) = F(d_2(a, -))(a) = 7, \end{aligned}$$

$F(d_2(a, -)) \notin d_{\tau_{d_2}}$. Hence $(d_{\tau_{d_2}}, F, G, d_{\tau_{d_2}})$ is not a residuated connection.

By Theorem 3.9(2), let $F_1(A)(y) = \bigwedge_{x\in X}(R_2(x, y) \nearrow A(x))$, $G_1(B)(x) = \bigvee_{y\in Y}(R_2(x, y) + B(y))$.

$$\begin{aligned} 6 &= G(d_2(a, -))(c) + d_2(c, a) = d_2^{-1}(a, c) + R_2(c, a) \\ &\not\geq R_2(a, a) = G(d_2(a, -))(a) = 7. \end{aligned}$$

Hence $(d_{\tau_{d_2}}, F, G, d_{\tau_{d_2}})$ is not a dual residuated connection.

(3) Let $(X = \{a, b, c\}, d_2)$ be a non-symmetric pseudo-metric space and $f : X \to X$ a function as $f(a) = f(b) = b, f(c) = c$. Then $d_2(x, y) \geq d_2(f(x), f(y))$ for all $x, y \in X$. Put $R_1(x, y) = d_2(y, f(x))$ and $R_2(x, y) = d_2(f(x), y)$. By Lemma 3.5(4), $d_2^{-1} \oplus R_1 \oplus d_2^{-1} \geq R_1$ and $d_2 \oplus R_2 \oplus d_2 \geq R_2$. Let $F_2, G_2 : \tau_{d_2} \to \tau_{d_2}$ be functions with $F_2(A)(y) = \bigwedge(A(x) + d_2(f(x), y))$ and $G_2(B)(x) = \bigvee_{y\in X}(d_2(f(x), y) \nearrow B(y))$. By Theorem 3.9(1), $(d_{\tau_{d_2}}, F_2, G_2, d_{\tau_{d_2}})$ is a residuated connection.

Let $F_1, G_1 : \tau_{d_2} \to \tau_{d_2}$ be a function with $F_1(A)(y) = \bigvee_{y\in X}(d_2(y, f(x)) \nearrow A(x))$ and $G_1(B)(x) = \bigwedge(d_2(y, f(x)) + B(y))$. By Theorem 3.9(2), $(d_{\tau_{d_2}}, F_1, G_1, d_{\tau_{d_2}})$ is a dual residuated connection.

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