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SOME FIXED POINT THEOREMS FOR MODIFIED JS-G-CONTRACTIONS AND AN APPLICATION TO INTEGRAL EQUATIONS

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Abstract. In this article, we establish some fixed point results in G-metric spaces using the modified JS-G-contractions and we provide some suitable examples to support the results. Also, we give an application to solve an integral equation.

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1. Introduction

The Banach contraction principle (BCP) is the tremendous, most versatile and fundamental result in fixed point theory. Many authors gave different contractions based on the BCP in the literature. Parvaneh et al. [15] initiated the modified JS-contractions and weakly JS-contractions. In this paper, we consider the related contractions in the setting of G-metric spaces (see, [1–8, 14, 16–21]).

Definition 1.1 ([14]). Let Y be a non-empty set and $G : Y \times Y \times Y \to \mathbb{R}^+$ be a function so that:

(1) $G(a, b, c) = 0$ if $a = b = c$;
(2) $G(a, a, b) > 0$ for all $a, b \in Y$, with $a \neq b$;
(3) $G(a, a, b) \leq G(a, b, c)$, for all $a, b, c \in Y$ with $b \neq c$;
(4) $G(a, b, c) = G(b, c, a) = G(c, a, b) = \cdots$ (symmetry in all three variables);
(5) $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$, for all $a, b, c, x \in Y$ (rectangular inequality).

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Then G is called a generalized metric (or a G-metric) on Y and the pair (Y, G) is a G-metric space.

Example 1.2 ([14]). Let Y be a non-empty subset of $\mathbb{R}$, then the function $G : Y \times Y \times Y \to [0, \infty)$ given by $G(a, b, c) = |a - b| + |b - c| + |c - a|$ for all $a, b, c \in Y$, is a G-metric on Y.

Now, let $\{y_n\}$ be a sequence in G-metric space (Y, G).

Definition 1.3 ([14]). An element $y \in Y$ is said to be limit of a sequence $\{y_n\}$, if $\lim_{n,m\to\infty} G(y, y_n, y_m) = 0$. Here, $\{y_n\}$ is G-convergent to y. That is, for any $\varepsilon > 0$, there is a positive integer N so that $G(y, y_n, y_m) < \epsilon$, for all $n, m \geq N$.

Definition 1.4 ([14]). $\{y_n\}$ is said to be G-Cauchy, if for each $\varepsilon > 0$, there is an integer N such that $G(y_n, y_m, y_l) < \varepsilon$ for all $n, m, l \geq N$.

Definition 1.5. (Y, G) is said to be G-complete, if every G-Cauchy sequence in (Y, G) is G-convergent in (Y, G).

Following [13], let $\vartheta : [0, \infty) \to [1, \infty)$ be a function satisfying the following conditions:

(ϑ_1) ϑ is nondecreasing and $\vartheta(k) = 1$ if and only if $k = 0$;
(ϑ_2) for each $\{k_h\} \subseteq (0, \infty), \lim_{h\to\infty} \vartheta(k_h) = 1$ if and only if $\lim_{h\to\infty} k_h = 0$;
(ϑ_3) there exist $n \in (0, 1)$ and $\Omega \in (0, \infty]$ so that $\lim_{k\to 0^+} \frac{\vartheta(k)-1}{k^n} = \Omega$;
(ϑ_4) ϑ is continuous.

For examples of such functions, one may consult the work in [9–12]. Denote by Δ the class of functions ϑ satisfying the conditions (ϑ_1)-(ϑ_3).

Theorem 1.6 ([13]). *Let β be a self-mapping on a complete metric space (Y, d). Suppose there exist $\vartheta \in \Delta$ and $r \in (0, 1)$ so that for $\tau, \sigma \in Y$, $d(\beta\tau, \beta\sigma) \neq 0$ implies $\vartheta(d(\beta\tau, \beta\sigma)) \leq [\vartheta(d(\tau, \sigma))]^r$. Then ϑ has a unique fixed point.*

Our goal is to make the conditions on the control function ϑ to be more restricted. Let $\mathcal{P}$ be the family of functions $\vartheta : [0, \infty) \to [1, \infty)$ satisfying the conditions (ϑ_1), (ϑ_2) and (ϑ_4).

Example 1.7. Some examples of functions in the set $\mathcal{P}$ are $g(s) = \cosh(s)$, $g(s) = e^{se^s}$, $g(s) = e^{\sqrt{s}e^{\sqrt{s}}}$, $g(s) = \frac{2\cosh(s)}{1+\cosh(s)}$, $g(s) = \frac{2e^{se^s}}{1+e^{se^s}}$, $g(s) = \frac{2e^{\sqrt{s}e^{\sqrt{s}}}}{1+e^{\sqrt{s}e^{\sqrt{s}}}}$, $g(s) = 1 + \ln(1 + s)$, $g(s) = e^{\sqrt{se^s}}$, $g(s) = \frac{2+2\ln(1+s)}{2+\ln(1+s)}$ for all $s > 0$.

2. Main results

Definition 2.1. Let β be a self-mapping on a complete G-metric space (Y, G). Then β is said to be a G-$\mathcal{P}$-contraction, whenever there are $\vartheta \in \mathcal{P}$ and $\hbar_1$, $\hbar_2$, $\hbar_3$, $\hbar_4 \geq 0$ with $\hbar_1 + \hbar_2 + \hbar_3 + \hbar_4 < 1$ such that

$$\vartheta(G(\beta\iota, \beta\sigma, \beta\rho)) \leq [\vartheta(G(\iota, \sigma, \rho))]^{\hbar_1} [\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_2} [\vartheta(G(\sigma, \beta\sigma, \beta\sigma))]^{\hbar_3} [\vartheta(\frac{G(\iota, \beta\sigma, \beta\sigma) + G(\sigma, \beta\iota, \beta\iota)}{2})]^{\hbar_4}, \quad (1)$$

for all $\iota, \sigma, \rho \in Y$.

Theorem 2.2. *Every G-$\mathcal{P}$-contraction mapping on a complete G-metric space possesses a unique fixed point.*

Proof. For $\iota_0 \in Y$, define $\{\iota_n\}$ by $\iota_n = \beta\iota_{n-1}$, $n \geq 1$. If there is $\iota_N = \iota_{N+1}$ for some N, nothing to prove. Suppose that $\iota_N \neq \iota_{N+1}$ for each $n \geq 0$. We claim that $\lim_{n\to\infty} G(\iota_n, \iota_{n+1}, \iota_{n+1}) = 0$. From (1), we have

$$\begin{aligned}
&\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1})) = \vartheta(G(\beta\iota_{n-1}, \beta\iota_n, \beta\iota_n))\\
&\quad\leq [\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_1}[\vartheta(G(\iota_{n-1}, \beta\iota_{n-1}, \beta\iota_{n-1}))]^{\hbar_2}[\vartheta(G(\iota_n, \beta\iota_n, \beta\iota_n))]^{\hbar_3}\\
&\qquad[\vartheta(\frac{G(\iota_{n-1}, \beta\iota_n, \beta\iota_n) + G(\iota_n, \beta\iota_{n-1}, \beta\iota_{n-1})}{2})]^{\hbar_4}\\
&\quad\leq [\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_1}[\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_2}[\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1}))]^{\hbar_3}\\
&\qquad[\vartheta(\frac{G(\iota_{n-1}, \iota_{n+1}, \iota_{n+1}) + G(\iota_n, \iota_n, \iota_n)}{2})]^{\hbar_4}\\
&\quad\leq [\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_1+\hbar_2}[\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1}))]^{\hbar_3}[\vartheta(\frac{G(\iota_{n-1}, \iota_{n+1}, \iota_{n+1})}{2})]^{\hbar_4}\\
&\quad\leq [\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_1+\hbar_2}[\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1}))]^{\hbar_3}\\
&\qquad[\vartheta(\max\{G(\iota_{n-1}, \iota_n, \iota_n), G(\iota_n, \iota_{n+1}, \iota_{n+1})\})]^{\hbar_4}. \qquad (2)
\end{aligned}$$

If for some N, we have $G(\iota_{N-1}, \iota_N, \iota_N) < G(\iota_N, \iota_{N+1}, \iota_{N+1})$, then in view of (ϑ_1), we get that

$$\vartheta(G(\iota_{N-1}, \iota_N, \iota_N)) < \vartheta(G(\iota_N, \iota_{N+1}, \iota_{N+1})). \qquad (3)$$

Using (2), we have

$$\vartheta(G(\iota_N, \iota_{N+1}, \iota_{N+1})) \leq [\vartheta(G(\iota_{N-1}, \iota_N, \iota_N))]^{\hbar_1+\hbar_2}[\vartheta(G(\iota_N, \iota_{N+1}, \iota_{N+1}))]^{\hbar_3+\hbar_4}. \qquad (4)$$

Therefore,

$$\vartheta(G(\iota_N, \iota_{N+1}, \iota_{N+1})) \leq [\vartheta(G(\iota_{N-1}, \iota_N, \iota_N))]^{\frac{\hbar_1+\hbar_2}{1-\hbar_3-\hbar_4}} \leq [\vartheta(G(\iota_{N-1}, \iota_N, \iota_N))],$$

which is a contradiction with respect to (3). Consequently, for all $n \geq 1$,

$$\max\{G(\iota_{n-1}, \iota_n, \iota_n), G(\iota_n, \iota_{n+1}, \iota_{n+1})\} = G(\iota_{n-1}, \iota_n, \iota_n),$$

which yields that

$$1 < \vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1})) \leq [\vartheta(G(\iota_0, \iota_1, \iota_1))]^{[\frac{\hbar_1+\hbar_2+\hbar_4}{1-\hbar_3}]^n}.$$

Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} \vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1})) = 1$. From (ϑ_2), we get

$$\lim_{n\to\infty} G(\iota_n, \iota_{n+1}, \iota_{n+1}) = 0. \qquad (5)$$

To prove that $\{\iota_n\}$ is a G-Cauchy sequence, suppose the contrary, that is, there is $\epsilon > 0$ for which there are m_i and n_i so that $n_i > m_i > i$,

$$G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}) \geq \epsilon, \tag{6}$$

and

$$G(\iota_{m_i}, \iota_{n_i-1}, \iota_{n_i-1}) < \epsilon. \tag{7}$$

From the rectangular inequality, we have

$$G(\iota_{m_i-1}, \iota_{n_i-1}, \iota_{n_i-1}) \leq G(\iota_{m_i-1}, \iota_{m_i}, \iota_{m_i}) + G(\iota_{m_i}, \iota_{n_i-1}, \iota_{n_i-1}).$$

In view of (5) and (7), we get

$$\limsup_{i\to\infty} G(\iota_{m_i-1}, \iota_{n_i-1}, \iota_{n_i-1}) \leq \epsilon. \tag{8}$$

Similarly,

$$\limsup_{i\to\infty} G(\iota_{m_i-1}, \iota_{n_i}, \iota_{n_i}) \leq \epsilon. \tag{9}$$

On the other hand, since $G(\beta\iota_{m_i-1}, \beta\iota_{n_i-1}, \beta\iota_{n_i-1}) > 0$, so one writes

$$\begin{aligned}
&\vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i})) = \vartheta(G(\beta\iota_{m_i-1}, \beta\iota_{n_i-1}, \beta\iota_{n_i-1})) \\
&\leq [\vartheta(G(\iota_{m_i-1}, \iota_{n_i-1}, \iota_{n_i-1}))]^{\hbar_1} [\vartheta(G(\iota_{m_i-1}, \beta\iota_{m_i-1}, \beta\iota_{m_i-1}))]^{\hbar_2} \\
&\quad [\vartheta(G(\iota_{n_i-1}, \beta\iota_{n_i-1}, \beta\iota_{n_i-1}))]^{\hbar_3} \\
&\quad [\vartheta(\frac{G(\iota_{m_i-1}, \beta\iota_{n_i-1}, \beta\iota_{n_i-1}) + G(\iota_{m_i-1}, \beta\iota_{m_i-1}, \beta\iota_{m_i-1})}{2})]^{\hbar_4}.
\end{aligned}$$

Using now (ϑ_4) and (5)-(8), we have

$$\begin{aligned}
\vartheta(\epsilon) &\leq [\vartheta(\limsup_{i\to\infty} G(\iota_{m_i-1}, \iota_{n_i-1}, \iota_{n_i-1}))]^{\hbar_1} [\vartheta(\limsup_{i\to\infty} G(\iota_{m_i-1}, \iota_{m_i}, \iota_{m_i}))]^{\hbar_2} \\
&\quad [\vartheta(\limsup_{i\to\infty} G(\iota_{n_i-1}, \iota_{n_i}, \iota_{n_i}))]^{\hbar_3} \\
&\quad [\vartheta(\limsup_{i\to\infty}(\frac{G(\iota_{m_i-1}, \iota_{n_i}, \iota_{n_i}) + G(\iota_{n_i-1}, \iota_{m_i}, \iota_{m_i})}{2}))]^{\hbar_4} \\
&\leq (\vartheta(\epsilon))^{\hbar_1} (\vartheta(\epsilon))^{\hbar_4}.
\end{aligned}$$

This implies that $1 < \vartheta(\epsilon) \leq (\vartheta(\epsilon))^{\hbar_1+\hbar_4}$, which is a contradiction. Thus, $\{\iota_n\}$ is a G-Cauchy sequence. The completeness of Y implies that there is $\iota \in Y$ so that $\iota_n \to \iota$ as $n \to \infty$. On the other hand,

$$\begin{aligned}
&\vartheta(G(\iota_n, \beta\iota, \beta\iota)) = \vartheta(G(\beta\iota_{n-1}, \beta\iota, \beta\iota)) \\
&\leq [\vartheta(G(\iota_{n-1}, \iota, \iota))]^{\hbar_1} [\vartheta(G(\iota_{n-1}, \beta\iota_{n-1}, \beta\iota_{n-1}))]^{\hbar_2} [\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_3} \\
&\quad [\vartheta(\frac{G(\iota_{n-1}, \beta\iota, \beta\iota) + G(\iota, \beta\iota_{n-1}, \beta\iota_{n-1})}{2})]^{\hbar_4} \\
&\leq [\vartheta(G(\iota_{n-1}, \iota, \iota))]^{\hbar_1} [\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))]^{\hbar_2} [\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_3} \\
&\quad [\vartheta(\frac{G(\iota_{n-1}, \beta\iota, \beta\iota) + G(\iota, \iota_n, \iota_n)}{2})]^{\hbar_4}.
\end{aligned}$$

Suppose that $\beta\iota \neq \iota$, so $G(\iota, \beta\iota, \beta\iota) > 0$. Taking $n \to \infty$ and using (ϑ_1) and (5), we have

$$\begin{aligned}\vartheta(G(\iota, \beta\iota, \beta\iota)) &\leq [\vartheta(G(\iota, \iota, \iota))]^{\hbar_1}[\vartheta(G(\iota, \iota, \iota))]^{\hbar_2}[\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_3}[\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_4}\\ &= [\vartheta(G(\iota, \beta\iota, \beta\iota))]^{\hbar_3+\hbar_4} < [\vartheta(G(\iota, \beta\iota, \beta\iota))],\end{aligned}$$

which is a contradiction. Hence, ι is a fixed point of β. Let ι, σ be two different fixed points of β. Then,

$$\begin{aligned}&\vartheta(G(\iota, \sigma, \sigma)) = \vartheta(G(\beta\iota, \beta\sigma, \beta\sigma))\\ &\leq [\vartheta(G(\iota, \sigma, \sigma))]^{\hbar_1}[\vartheta(G(\iota, \iota, \iota))]^{\hbar_2}[\vartheta(G(\sigma, \sigma, \sigma))]^{\hbar_3}[\vartheta(\frac{G(\iota, \sigma, \sigma) + G(\sigma, \iota, \iota)}{2})]^{\hbar_4}\\ &= [\vartheta(G(\iota, \sigma, \sigma)]^{\hbar_1+\hbar_4} < \vartheta(G(\iota, \sigma, \sigma),\end{aligned}$$

which is a contradiction. Therefore, the fixed point of β is unique. □

Remark 2.1. In Theorem 2.2, we can substitute the continuity of ϑ by the continuity of β.

By setting $\vartheta(s) = e^{\sqrt{s}}$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let β be a self-mapping on a complete G-metric space (Y, G) so that*

$$\begin{aligned}\sqrt{G(\beta\iota, \beta\sigma, \beta\rho)} \leq &\hbar_1\sqrt{G(\iota, \sigma, \rho)} + \hbar_2\sqrt{G(\iota, \beta\iota, \beta\iota)} + \hbar_3\sqrt{G(\sigma, \beta\sigma, \beta\sigma)}\\ &+ \hbar_4\sqrt{\frac{G(\iota, \beta\sigma, \beta\sigma) + G(\sigma, \beta\iota, \beta\iota)}{2}},\end{aligned}$$

for all $\iota, \sigma, \rho \in Y$, where $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \geq 0$ with $\hbar_1 + \hbar_2 + \hbar_3 + \hbar_4 < 1$. Then β has a unique fixed point.

Setting $\vartheta(s) = e^{\sqrt[n]{s}}$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *Let $\beta : Y \to Y$ be a mapping on a complete G-metric space (Y, G) such that*

$$\begin{aligned}\sqrt[n]{G(\beta\iota, \beta\sigma, \beta\rho)} \leq &\hbar_1\sqrt[n]{G(\iota, \sigma, \rho)} + \hbar_2\sqrt[n]{G(\iota, \beta\iota, \beta\iota)} + \hbar_3\sqrt[n]{G(\sigma, \beta\sigma, \beta\sigma)}\\ &+ \hbar_4\sqrt[n]{\frac{G(\iota, \beta\sigma, \beta\sigma) + G(\sigma, \beta\iota, \beta\iota)}{2}},\end{aligned}$$

for all $\iota, \sigma, \rho \in Y$, where $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \geq 0$ with $\hbar_1 + \hbar_2 + \hbar_3 + \hbar_4 < 1$. Then β has a unique fixed point.

Corollary 2.5. *Let $\beta : Y \to Y$ be a mapping on a complete G-metric space (Y, G) such that*

$$\begin{aligned}&[1 + ln(1 + G(\beta\iota, \beta\sigma, \beta\rho))]\\ &\quad\leq [1 + \ln(1 + G(\iota, \sigma, \rho))]^{\hbar_1}[1 + \ln(1 + G(\iota, \beta\iota, \beta\iota))]^{\hbar_2}\\ &\qquad[1 + \ln(1 + G(\sigma, \beta\sigma, \beta\sigma))]^{\hbar_3}[1 + \ln(1 + \frac{G(\iota, \beta\sigma, \beta\sigma) + G(\sigma, \beta\iota, \beta\iota)}{2})]^{\hbar_4},\end{aligned}$$

for all $\iota, \sigma, \rho \in Y$, where $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \geq 0$ so that $\hbar_1 + \hbar_2 + \hbar_3 + \hbar_4 < 1$. Then β has a unique fixed point.

Example 2.6. Let $Y = [0, \infty)$ and $G(\iota, \sigma, \rho) = \max\{|\iota - \sigma|, |\sigma - \rho|, |\rho - \iota|\}$. Define $\beta : Y \to Y$ by $\beta(t) = \frac{t}{6}$ where $t \in Y$ and $\vartheta(s) = e^{\sqrt{s}}$ and $\hbar_i = \frac{1}{5}$; $i = 1, 2, 3, 4$ and $t = 0$ is a unique fixed point of β.

3. Weakly JS-G-contractive conditions

Let Φ be the set of functions $\theta : [1, \infty) \to [0, \infty)$ so that:

(θ_1) θ is continuous;

(θ_2) $\theta(1) = 0$;

(θ_3) For each $\{a_n\} \subseteq (1, \infty)$, $\lim_{n\to\infty} \theta(a_n) = 0$ iff $\lim_{n\to\infty} a_n = 1$.

Remark 3.1. Note that $\theta(s) = s - \sqrt[n]{s}$ $(n \geq 1)$ belongs to Φ. Other examples are $\theta(s) = e^{s-1} - 1$ and $\theta(s) = \ln(s)$.

Definition 3.1. Let (Y, G) be a G-metric space and β be a self-mapping on Y. We say that β is a weakly JS-G-contraction if for all $\iota, \sigma, \rho \in Y$, we have

$$\vartheta(G(\beta\iota, \beta\sigma, \beta\rho)) \leq \vartheta(G(\iota, \sigma, \rho)) - \theta(\vartheta(G(\iota, \sigma, \rho))), \tag{10}$$

where $\theta \in \Phi$ and $\vartheta \in \mathcal{P}$.

Theorem 3.2. *Let (Y, G) be a complete G-metric space and β be a self-mapping on Y such that*

(1) β is a weakly JS-G-contraction;

(2) β is continuous.

Then β admits a unique fixed point.

Proof. For $\iota_0 \in Y$, define $\{\iota_n\}$ by $\iota_n = \beta^n \iota_0 = \beta\iota_{n-1}$. Without loss of generality, assume that $\iota_n \neq \iota_{n+1}$ for each $n \geq 0$. Since β is a weakly JS-G-contraction, we derive

$$\begin{aligned}\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1})) =& \vartheta(G(\beta\iota_{n-1}, \beta\iota_n, \beta\iota_n)) \\ &\leq \vartheta(G(\iota_{n-1}, \iota_n, \iota_n)) - \theta(\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))).\end{aligned} \tag{11}$$

Thus, $\{\vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1}))\}$ is decreasing, and so there is $r \geq 1$ such that

$$\lim_{n\to\infty} \vartheta(G(\iota_n, \iota_{n+1}, \iota_{n+1})) = r.$$

Now, we will prove that $r = 1$. Taking $n \to \infty$ in (11), we have

$$r \leq r - \theta(r), \tag{12}$$

and hence

$$\lim_{n\to\infty} \theta(\vartheta(G(\iota_{n-1}, \iota_n, \iota_n))) = 0. \tag{13}$$

That is,

$$\lim_{n\to\infty} \vartheta(G(\iota_{n-1}, \iota_n, \iota_n)) = 1, \tag{14}$$

and so

$$\lim_{n\to\infty} G(\iota_{n-1}, \iota_n, \iota_n) = 0. \tag{15}$$

We now show that $\{\iota_n\}$ is a G-Cauchy sequence. We argue by contradiction. Suppose there is $\epsilon > 0$ for which there are $\{\iota_{m_i}\}$ and $\{\iota_{n_i}\}$ of $\{\iota_n\}$ so that $n_i > m_i > i$ and

$$G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}) \geq \epsilon \quad \text{and} \quad G(\iota_{m_i}, \iota_{n_i-1}, \iota_{n_i-1}) < \epsilon. \tag{16}$$

Using (16) and the rectangular inequality, we get

$$\begin{aligned}\epsilon &\leq G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i})\\ &\leq G(\iota_{m_i}, \iota_{m_i+1}, \iota_{m_i+1}) + G(\iota_{m_i+1}, \iota_{n_i}, \iota_{n_i})\\ &\leq G(\iota_{m_i}, \iota_{m_i+1}, \iota_{m_i+1}) + G(\iota_{m_i+1}, \iota_{n_i+1}, \iota_{n_i+1}) + G(\iota_{n_i+1}, \iota_{n_i}, \iota_{n_i}).\end{aligned}$$

Taking $i \to \infty$, and using (15), we get

$$\epsilon \leq \limsup_{i\to\infty} G(\iota_{m_i+1}, \iota_{n_i+1}, \iota_{n_i+1}). \tag{17}$$

Also,

$$G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}) \leq G(\iota_{m_i}, \iota_{n_i-1}, \iota_{n_i-1}) + G(\iota_{n_i-1}, \iota_{n_i}, \iota_{n_i}).$$

Then from (15) and (16), we have

$$\limsup_{i\to\infty} G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}) \leq \epsilon. \tag{18}$$

By (10), we deduce

$$\begin{aligned}\vartheta(G(\iota_{m_i+1}, \iota_{n_i+1}, \iota_{n_i+1})) =&\vartheta(G(\beta\iota_{m_i}, \beta\iota_{n_i}, \beta\iota_{n_i}))\\ &\leq \vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i})) - \theta(\vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}))).\end{aligned}$$

Now, taking $i \to \infty$, using (ϑ_1), (ϑ_4), (17) and (18), we have

$$\begin{aligned}\vartheta(\epsilon) &\leq \vartheta(\limsup_{i\to\infty} G(\iota_{m_i+1}, \iota_{n_i+1}, \iota_{n_i+1}))\\ &\leq \vartheta(\limsup_{i\to\infty} G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i})) - \liminf_{i\to\infty} \theta(\vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i})))\\ &\leq \vartheta(\epsilon) - \liminf_{i\to\infty} \theta(\vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}))),\end{aligned}$$

which implies that

$$\liminf_{i\to\infty} \theta(\vartheta(G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}))) = 0.$$

Thus,

$$\liminf_{i\to\infty} G(\iota_{m_i}, \iota_{n_i}, \iota_{n_i}) = 0.$$

which contradicts with (16). Hence, $\{\iota_n\}$ is a G-Cauchy sequence in the complete G-metric space (Y, G). Therefore, there exists $\iota \in Y$ such that

$$\lim_{n\to\infty} G(\iota_n, \iota, \iota) = 0.$$

Now, since β is continuous, $\iota_{n+1} = \beta\iota_n \to \beta\iota$ as $n \to \infty$. From the uniqueness of limit, we obtain $\iota = \beta\iota$. Thus, β admits a fixed point. Let ι, σ be two different fixed points of β. Then,

$$\begin{aligned}\vartheta(G(\iota,\sigma,\sigma)) &= \vartheta(G(\beta\iota,\beta\sigma,\beta\sigma)) \\ &\leq \vartheta(G(\iota,\sigma,\sigma)) - \theta(\vartheta(G(\iota,\sigma,\sigma))),\end{aligned}$$

which implies that $G(\iota,\sigma,\sigma) = 0$, and hence $\iota = \sigma$. $\square$

Without the continuity assumption of β, we have the following result.

Theorem 3.3. *Let (Y, G) be a complete G-metric space and β be a self-mapping on Y. Suppose that*

$$\vartheta(G(\beta\iota,\beta\sigma,\beta\rho)) \leq \vartheta(G(\iota,\sigma,\rho)) - \theta(\vartheta(G(\iota,\sigma,\rho))), \tag{19}$$

for all $\iota,\sigma,\rho \in Y$, where $\vartheta \in \mathcal{P}$ and $\theta \in \Phi$. Then β admits a unique fixed point.

Proof. For $\iota_0 \in Y$, let $\{\iota_n\}$ be defined by $\iota_{n+1} = \beta\iota_n$ for $n \geq 0$. Arguing similar lines in Theorem 3.2, there exists $\iota \in Y$ such that $\lim_{n\to\infty} G(\iota_n,\iota,\iota) = 0$. On the other hand,

$$G(\iota,\beta\iota,\beta\iota) \leq G(\iota,\beta\iota_n,\beta\iota_n) + G(\beta\iota_n,\beta\iota,\beta\iota). \tag{20}$$

From (19), we have

$$1 \leq \vartheta(G(\beta\iota_n,\beta\iota,\beta\iota)) \leq \vartheta(G(\iota_n,\iota,\iota)) - \theta(\vartheta(G(\iota_n,\iota,\iota))). \tag{21}$$

Hence, we get that

$$\lim_{n\to\infty} \vartheta(G(\beta\iota_n,\beta\iota,\beta\iota)) = 1.$$

Thus, we have $\lim_{n\to\infty} G(\beta\iota_n,\beta\iota,\beta\iota) = 0$ which by (20), implies that $\beta\iota = \iota$. $\square$

Example 3.4. Let $Y = [2,\infty)$ and consider $G(p,q,r) = |p-q| + |q-r| + |r-p|$ for all $p,q,r \in Y$. Define $\beta : Y \to Y, \theta : [1,\infty) \to [0,\infty)$ and $\vartheta : [0,\infty) \to [1,\infty)$ by $\beta(p) = \ln(10+p), \theta(p) = \ln(p)$ and $\vartheta(p) = e^p$. Note that for all $z \geq 0$, $e^{\frac{z}{10}} \leq e^z - z$. Now, for all $p,q,r \in Y$, we have

$$\begin{aligned}\vartheta(G(\beta p,\beta q,\beta r)) &= e^{G(\beta p,\beta q,\beta r)} \\ &= e^{(|\beta p-\beta q|+|\beta q-\beta r|+|\beta r-\beta p|)} \\ &= e^{(|ln(10+p)-ln(10+q)|+|ln(10+q)-ln(10+r)|+|ln(10+r)-ln(10+p)|)} \\ &= e^{|ln(10+p)-ln(10+q)|}e^{|ln(10+q)-ln(10+r)|}e^{|ln(10+r)-ln(10+p)|} \\ &\leq e^{\frac{|p-q|}{10}}e^{\frac{|q-r|}{10}}e^{\frac{|r-p|}{10}} \\ &= e^{\frac{|p-q|+|q-r|+|r-p|}{10}} \\ &\leq e^{|p-q|+|q-r|+|r-p|} - (|p-q|+|q-r|+|r-p|) \\ &\leq e^{G(p,q,r)} - G(p,q,r) \\ &\leq \vartheta(G(p,q,r)) - \phi(\vartheta(G(p,q,r))).\end{aligned}$$

Thus, β is a weakly JS-G-contraction. All the hypotheses of Theorem 3.2 are satisfied, and so β possesses a unique fixed point, which is, $\rho = \frac{252}{100} = 2.52$.

4. Application to integral equations

Consider the following nonlinear integral equation

$$\iota(t) = \phi(t) + \int_a^b \eta(t, s, \iota(s))ds, \tag{22}$$

where $a, b \in \mathbb{R}, \iota \in C[a, b]$ (the set of continuous functions from $[a, b]$ to $\mathbb{R}$), $\phi : [a, b] \to \mathbb{R}$ and $\eta : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given functions.

Theorem 4.1. *Suppose that*

(1) $\eta : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous and there is* $\vartheta \in \mathcal{P}$ *such that* $\vartheta(\sup_{t\in[a,b]} g(t)) \leq \sup_{t\in[a,b]} \vartheta(g(t))$ *for arbitrary function* g *with*

$$\begin{aligned}&\vartheta\Big(\int_a^b \Big[|\eta(t, s, \iota(s)) - \eta(t, s, \sigma(s))| + |\eta(t, s, \sigma(s)) - \eta(t, s, \rho(s))| \\ &\quad + |\eta(t, s, \rho(s)) - \eta(t, s, \iota(s)|)\Big] ds\Big) \\ &\leq \int_a^b \vartheta\Big(|\eta(t, s, \iota(s)) - \eta(t, s, \sigma(s))| + |\eta(t, s, \sigma(s)) - \eta(t, s, \rho(s))| \\ &\quad + |\eta(t, s, \rho(s)) - \eta(t, s, \iota(s))|\Big) ds;\end{aligned}$$

(2) *there exist* $\hbar_i \in (0, 1)$ $(i \in \{1, 2, 3, 4\})$ *such that*

$$\begin{aligned}&\vartheta\Big(|\eta(t, s, \iota(s)) - \eta(t, s, \sigma(s))| + |\eta(t, s, \sigma(s)) - \eta(t, s, \rho(s))| \\ &\quad + |\eta(t, s, \rho(s)) - \eta(t, s, \iota(s))|\Big) \\ &\leq \frac{1}{b-a}[\vartheta(|\iota(t) - \sigma(t)| + |\sigma(t) - \rho(t)| + |\rho(t) - \iota(t))|]^{\hbar_1} \\ &\quad [\vartheta(|\iota(t) - \int_a^b \eta(t, s, \iota(s))ds|)]^{\hbar_2} [\vartheta(|\sigma(t) - \int_a^b \eta(t, s, \sigma(s))ds|)]^{\hbar_3} \\ &\quad [\vartheta(|\sigma(t) - \int_a^b \eta(t, s, \iota(s))ds|)]^{\hbar_4},\end{aligned}$$

for all $\iota, \sigma, \rho \in C[a, b]$, *and* $t, s \in [a, b]$. *Then (22) has a unique solution.*

Proof. Let $Y = C[a, b]$. Define G-metric G on Y by

$$G(\iota, \sigma, \rho) = \sup_{t\in[a,b]} \{|\iota(t) - \sigma(t)| + |\sigma(t) - \rho(t)| + |\rho(t) - \iota(t)|\}.$$

Note that (Y, G) is a complete G-metric space. Consider $\beta : Y \to Y$ by

$$\beta\iota(t) = \phi(t) + \int_a^b \eta(t, s, \iota(s))ds.$$

For $\iota, \sigma, \rho \in Y$ and $t \in [a,b]$, we have

$$\begin{aligned}
&\vartheta(|\beta\iota(t) - \beta\sigma(t)| + |\beta\sigma(t) - \beta\rho(t)| + |\beta\rho(t) - \beta\iota(t)|) \\
&= \vartheta\Big(|\int_a^b \eta(t,s,\iota(s))ds - \int_a^b \eta(t,s,\sigma(s))ds| \\
&\quad + |\int_a^b \eta(t,s,\sigma(s))ds - \int_a^b \eta(t,s,\rho(s))ds| \\
&\quad + |\int_a^b \eta(t,s,\rho(s))ds - \int_a^b \eta(t,s,\iota(s))ds|\Big) \\
&\leq \int_a^b \vartheta\Big(|\eta(t,s,\iota(s)) - \eta(t,s,\sigma(s))| + |\eta(t,s,\sigma(s)) - \eta(t,s,\rho(s))| \\
&\quad + |\eta(t,s,\rho(s)) - \eta(t,s,\iota(s))|\Big)ds \\
&\leq \frac{1}{b-a}\int_a^b [\vartheta(|\iota(t) - \sigma(t)| + |\sigma(t) - \rho(t)| + |\rho(t) - \iota(t)|)]^{\hbar_1} \\
&\quad [\vartheta(|\iota(t) - \int_a^b \eta(t,s,\iota(s))ds|)]^{\hbar_2}[\vartheta(|\sigma(t) - \int_a^b \eta(t,s,\sigma(s))ds|)]^{\hbar_3} \\
&\quad [\vartheta(|\sigma(t) - \int_a^b \eta(t,s,\iota(s))ds|)]^{\hbar_4}ds \\
&\leq \frac{1}{b-a}\int_a^b [\vartheta(G(\iota,\sigma,\rho))]^{\hbar_1}[\vartheta(G(\iota,\beta\iota,\beta\iota))]^{\hbar_2}[\vartheta(G(\sigma,\beta\sigma,\beta\sigma))]^{\hbar_3} \\
&\qquad [\vartheta(G(\sigma,\beta\iota,\beta\iota))]^{\hbar_4}ds \\
&= [\vartheta(G(\iota,\sigma,\rho))]^{\hbar_1}[\vartheta(G(\iota,\beta\iota,\beta\iota))]^{\hbar_2}[\vartheta(G(\sigma,\beta\sigma,\beta\sigma))]^{\hbar_3}[\vartheta(G(\sigma,\beta\iota,\beta\iota))]^{\hbar_4}.
\end{aligned}$$

Thus β is a G-$\mathcal{P}$-contraction. All the properties of Theorem 2.2 are satisfied and so β has a unique fixed point, that is, (22) possesses a unique solution. □

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