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FAST MATRIX SPLITTING ITERATION METHOD FOR THE LINEAR SYSTEM FROM SPATIAL FRACTIONAL DIFFUSION EQUATIONS

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Abstract. The spatial fractional diffusion equation can be discretized by employing the implicit finite difference scheme using the shifted Grünwald formula. The discretized linear system is obtained, whose the coefficient matrix has a diagonal-plus-Toeplitz structure. In order to solve the diagonal-plus-Toeplitz linear system, on the basis of circulant and skew-circulant splitting (CSCS splitting), we construct a new and efficient iterative method, called DSCS iterative methods, which have two parameters. Than we prove the convergence of DSCS methods. As a focus, we derive the simple and effective values of two optimal parameters under some restrictions. Some numerical experiments are carried out to illustrate the validity and accuracy of the new methods.

AMS Mathematics Subject Classification : 65H05, 65F10.
Key words and phrases : spatial fractional diffusion equation, diagonal-plus-Toeplitz structure, CSCS splitting, iterative method.

1. Introduction

We consider the numerical solution of the spatial fractional diffusion equation and the equation form is:

$$\begin{cases} d(x,t)\frac{\partial u(x,t)}{\partial t} - \frac{\partial^\beta u(x,t)}{\partial_+ x_\beta} - \frac{\partial^\beta u(x,t)}{\partial_- x_\beta} = f(x,t), \\ \\ u(x_L,t) = u(x_R,t) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

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where the function $d(x,t)$ is nonnegative , $\frac{\partial^\beta u(x,t)}{\partial_+ x_\beta}$ and $\frac{\partial^\beta u(x,t)}{\partial_- x_\beta}$ are the left and right Riemann-Liouville fractional derivatives [1,2] of order $\beta \in (1,2)$ defined by

$$\frac{\partial^\beta u(x,t)}{\partial_+ x_\beta} = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_{x_L}^{x} (x-\xi)^{1-\beta} u(\xi,t) d\xi,$$

$$\frac{\partial^\beta u(x,t)}{\partial_- x_\beta} = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_{x}^{x_{\mathrm{R}}} (\xi-x)^{1-\beta} u(\xi,t) d\xi,$$

where $\Gamma(\cdot)$ is the Gamma function,$(x,t) \in (x_L, x_R) \times (0, +\infty)$.

Recently, compared with the classical second-order diffusion equations, fractional diffusion models provide a more adequate and accurate description of memory and hereditary characteristics of anomalous transport process [3–5].Hence, fractional differential equations have received mainstream concern in many fields involving physics [6,7], chemistry [8,9], finance [10,11] and so on.There are many analytical methods have been developed to seek closed-form analytical solutions of fractional differential equations, such as the Fourier transform methods, the Laplace transform methods, and the Mellin transform method [12]. However, we usually cannot find an analytical solution for FDE closed-form solution. Therefore numerical methods have become significant ways to calculate approximate solutions efficiently and reliably.

The FDEs have the following features:

(i) Fractional differential operators are nonlocal, causing subtle stability problems on the corresponding numerical approximations

(ii)The coefficient matrices of numerical methods for FDEs are full, which is different from numerical matrices of second-order diffusion equations.

How to solve the discretized fractional diffusion equations has been analyzed and constructed by various direct methods or iterative methods in previous studies [13–18].

By a suitable finite difference discretization, the spatial FDE in Equation (1) leads to a linear system at every temporal step. The coefficient matrix of linear system is $D + T \in \mathbb{R}^{n \times n}$, where D is a diagonal matrix with nonnegative elements and T is a symmetric positive definite Toeplitz matrix. This class of linear system cannot be simply solved by Toeplitz structure on account of the addition of the diagonal matrix, although the pure Toeplitz linear system can be solved in $O(n^2)$ or even $O(n \log^2 n)$ or $O(n \log n)$,see the previous studies [19].The Krylov subspace iteration method(the conjugate gradient (CG) [20] and preconditioned conjugate gradient (PCG) [20] and so on) is a kind of common and valid solution methods for solving these linear systems [21, 22]. We can economically compute the product of the diagonal-plus-Toeplitz matrix $A = D + T$, with a vector $v \in R^n$ in $O(n \log n)$ operations by the use of the fast Fourier transform(FFT) [23]. However, these methods confront with deterioration of convergence rate and other problems [24].For better solving, Bai and Lu proposed an iterative method called DTS and the preconditioner of DTS to solve this linear system [24].Ran and Wang proposed an ADI-like method in view of

the classical ADI iteration, by which we need to solve two linear systems with Hessenberg coefficient matrices through Hessenberg LU factorization [25].Dai and Wu constructed a type of the quasi-Toeplitz splitting iteration method and developed a preconditioner by appropriating circular matrix [26]. Zhang and Zeng constructed an incomplete circulant and skew-circulant splitting iteration method, which introduces a new matrix to simplify Toeplitz-like structure [27]. Wang and Huang proposed a preconditioning method by doing equivalent transformation for the original discretized system and left multiplying a permutation matrix, then solved the equivalent system of linear equations using BiCGT and BiCRT [28].

Toeplitz matrices T can be splitted into a circulant matrix C and a skew-circulant matrix S [29] that is $T = C + S$ (expressed by CSCS splitting).On this basis, CSCS iteration method for solving Toeplitz system is given by Ng in 2003 [30]. N. Akhondi and F. Toutounian proposed an accelerated CSCS iterative method for Hermitian positive definite Toeplitz linear system, which has two parameters [31].In addition, a shifted CSCS iteration method was proposed according to the variant of such CSCS splitting by Liu and Qin [32]. In this paper, we propose a new iterative solver for the linear system of which the coefficient matrix has $D+T \in \mathbb{R}^{n\times n}$ structure, based on classical CSCS splitting.

The organization of this paper is listed below. In Section 2, the discretized linear system from spatial FDE (1) is obtained by finite difference discretization. In Section 3, we present the DSCS iteration method and establish its convergence theory. In Section 4, we derive the optimal value of the two parameters having a great impact on convergence. In Section 5, the numerical experiments are presented to prove the effectiveness of our methods. At last, we give some conclusions and remarks at the end of the paper.

2. Finite difference discretization of FDE

Let $(\cdot)^*$ indicate the transpose of either a matrix or vector, $\lambda(\cdot)$ represent eigenvalues of the matrix, and $sp(\cdot)$ indicate the spectral set of the corresponding matrix. For the spatial FDE (1),we suppose that $x_L = 0$ and $x_R = 1$. Besides, we define a spatial partition $x_i = ih(i = 0, 1, ..., n + 1)$ in the interval $[x_L, x_R]$, a temporal partition $t_\ell = \ell\Delta t(\ell = 0, 1, ..., m)$, and denote by $u_i^\ell \approx u(x_i, t_\ell)$, $d_i^\ell = d(x_i, t_\ell)$ and $f_i^\ell = f(x_i, t_\ell)$, $h = \frac{1}{n+1}$.

We employ the shifted Grünwald-Letnikov [33] approximation type to discrete the fractional spatial derivative.

$$\frac{\partial^\beta u(x,t)}{\partial_+ x_\beta} = \lim_{h\to 0} \frac{1}{h^\beta} \sum_{k=0}^{\lfloor x/h \rfloor} g_k^{(\beta)} u(x-(k-1)h, t)$$

$$\frac{\partial^\beta u(x,t)}{\partial_- x_\beta} = \lim_{h\to 0} \frac{1}{h^\beta} \sum_{k=0}^{\lfloor (1-x)/h \rfloor} g_k^{(\beta)} u(x+(k-1)h, t)$$

where $\lfloor x \rfloor$ denotes the floor of x. So we obtain the implicit finite-difference scheme of FDE (1).

$$\begin{cases} d_i^{\ell+1} \frac{u_i^{\ell+1}-u_i^{\ell}}{\Delta t} - \frac{1}{h^\beta} \sum\limits_{k=0}^{i+1} g_k^{(\beta)} u_{i-k+1}^{\ell+1} - \frac{1}{h^\beta} \sum\limits_{k=0}^{n-i+2} g_k^{(\beta)} u_{i-k+1}^{\ell+1} = f_i^{\ell+1}, \\ i = 1, 2, ..., n, \ell = 0, 1, ..., m-1, \end{cases} \tag{2}$$

where $g_k^{(\beta)}$ is given as $g_k^{(\beta)} = (-1)^k \begin{pmatrix} \beta \\ k \end{pmatrix}$, where $\begin{pmatrix} \beta \\ k \end{pmatrix}$ is the fractional binomial coefficients. It have the following properties:

$$g_k^{(\beta)} = (-1)^k \frac{\alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdot \cdots \cdot (\alpha-k+1)}{k!}$$

$$g_0^{(\beta)} = 1, g_1^{(\beta)} = -\beta, 1 \geq g_2^{(\beta)} \geq g_3^{(\beta)} \geq \cdots \geq 0,$$

$$\sum_{k=0}^{\infty} g_k^{(\beta)} = 0, \sum_{k=0}^{\ell} g_0^{(\beta)} < 0, \forall \ell \geq 1.$$

The finite difference form in (2) can be rewritten as following:

$$(D^{\ell+1} + T) u^{\ell+1} = \Delta t f^{\ell+1} + D^{\ell+1} u^{\ell}, \ell = 0, 1, ..., \mathrm{m}-1, \tag{3}$$

where $u^\ell = [u_1^\ell, u_2^\ell, ..., u_n^\ell]^*$ and $f^\ell [f_1^\ell, f_2^\ell, ..., f_n^\ell]^*$ at the $(\ell+1) - th$ temporal level , where $D^{\ell+1} = diag(d_1^{\ell+1}, d_2^{\ell+1}, ..., d_n^{\ell+1})$ is a diagonal matrix and $T = \frac{\Delta t}{h^\beta}(T_\beta + T_\beta^*)$ is a Toeplitz matrix , with

$$T_\beta = - \begin{pmatrix} g_1^{(\beta)} & g_0^{(\beta)} & 0 & \cdots & 0 & 0 \\ g_2^{(\beta)} & g_1^{(\beta)} & g_0^{(\beta)} & \ddots & \ddots & 0 \\ \vdots & g_2^{(\beta)} & g_1^{(\beta)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ g_{n-1}^{(\beta)} & \ddots & \ddots & \ddots & g_1^{(\beta)} & g_0^{(\beta)} \\ g_n^{(\beta)} & g_{n-1}^{(\beta)} & \cdots & \cdots & g_2^{(\beta)} & g_1^{(\beta)} \end{pmatrix}$$

Consequently, the FDE transforms into a discretized linear system (3), with the coefficient matrix $A^{\ell+1} := D^{\ell+1} + T$ being a diagonal-plus-Toeplitz matrix at each temporal step.

Because of the decay property of $g_k^{(\beta)}$, Toeplitz matrix T is a strictly diagonally dominant M-matrix [34]. Therefore T is a symmetric positive definite matrix.

Next, we know that the Toeplitz linear system can be solved by the CSCS iterative method. The theoretical analysis proves that when both the circulant matrix and the skew-circulant matrix are positive definite matrices, the CSCS iterative method converges to the solution of the Toeplitz linear system [30].

The CSCS iterative method [30] is based on a splitting (C, S) of T , i.e. $T = C + S$, where T is a Toeplitz matrix, C is a circulant matrix and S is a

skew-circulant matrix.

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \ddots & t_{2-n} \\ \vdots & t_1 & \ddots & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix},$$

$$C = \frac{1}{2}\begin{pmatrix} t_0 & t_{-1}+t_{n-1} & \cdots & t_{2-n}+t_2 & t_{1-n}+t_1 \\ t_1+t_{1-n} & t_0 & t_{-1}+t_{n-1} & \ddots & t_{2-n}+t_2 \\ \vdots & t_1+t_{1-n} & \ddots & \ddots & \vdots \\ t_{n-2}+t_{-2} & \ddots & \ddots & \ddots & t_{-1}+t_{n-1} \\ t_{n-1}+t_{-1} & t_{n-2}+t_{-2} & \cdots & t_1+t_{1-n} & t_0 \end{pmatrix},$$

$$S = \frac{1}{2}\begin{pmatrix} t_0 & t_{-1}-t_{n-1} & \cdots & t_{2-n}-t_2 & t_{1-n}-t_1 \\ t_1-t_{1-n} & t_0 & t_{-1}-t_{n-1} & \ddots & t_{2-n}-t_2 \\ \vdots & t_1-t_{1-n} & \ddots & \ddots & \vdots \\ t_{n-2}-t_{-2} & \ddots & \ddots & \ddots & t_{-1}-t_{n-1} \\ t_{n-1}-t_{-1} & t_{n-2}-t_{-2} & \cdots & t_1-t_{1-n} & t_0 \end{pmatrix}.$$

The CSCS iteration method [30] :

Given an initial guess $x^{(0)}$, The sequence $\{x^{(k)}\}$ for $k = 0, 1, 2\cdots$ until converges, compute

$$\begin{cases} (\alpha I + C)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b \\ (\alpha I + S)x^{(k+1)} = (\alpha I - C)x^{(k+\frac{1}{2})} + b \end{cases}$$

where α is positive constant.

According to the CSCS iteration method, the matrix T generated by the discretization of FDEs can be split into the circulant matrix C and the skew-circulant matrix S. In this paper, the Toeplitz matrix $T = \frac{\Delta t}{h^\beta}(T_\beta + T_\beta^*)$ is a real symmetric positive definite matrix. In the light of the $g_k^{(\beta)}$ property, we can draw the conclusion that the circulant matrix C and the skew-circulant matrix S are the real symmetric positive definite matrices.

3. The DSCS iteration method

Based on the above knowledge, we propose a double-split iterative method in order to solve Hermitian positive definite linear systems:

$$Au = b, A \in \mathbb{C}^{n\times n}, b \in \mathbb{C}^n \tag{4}$$

where $A = D + T$, $D \in \mathbb{C}^{n\times n}$ is a diagonal matrix which the diagonal elements are non-negative, $T \in \mathbb{C}^{n\times n}$ is Hermitian positive definite Toeplitz matrix.

The coefficient matrix A from the linear system is split two times. Firstly, we introduce a splitting parameter ω , then the diagonal matrix is split into ωD and $(1-\omega)D$. Secondly, the Toeplitz matrix performs the CSCS splitting. That is, the split format is as follows:

$$\begin{aligned} A &= D + T = \omega D + (1-\omega)D + C + S \\ &= \omega D + C + (1-\omega)D + S \\ &= C_\omega + S_{1-\omega} \end{aligned} \tag{5}$$

where $C_\omega = \omega D + C, S_{1-\omega} = (1-\omega)D + S$, C, S are respectively the circulant matrix and the skew-circulant matrix from CSCS splitting.

We introduce a positive parameter α and apply the ADI format to process the coefficient matrix. Thus we construct the following two-step split iterative framework:

$$\begin{cases} (\alpha I + C_\omega)u = (\alpha I - S_{1-\omega})u + b \\ (\alpha I + S_{1-\omega})u = (\alpha I - C_\omega)u + b \end{cases} \tag{6}$$

Therefore we get the DSCS iteration method.

3.1. The DSCS Iteration Method. Given an initial guess $u^{(0)} \in \mathbb{C}^n$, calculate the next iteration vector $u^{(k+1)} \in \mathbb{C}^n$ by the following format

$$\begin{cases} (\alpha I + C_\omega)u^{(k+\frac{1}{2})} = (\alpha I - S_{1-\omega})u^{(k)} + b \\ (\alpha I + S_{1-\omega})u^{(k+1)} = (\alpha I - C_\omega)u^{(k+\frac{1}{2})} + b \end{cases}$$

until the vector sequence $\left\{u^{(k)}\right\} \in \mathbb{C}^n (k = 1, 2, 3...)$ converges, where α is a positive constant.

By direct calculation, this two-step iterative equation can be transformed into a standard equation:

$$M(\alpha)u^{(k+1)} = N(\alpha)u^{(k)} + b \tag{7}$$

where $M(\alpha) = \frac{1}{2\alpha}(\alpha I + C_\omega)(\alpha I + S_{1-\omega})$, $N(\alpha) = \frac{1}{2\alpha}(\alpha I - C_\omega)(\alpha I - S_{1-\omega})$

In fact, we give the new matrix splitting form for the coefficient matrix in (3), that is, $A = M(\alpha) - N(\alpha)$, called as DSCS splitting. The iteration matrix of the DSCS iteration method is:

$$L(\alpha) = M(\alpha)^{-1}N(\alpha) = (\alpha I + S_{1-\omega})^{-1}(\alpha I + C_\omega)^{-1}(\alpha I - C_\omega)(\alpha I - S_{1-\omega}) \tag{8}$$

According to previous studies [33,34], We conclude that the necessary and sufficient condition for the convergence of the DSCS iterative method is the spectral radius $\rho(L(\alpha))$ of its iteration matrix $L(\alpha)$ (Equation 8) is less than 1, that is, $\rho(L(\alpha)) < 1$.

The following theorem demonstrates that the DSCS iteration method is unconditional convergence.

3.2. DSCS convergence analysis.

Theorem 3.1. *: Let*

$$A = D + T = D + C + S = (\omega D + C) + ((1-\omega)D + S) = C_\omega + S_{1-\omega} \in C^{n\times n}$$

where $\omega \in [0,1], D \in \mathbb{C}^{n\times n}$ is a diagonal matrix which the elements are nonnegative and $T \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite Toeplitz matrix, $C \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ are respectively the circulant matrix and the skew-circulant matrix through CSCS splitting of Toeplitz matrix T. The iteration matrix of DSCS method is:

$$L(\alpha) = (\alpha I + S_{1-\omega})^{-1}(\alpha I + C_\omega)^{-1}(\alpha I - C_\omega)(\alpha I - S_{1-\omega})$$

Then the DSCS iteration method can unconditionally converge to the exact solution of $Ax = b$.

If the spectral radius of the iteration matrix $L(\alpha)$ denotes $\rho(L(\alpha))$, we obtain $\rho(L(\alpha)) \leq \sigma(\alpha) < 1$ for any positive constant α and initial guess $u^{(0)} \in \mathbb{C}^n$, where

$$\sigma(\alpha) = \max_{\xi_j \in \lambda(C_\omega)} \left\{ \frac{|\alpha - \xi|}{\alpha + \xi} \right\} \cdot \max_{\eta_j \in \lambda(S_{1-\omega})} \left\{ \frac{|\alpha - \eta|}{\alpha + \eta} \right\}$$

Proof. By the similar transformation of the matrix, the iteration matrix $L(\alpha)$ of the DSCS iterative method is transformed into the matrix

$$\hat{L}(\alpha) = (\alpha I + C_\omega)^{-1}(\alpha I - C_\omega)(\alpha I - S_{1-\omega})(\alpha I + S_{1-\omega})^{-1}$$

According to the relationship between the matrix 2-norm and the spectral radius, it holds that

$$\begin{aligned}
&\rho(L(\alpha)) = \rho(\hat{L}(\alpha)) \\
&\leq \left\| (\alpha I + C_\omega)^{-1}(\alpha I - C_\omega)(\alpha I - S_{1-\omega})(\alpha I + S_{1-\omega})^{-1} \right\|_2 \\
&\leq \left\| (\alpha I + C_\omega)^{-1}(\alpha I - C_\omega) \right\|_2 \left\| (\alpha I - S_{1-\omega})(\alpha I + S_{1-\omega})^{-1} \right\|_2 \\
&= \max_{\xi_j \in \lambda(C_\omega)} \left\{ \tfrac{|\alpha - \xi|}{\alpha + \xi} \right\} \cdot \max_{\eta_j \in \lambda(S_{1-\omega})} \left\{ \tfrac{|\alpha - \eta|}{\alpha + \eta} \right\} \\
&= \sigma(\alpha)
\end{aligned}$$

Because D is a positive semidefinite diagonal matrix, and the circulant matrix C and the skew-circulant matrix S are Hermitian positive definite. Consequently, the eigenvalues of $C_\omega \in \mathbb{C}^{n\times n}$ and $S_{1-\omega} \in \mathbb{C}^{n\times n}$ are positive, which readily implies $\sigma(\alpha) < 1$ for any positive constant α. □

The DSCS method divides the diagonal matrix D into ωD and $(1-\omega)D$, splits the Toeplitz matrix into the circulant matrix C and the skew-circulant matrix S, which introduces an unknown parameter ω to make adjustments. As we all know, the parameters have an important impact on the iterative method. Next we will discuss how the two parameters α, ω affect the spectral radius of $L(\alpha)$ in the DSCS iteration method.

4. The optimal α and ω

As mentioned above, the DSCS iteration method has two parameters and how to choose them has a huge impact on convergence rate.

Firstly, we discuss the optimal choice of the parameter α .

Theorem 4.1. *Let*

$A = D+T = \omega D+(1-\omega)D+C+S = (\omega D+C)+((1-\omega)D+S) = C_\omega + S_{1-\omega}$

$\omega \in [0,1]$, $\xi_{\min}$ *and* $\xi_{\max}$ *are the smallest and the largest eigenvalues of* C_ω, $\eta_{\min}$ *and* $\eta_{\max}$ *are the smallest and the largest eigenvalues of* $S_{1-\omega}$. *Let*

$$\xi_* \triangleq \sqrt{\xi_{\max}\xi_{\min}}, \eta_* \triangleq \sqrt{\eta_{\min}\eta_{\max}}$$

then the optimal parameter α_* *which minimizes the function* $\sigma(\alpha)$ *satisfies the following conclusions:*

(I) If $\xi_* = \eta_*$ *,then*

$$\alpha_* = \xi_* = \eta_*$$

$$\min_\alpha \sigma(\alpha) = \sigma(\alpha_*) = \frac{\sqrt{\xi_{\max}} - \sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}} + \sqrt{\xi_{\min}}} \frac{\sqrt{\eta_{\max}} - \sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}} + \sqrt{\eta_{\min}}}$$

(II) If $\xi_* < \eta_*$ *:*

(1) For $\xi_{\min} \geq \eta_{\min}$, *or for* $\xi_{\max} < \eta_{\max}, \xi_{\min} < \eta_{\min}$, $\frac{\xi_{\max}+\eta_{\max}}{\xi_{\min}+\eta_{\min}} \geq \sqrt{\frac{\xi_{\max}\eta_{\max}}{\xi_{\min}\eta_{\min}}}$. *we conclude that* $\alpha_* = \xi_*$ *and*

$$\min_\alpha \sigma(\alpha) = \sigma(\xi_*) = \frac{\eta_{\max} - \xi_*}{\eta_{\max} + \xi_*} \frac{\sqrt{\xi_{\max}} - \sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}} + \sqrt{\xi_{\min}}}$$

(2) For $\xi_{\max} \geq \eta_{\max}$, *or for* $\xi_{\max} < \eta_{\max}, \xi_{\min} < \eta_{\min}$, $\frac{\xi_{\min}+\eta_{\min}}{\xi_{\max}+\eta_{\max}} > \sqrt{\frac{\xi_{\min}\eta_{\min}}{\xi_{\max}\eta_{\max}}}$. *we conclude that* $\alpha_* = \eta_*$ *and*

$$\min_\alpha \sigma(\alpha) = \sigma(\eta_*) = \frac{\eta_* - \xi_{\min}}{\eta_* + \xi_{\min}} \frac{\sqrt{\eta_{\max}} - \sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}} + \sqrt{\eta_{\min}}}$$

(III) If $\xi_* > \eta_*$ *:*

(1) For $\xi_{\max} \leq \eta_{\max}$, *or for* $\xi_{\max} > \eta_{\max}, \xi_{\min} > \eta_{\min}$, $\frac{\xi_{\min}+\eta_{\min}}{\xi_{\max}+\eta_{\max}} \geq \sqrt{\frac{\xi_{\min}\eta_{\min}}{\xi_{\max}\eta_{\max}}}$. *we conclude that* $\alpha_* = \xi_*$ *and*

$$\min_\alpha \sigma(\alpha) = \sigma(\xi_*) = \frac{\xi_* - \eta_{\min}}{\xi_* + \eta_{\min}} \frac{\sqrt{\xi_{\max}} - \sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}} + \sqrt{\xi_{\min}}}$$

(2) For $\xi_{\min} \leq \eta_{\min}$, *or for* $\xi_{\max} > \eta_{\max}, \xi_{\min} > \eta_{\min}$, $\frac{\xi_{\max}+\eta_{\max}}{\xi_{\min}+\eta_{\min}} > \sqrt{\frac{\xi_{\max}\eta_{\max}}{\xi_{\min}\eta_{\min}}}$. *we conclude that* $\alpha_* = \eta_*$ *and*

$$\min_\alpha \sigma(\alpha) = \sigma(\eta_*) = \frac{\xi_{\max} - \eta_*}{\xi_{\max} + \eta_*} \frac{\sqrt{\eta_{\max}} - \sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}} + \sqrt{\eta_{\min}}}$$

Proof. Because the unary function $f(x) = \frac{\alpha - x}{\alpha + x} (x \in [0, \infty))$ s monotonically decreasing, then

$$\begin{aligned} \sigma_{C_\omega}(\alpha) &= \max_{\xi_j \in \lambda(C_\omega)} \left\{ \frac{|\alpha-\xi|}{\alpha+\xi} \right\} = \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} \left\{ \frac{|\alpha-\xi|}{\alpha+\xi} \right\} \\ &= \max \left\{ \frac{|\alpha-\xi_{\min}|}{\alpha+\xi_{\min}}, \frac{|\alpha-\xi_{\max}|}{\alpha+\xi_{\max}} \right\}, \end{aligned}$$

$$\sigma_{S_{1-\omega}}(\alpha) = \max_{\eta_j \in \lambda(S_{1-\omega})} \left\{ \frac{|\alpha-\eta|}{\alpha+\eta} \right\} = \max_{\eta_{\min} \le \eta \le \eta_{\max}} \left\{ \frac{|\alpha-\eta|}{\alpha+\eta} \right\}$$
$$= \max \left\{ \frac{|\alpha-\eta_{\min}|}{\alpha+\eta_{\min}}, \frac{|\alpha-\eta_{\max}|}{\alpha+\eta_{\max}} \right\}.$$

By straightforward computing, we have some conclusions that

$$\min \sigma_{C_\omega}(\alpha) = \sigma_{C_\omega}(\xi_*) = \frac{\sqrt{\xi_{\max}}-\sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}}+\sqrt{\xi_{\min}}}, \xi_* = \sqrt{\xi_{\max}\xi_{\min}},$$
$$\min \sigma_{S_{1-\omega}}(\alpha) = \sigma_{S_{1-\omega}}(\eta_*) = \frac{\sqrt{\eta_{\max}}-\sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}}+\sqrt{\eta_{\min}}}, \eta_* = \sqrt{\eta_{\max}\eta_{\min}}.$$

Next we distinguish the values of ξ_* and η_* , so that we get the optimal parameter α_*.

1. If $\xi_* = \eta_*$, then

$$\min_\alpha \sigma(\alpha) = \min_\alpha \left\{ \sigma_{C_\omega}(\alpha) \cdot \sigma_{S_{1-\omega}}(\alpha) \right\}$$
$$= \sigma(\xi_*) = \sigma(\eta_*)$$
$$= \frac{\sqrt{\xi_{\max}}-\sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}}+\sqrt{\xi_{\min}}} \frac{\sqrt{\eta_{\max}}-\sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}}+\sqrt{\eta_{\min}}}$$

2. If $\xi_* < \eta_*$, then$\xi_{\min} = \eta_{\max}$, Hence, we can obtain the function $\sigma(\alpha)$:

$$\sigma(\alpha) = \begin{cases} \frac{\xi_{\max}-\alpha}{\xi_{\max}+\alpha} \frac{\eta_{\max}-\alpha}{\eta_{\max}+\alpha}, \alpha < \xi_* \\ \frac{\alpha-\xi_{\min}}{\alpha+\xi_{\min}} \frac{\eta_{\max}-\alpha}{\eta_{\max}+\alpha}, \xi_* < \alpha < \eta_* \\ \frac{\alpha-\xi_{\min}}{\alpha+\xi_{\min}} \frac{\alpha-\eta_{\min}}{\alpha+\eta_{\min}}, \alpha > \eta_* \end{cases}$$

The function $\sigma(\alpha)$ is continuous. By calculation, we know that $\sigma(\alpha)$ is monotonically decreasing when $\alpha < \xi_*$, and $\sigma(\alpha)$ is monotonically increasing when $\alpha > \eta_*$. We need to discuss the monotonicity of the function $\sigma(\alpha)$ when $\xi_* < \alpha < \eta_*$.

i). If $\sqrt{\xi_{\min}\eta_{\max}} \le \xi_*, i.e., \eta_{\max} \le \xi_{\max}$, then $\sigma(\alpha)$ is monotonically decreasing in $[\xi_*, \eta_*]$, and $\min_\alpha \sigma(\alpha) = \sigma(\eta_*)$

ii). If $\sqrt{\xi_{\min}\eta_{\max}} \ge \eta_*, i.e., \xi_{\min} \ge \eta_{\min}$, then $\sigma(\alpha)$ is monotonically increasing in $[\xi_*, \eta_*]$, and $\min_\alpha \sigma(\alpha) = \sigma(\xi_*)$

iii). If $\xi_* < \sqrt{\xi_{\min}\eta_{\max}} < \eta_*, i.e., \xi_{\max} < \eta_{\max} and \xi_{\min} < \eta_{\min}$, then

$$\min_\alpha \sigma(\alpha) = \min\{\sigma(\xi_*), \sigma(\eta_*)\}$$

where

$$\sigma(\xi_*) = \frac{\eta_{\max}-\xi_*}{\eta_{\max}+\xi_*} \frac{\sqrt{\xi_{\max}}-\sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}}+\sqrt{\xi_{\min}}},$$
$$\sigma(\eta_*) = \frac{\eta_*-\xi_{\min}}{\eta_*+\xi_{\min}} \frac{\sqrt{\eta_{\max}}-\sqrt{\eta_{\min}}}{\sqrt{\eta_{\max}}+\sqrt{\eta_{\min}}}.$$

The inequality $\sigma(\xi_*) < \sigma(\eta_*)$ holds if and only if

$$\frac{\xi_{\max}+\eta_{\max}}{\xi_{\min}+\eta_{\min}} > \sqrt{\frac{\xi_{\max}\eta_{\max}}{\xi_{\min}\eta_{\min}}}.$$

In summary, when $\xi_* < \eta_*$, the arguments about the optimal and show that the conclusions in theorem 4.1(II) are valid.

Similarly, when $\xi_* > \eta_*$, we can get similar conclusions. □

Lemma 4.2 ([32]). *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices, the eigenvalues of $A, B, A+B$ are $\lambda_i(A), \lambda_i(B), \lambda_i(A+B)$ (in ascending order). For each $i = 1, 2..., n$, we have*

$$1.\lambda_i(A) + \lambda_1(B) \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_n(B)$$
$$2.\lambda_1(A) + \lambda_i(B) \le \lambda_i(A+B) \le \lambda_n(A) + \lambda_i(B)$$

We employ two parameters α, ω in DSCS iteration method, and we get the optimal parameter α that $\min\limits_{\alpha} \sigma(\alpha)$ in the discussion above. Therefore we try to find an estimate of another optimal parameter ω to accelerate the convergence of DSCS, which is hard to get actually. DSCS iteration method splits the coefficient matrix into the form $A = D + C + S = (\omega D + C) + ((1-\omega)D + S) = C_\omega + S_{1-\omega}$ Due to the special structure of the matrix and the actual situation of equation (1), we make three reasonable assumptions for the convenience of calculation.

1. The eigenvalues of circulant matrix C and skew-circulant matrix S belong to $\Omega = [\lambda_{\min}, \lambda_{\max}]$

2. The eigenvalues of diagonal matrix D which are diagonal elements d_i are much larger than the eigenvalues of the circulant matrix C and the skew-circulant matrix S, that is $d_{\min} \ge \lambda_{\max}$.

3. In the base of Lemma 4.2 and Hypothesis 2, we can assume the smallest and largest eigenvalues of C_ω are defined as:

$\xi_{\min} = \omega d_{\min} + \lambda_{\min}(C) = \omega d_{\min} + \lambda_{\min}; \xi_{\max} = \omega d_{\max} + \lambda_{\max}(C) = \omega d_{\max} + \lambda_{\max}$
Similarly, the smallest and largest eigenvalues of $S_{1-\omega}$ are defined as:

$$\eta_{\min} = (1-\omega)d_{\min} + \lambda_{\min}; \eta_{\max} = (1-\omega)d_{\max} + \lambda_{\max}$$

Theorem 4.3. *Under the above three assumptions, the upper bound function $\sigma(\omega)$ of spectral radius of DSCS iterative method is*

$$\sigma(\omega) = \max_{\xi_j \in \lambda(C_\omega)} \left\{ \frac{|\alpha - \xi|}{\alpha + \xi} \right\} \cdot \max_{\eta_j \in \lambda(S_{1-\omega})} \left\{ \frac{|\alpha - \eta|}{\alpha + \eta} \right\},$$

where $\omega \in (0,1)$. $\xi_j \in \lambda(C_\omega)$ and $\eta_j \in \lambda(S_{1-\omega})$ are related to parameter ω. When

$$\phi = \frac{|\xi_* - (1-\omega)d_{\max} - \lambda_{\max}|}{\xi_* + (1-\omega)d_{\max} + \lambda_{\max}} = 0 \text{ or } \varphi = \frac{|\xi_* - (1-\omega)d_{\min} - \lambda_{\min}|}{\xi_* + (1-\omega)d_{\min} + \lambda_{\min}} = 0$$

the optimal parameter ω_* which minimizes the function $\sigma(\omega)$ is as following:

$$\omega_* = \frac{\sqrt{d_{\max}}}{\sqrt{d_{\max}} + \sqrt{d_{\min}}} \text{ or } \omega_* = \frac{\sqrt{d_{\min}}}{\sqrt{d_{\max}} + \sqrt{d_{\min}}}$$

Proof.

$$\sigma(\omega) = \max_{\xi_j \in \lambda(C_\omega)} \left\{ \frac{|\alpha-\xi|}{\alpha+\xi} \right\} \cdot \max_{\eta_j \in \lambda(S_{1-\omega})} \left\{ \frac{|\alpha-\eta|}{\alpha+\eta} \right\}$$
$$= \max \left\{ \left(\frac{|\alpha-\xi_{\max}|}{\alpha+\xi_{\max}}, \frac{|\alpha-\xi_{\min}|}{\alpha+\xi_{\min}} \right) \times \left(\frac{|\alpha-\eta_{\max}|}{\alpha+\eta_{\max}}, \frac{|\alpha-\eta_{\min}|}{\alpha+\eta_{\min}} \right) \right\}$$

where $\omega \in (0,1)$, $\xi_j \in \lambda(C_\omega)$, $\eta_j \in \lambda(S_{1-\omega})$, $\xi_{\min}$, $\xi_{\max}$, $\eta_{\min}$ and $\eta_{\max}$ are defined in assumption 3.

Theorem 4.1 shows that there are only two choice of the parameter α. This means that $\alpha_* = \xi_* = \sqrt{\xi_{\max}\xi_{\min}}$ or $\alpha_* = \eta_* = \sqrt{\eta_{\max}\eta_{\min}}$. Therefore, we analysis the parameter ω in two cases.

(I) If $\alpha_* = \xi_* = \sqrt{\xi_{\min}\xi_{\max}}$, then

$$\begin{aligned}\sigma(\omega) &= \frac{\sqrt{\xi_{\max}}-\sqrt{\xi_{\min}}}{\sqrt{\xi_{\max}}+\sqrt{\xi_{\min}}} \cdot \left\{\max\left(\frac{|\alpha-\eta_{\max}|}{\alpha+\eta_{\max}}, \frac{|\alpha-\eta_{\min}|}{\alpha+\eta_{\min}}\right)\right\} \\ &= \frac{\sqrt{\omega d_{\max}+\lambda_{\max}}-\sqrt{\omega d_{\min}+\lambda_{\min}}}{\sqrt{\omega d_{\max}+\lambda_{\max}}+\sqrt{\omega d_{\min}+\lambda_{\min}}} \cdot \left\{\max\left(\frac{|\xi_*-(1-\omega)d_{\max}-\lambda_{\max}|}{\xi_*+(1-\omega)d_{\max}+\lambda_{\max}}, \frac{|\xi_*-(1-\omega)d_{\min}-\lambda_{\min}|}{\xi_*+(1-\omega)d_{\min}+\lambda_{\min}}\right)\right\} \\ &= \frac{\sqrt{\omega d_{\max}+\lambda_{\max}}-\sqrt{\omega d_{\min}+\lambda_{\min}}}{\sqrt{\omega d_{\max}+\lambda_{\max}}+\sqrt{\omega d_{\min}+\lambda_{\min}}} \cdot \{\max(\phi,\varphi)\}\end{aligned}$$

where $\phi = \frac{|\xi_*-(1-\omega)d_{\max}-\lambda_{\max}|}{\xi_*+(1-\omega)d_{\max}+\lambda_{\max}}$, $\varphi = \frac{|\xi_*-(1-\omega)d_{\min}-\lambda_{\min}|}{\xi_*+(1-\omega)d_{\min}+\lambda_{\min}}$

Because φ is a monotonically increasing function on the parameter ω for $\omega \in (0,1)$, we know the function φ and ϕ are equal when $\omega = 0.5$. The function $\max(\phi,\varphi)$ must take one choice from the function φ or ϕ when the parameter $\omega \neq 0.5$. We distinguish the function of φ and ϕ:

1. $\sigma(\omega) = \frac{\sqrt{\omega d_{\max}+\lambda_{\max}}-\sqrt{\omega d_{\min}+\lambda_{\min}}}{\sqrt{\omega d_{\max}+\lambda_{\max}}+\sqrt{\omega d_{\min}+\lambda_{\min}}} \frac{|\xi_*-(1-\omega)d_{\max}-\lambda_{\max}|}{\xi_*+(1-\omega)d_{\max}+\lambda_{\max}}$

No matter how the parameter ω is selected in the interval $(0,1)$, $\xi_* + (1-\omega)d_{\max}+\lambda_{\max}$, $\frac{\sqrt{\omega d_{\max}+\lambda_{\max}}-\sqrt{\omega d_{\min}+\lambda_{\min}}}{\sqrt{\omega d_{\max}+\lambda_{\max}}+\sqrt{\omega d_{\min}+\lambda_{\min}}}$ and $|\xi_* - (1-\omega)d_{\max} - \lambda_{\max}|$ are positive. If the function $\sigma(\omega)$ can get the minimum, there can only be one situation $\sigma(\omega) = 0$, that is:

$$\xi_* - (1-\omega)d_{\max} - \lambda_{\max} = 0,$$

where $\xi_* = \sqrt{(\omega d_{\max} + \lambda_{\max})(\omega d_{\min} + \lambda_{\min})}$, $\omega \in (0,1)$.

Because $d_{\min} \gg \lambda_{\max}$, the optimal value of ω is as following:

$$\omega_* = \frac{\sqrt{d_{\max}}}{\sqrt{d_{\max}} + \sqrt{d_{\min}}}$$

2. $\sigma(\omega) = \frac{\sqrt{\omega d_{\max}+\lambda_{\max}}-\sqrt{\omega d_{\min}+\lambda_{\min}}}{\sqrt{\omega d_{\max}+\lambda_{\max}}+\sqrt{\omega d_{\min}+\lambda_{\min}}} \frac{|\xi_*-(1-\omega)d_{\min}-\lambda_{\min}|}{\xi_*+(1-\omega)d_{\min}+\lambda_{\min}}$

Similarly, if the minimum of the function $\sigma(\omega)$ can be obtained, there only is one situation such that $\sigma(\omega) = 0$. So we have

$$\xi_* - (1-\omega)d_{\min} - \lambda_{\min} = 0$$

where $\xi_* = \sqrt{(\omega d_{\max} + \lambda_{\max})(\omega d_{\min} + \lambda_{\min})}$, $\omega \in (0,1)$.

Because $d_{\min} \gg \lambda_{\max}$, the optimal value of ω is as following:

$$\omega_* = \frac{\sqrt{d_{\min}}}{\sqrt{d_{\max}} + \sqrt{d_{\min}}}$$

Similarly, we can get the conclusions when $\alpha_* = \eta_* = \sqrt{\eta_{\max}\eta_{\min}}$.

In summary, we obtain the optimal parameter ω_* when $\varphi = 0$. However $\varphi > \phi$ in actual experiment, ω_* is not real theoretical optimal parameter. In fact, ω_* is infinitely close to the optimal numerical parameter as the matrix order n increases. So we consider ω_* is the better estimated value of the theoretical optimal value. □

5. Numerical examples

In this section, we prove that DSCS methods are more effective than HHS methods, DTS methods, and the Krylov subspace iteration methods like CG. We consider CPU time (in seconds) for solving the linear system (record as CPU) and the iteration steps (record as IT).

We start experiments from the initial vectors $u^{(0)} = 0$, when the relative residual error meets $\left\|b - Au^{(k)}\right\| \leq 10^{-5} \|b\|$, or the number of iteration steps is greater than 10000, the iteration terminates. All numerical experiments are performed in MATLAB (version R2016b) on a personal computer with Intel(R) Core(TM) i5-7200U CPU @ 2.50 GHz 2.70 GHz, 4.00 GB memory and Windows 10 operating system. Actually, we consider the solution to the discretized linear system at the initial time level.

$$A = D^1 + T, b = \Delta t f^1 + D^1 u^0$$

where $\Delta t = h, f = 0$.

Example 5.1 We consider the spatial fractional diffusion equation (1) defined on the spatial domain $(0, 1)$ and time field $(0, +\infty)$, in which $u_0 = x^2(1 - x)$, and $d(x, t) = \frac{1}{x^2(1-x)^2}$ [24].

Table 1: The numerical results of Example 5.1 when $\beta = 1.2$

Method	Index	2^6	2^7	2^8	2^9	2^10	2^11	2^12
HSS	IT	25	36	50	70	99	139	197
	CPU	0.0029	0.0062	0.0169	0.1034	0.6049	2.8372	14.8313
DTS	IT	25	36	50	70	99	139	197
	CPU	0.0053	0.0084	0.0204	0.1121	0.614	3.0424	15.6342
CG	IT	22	33	48	69	98	148	212
	CPU	0.0017	0.0045	0.0073	0.0344	0.1578	0.9123	4.6492
DSCS	IT	11	13	16	20	25	30	37
	CPU	0.0009	0.0017	0.0046	0.0195	0.1476	0.6371	3.0924

In all experiments, the HSS, DTS, and DSCS methods use theoretically optimal iterative parameter values. In particular, the optimal parameter of the HSS is selected $\sqrt{\lambda_{\max} * \lambda_{\min}}$ [35], where $\lambda \in \rho(H)$, the value of optimal parameter for the DTS methods is $\sqrt{d_{\max} * d_{\min}}$ [22], where $d \in \rho(D)$, the optimal parameters α_*, ω_* of DSCS are defined in section 4.

In the Table 1 and Table 2, we list the number of iteration steps (express as IT) and the computing time (express as IT) of the methods applied to the experiments. From the above tables, we notice that DSCS significantly outperform than HSS, DTS, and CG in the field of iteration steps as well as CPU time.

Table 2: The numerical results of Example 5.1 when $\beta = 1.8$

Method	Index	2^6	2^7	2^8	2^9	2^10	2^11	2^12
HSS	IT	26	36	50	70	99	140	197
	CPU	0.0021	0.0054	0.0153	0.1158	0.5899	2.7871	15.3897
DTS	IT	25	36	50	70	99	139	197
	CPU	0.006	0.0098	0.0155	0.1149	0.6076	2.7776	15.5034
CG	IT	23	34	48	68	100	148	217
	CPU	0.0055	0.006	0.0117	0.0374	0.198	0.8846	4.3161
DSCS	IT	13	16	20	24	30	36	44
	CPU	0.0014	0.0021	0.0061	0.035	0.1817	0.7282	3.4438

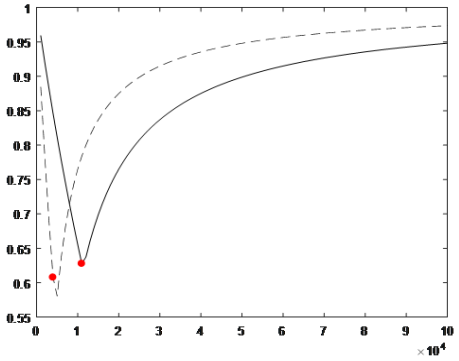
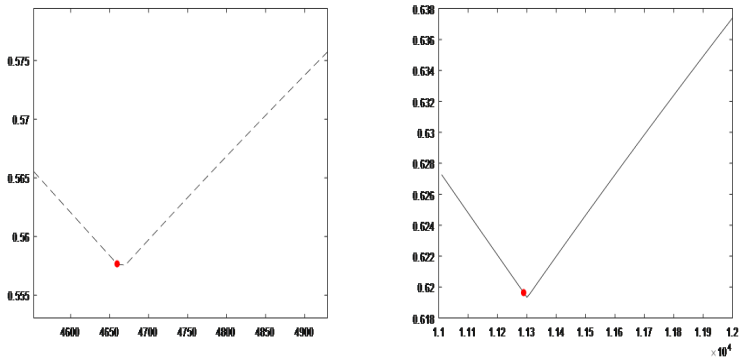


Figure 1: The curves of $\rho(L(\alpha))$ when $\beta = 1.2$

Figure.1 and Figure.2 depict the spectral radius curve of $\rho(L(\alpha))$ for the parameter α with $n = 2^9$ ("–" curve) and $n = 2^{10}$ ("-" curve), respectively. And the theoretical optimal parameters α_* of DSCS iterative method is marked by the point "•". From the Figure.1, we see that the theoretically optimal value α_* almost coincides with the experimentally optimal value α_{exp} because the abscissa axis of parameter α span is too large. Therefore we intercept the interval which contains the optimal α and narrow the span of abscissa axis, so that we can observe more clearly in Figure.2. The case of $n = 2^9$ is the left figure and another case of $n = 2^{10}$ is the right figure in the Figure.2. The results of Figure.2 indicates that the theoretical optimal value is close to the experimental optimal value, proving that our estimated parameters are valid.



(a) (b)

Figure 2: The interval near α_* from Figure.1

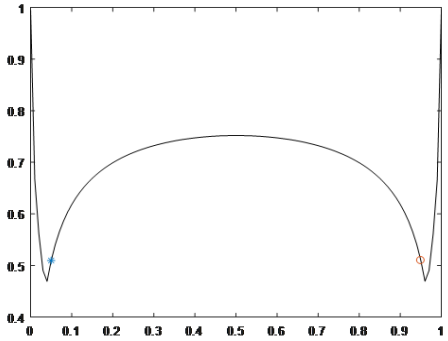


Figure 3: The curves of $\rho(L_{DCCS}(\omega))$ when $\beta = 1.2$

Figure.3 depict the spectral radius curve of $\rho(L_{DCCS}(\omega))$ for the parameter ω when $\beta = 1.2$ and $n = 2^9$. And the one optimal parameter ω_* of DSCS iteration method is marked by the symbol "$*$", the other optimal parameter ω_* is marked by "$\circ$". Obviously, these two optimal parameters look symmetrical as shown on the Figure.3. Actually, under the assumptions in section 4, they have almost the same effect on the optimization of the spectral radius $\rho(L_{DCCS}(\omega))$ because we split the diagonal matrix D into ωD and $(1-\omega)D$ in the estimation process of ω_*.

Example 5.1 We consider the spatial fractional diffusion equation (1) defined on the spatial domain $(0,1)$ and time field $(0,+\infty)$, in which $u_0 = x(1-x)$, and $d(x,t) = \frac{512}{x^3(1+8x)^3}$ [24].

Table 3: The numerical results of Example 5.2 when $\beta = 1.2$

Method	Index	2^6	2^7	2^8	2^9	2^10	2^11	2^12
HSS	IT	72	99	139	195	275	388	548
	CPU	0.0068	0.0128	0.0422	0.4043	1.6552	8.064	45.4345
DTS	IT	72	99	139	195	275	388	548
	CPU	0.0063	0.0089	0.0765	0.432	1.6824	8.2616	45.1749
CG	IT	63	127	255	459	698	2047	4095
	CPU	0.0046	0.0103	0.0478	0.18	1.1381	11.9645	88.3875
DSCS	IT	23	27	33	39	47	57	69
	CPU	0.0023	0.0035	0.0084	0.0748	0.2905	1.1796	5.1749

Table 4: The numerical results of Example 5.2 when $\beta = 1.8$

Method	Index	2^6	2^7	2^8	2^9	2^10	2^11	2^12
HSS	IT	72	99	139	195	275	388	548
	CPU	0.0062	0.0092	0.0381	0.3945	1.6881	7.9599	41.2243
DTS	IT	72	99	139	195	275	388	548
	CPU	0.0074	0.0135	0.0469	0.3761	1.6607	7.8374	41.5969
CG	IT	63	127	255	454	707	2047	4095
	CPU	0.0113	0.0171	0.0511	0.266	0.133	12.0178	79.8445
DSCS	IT	23	27	33	39	47	57	69
	CPU	0.0022	0.003	0.0091	0.0779	0.2831	1.2126	5.5463

In Table 3, 4, we show the iteration steps (express as "IT") and the computing time (express as "CPU") of methods applied to the experiments, which the parameters of all the methods are theoretically optimal. From tables 3 and Table 4, we observe again that DSCS significantly outperform than HSS, DTS, and CG in the field of iteration steps as well as CPU time when $\beta = 1.2$ and $\beta = 1.8$ for Example 5.2.

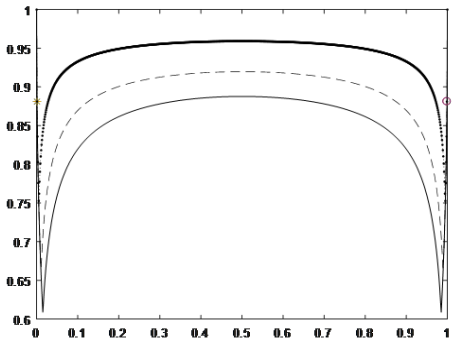


Figure 4: The curves of $\rho(L_{DCCS}(\omega))$ when $\beta = 1.2$

We drew the curves of $\rho(L_{DCCS}(\omega))$ for the parameter ω when $n = 2^7, n = 2^8, n = 2^{10}$ (from bottom to top)in the Figure.4. And the one optimal parameter ω_* of DSCS iteration method is marked by the symbol "$*$", the other optimal parameter ω_* is marked by "$\circ$" From Figure.4, we could see the fact that the larger the matrix order, the smaller the error between ω_* and ω_{exp}. However, with the matrix order increasing, the marking points of ω_* almost coincide in the Figure.4 due to the scale of the abscissa axis is relatively large.

6. Concluding remarks

In the paper, we propose the DSCS iterative method to solve diagonal-plus-Toeplitz linear systems obtained by the discretized spatial fractional diffusion equations. The convergence of the DSCS iterative method is analyzed and we show the conditions that the DSCS method converges to the unique solution of linear system. Then we analyze and derive an upper bound $\sigma(\alpha)$ of the spectral radius of the DSCS iterative matrix. As the main part of the article, there are two parameters which have a large influence on the convergence speed of the DSCS method. Based on reasonable assumptions, we consider and obtain the optimal form of these parameters, which are both concise and well-calculated.

At last, the numerical experiments illustrate that the DSCS method is more effective than the HSS and DTS because the spectral radius of the DSCS method is much lower. Compared with the CG methods, the DSCS is also more effective, especially when the matrix is larger. In a word, the DSCS has a better numerical result than HSS, DTS and CG.

However, in the proof process of the optimal parameters ω shown in the previous sections, we ignore some factors and make some hypotheses in order to facilitate calculations, which result in the error between the theoretical optimal value and the experimental optimal value of the parameter ω. Besides, we don't

consider the preconditioner according to the DSCS iterative method. The preconditioning methods of DSCS need further in-depth study from both theoretical and computational perspectives. This will be the further study in future.

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