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A NEW THEOREM ON GENERALIZED ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper, we have proved a new theorem dealing with $\varphi - |C, \alpha|_k$ summability factors of infinite series under weaker conditions. Also, some new and known results are obtained.

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1. Introduction

A positive sequence (X_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq X_n \leq Nc_n$ (see [2]). A positive sequence (X_n) is said to be quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$ (see [10]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the nth Cesàro means of order α $(\alpha > -1)$ of the sequences (s_n) and (na_n), respectively, that is (see [6])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \texttt{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \texttt{ and } A_{-n}^\alpha = 0 \texttt{ for } n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$, if (see [1, 9])

$$\sum_{n=1}^{\infty} \left| \varphi_n (u_n^\alpha - u_{n-1}^\alpha) \right|^k = \sum_{n=1}^{\infty} n^{-k} \left| \varphi_n t_n^\alpha \right|^k < \infty. \tag{3}$$

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In the special case when $\varphi_n = n^{1-1/k}$ (resp. $\varphi_n = n^{\delta+1-1/k}$) $\varphi-|C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ (see [7]) (resp. $|C,\alpha;\delta|_k$ (see [8])) summability.
The following theorems are known dealing with $\varphi-\mid C,\alpha\mid_k$ summability factors of infinite series.
Theorem 1.1 ([3]). Let $0<\alpha\leq 1$. Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid\leq \beta_n \tag{4}$$

$$\beta_n \to 0 \quad as \quad n\to\infty \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{6}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n\to\infty. \tag{7}$$

If there exists an $\epsilon>0$ such that the sequence $(n^{\epsilon-k}\,|\varphi_n|^k)$ is non increasing and if the sequence (ω_n^α) defined by (see [12])

$$\omega_n^\alpha = \begin{cases} |t_n^\alpha| & (\alpha=1) \\ \max_{1\leq v\leq n}|t_v^\alpha| & (0<\alpha<1) \end{cases} \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m}\frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k} = O(X_m) \quad \text{as} \quad m\to\infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi-\mid C,\alpha\mid_k$, $k\geq 1$ and $(\alpha+\epsilon)>1$.
Theorem 1.2 ([4]). Let $0<\alpha\leq 1$ and let (X_n) be an almost increasing sequence. If the sequences (β_n) and (λ_n) satisfy the the conditions (4)-(7) of Theorem A. If there exists an $\epsilon>0$ such that the sequence $(n^{\epsilon-k}\,|\varphi_n|^k)$ is non increasing and if the sequence (ω_n^α) defined by (8)satisfies the condition

$$\sum_{n=1}^{m}\frac{(\mid \varphi_n \mid w_n^\alpha)^k}{n^k {X_n}^{k-1}} = O(X_m) \quad \text{as} \quad m\to\infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi-\mid C,\alpha\mid_k$, $k\geq 1$ and $(1+\alpha k+\epsilon-k)>1$.
Remark 1.3. It should be noted that every increasing sequence is an almost increasing. However, the converse need not be true (see [11]).

2. Main result

The aim of this paper is to prove Theorem 1.2 under weaker conditions. Now we shall prove the following main theorem.
Theorem 2.1. Let $0<\alpha\leq 1$ and let (X_n) be a quasi-σ-power increasing sequence. If the sequences (β_n) and (λ_n) satisfy the the conditions (4)-(7) of Theorem A. If there exists an $\epsilon>0$ such that the sequence $(n^{\epsilon-k}\,|\varphi_n|^k)$ is non increasing and if the condition (10) holds, then the series $\sum a_n\lambda_n$ is summable

$\varphi - \mid C, \alpha \mid_k$, $k \geq 1$ and $(1 + \alpha k + \epsilon - k) > 1$.

Remark 2.2. It should be noted that every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [10]).

We need the following lemmas for the proof of our theorem.

Lemma 2. 3 ([5]). If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then

$$\left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=1}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{11}$$

Lemma 2.4 ([10]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (6) is satisfied

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty \tag{12}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{13}$$

Proof. Let (T_n^α) be the nth (C, α) mean, with $0 < \alpha \leq 1$, of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{14}$$

Applying Abel's transformation first and then using Lemma 2.3, we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

$$\begin{aligned} \mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right| \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha \omega_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \omega_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha. \end{aligned}$$

To complete the proof of Theorem 2.1, by Minkowski's inequality for $k > 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \mid \varphi_n T_{n,r}^\alpha \mid^k < \infty, \quad \texttt{for} \quad r = 1, 2.$$

For $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=2}^{m+1} n^{-k} \mid \varphi_n T_{n,1}^\alpha \mid^k \leq \sum_{n=2}^{m+1} n^{-k} (A_n^\alpha)^{-k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} A_v^\alpha \omega_v^\alpha \beta_v \right\}^{k-1}$$

$$\begin{aligned}
&\leq \sum_{n=2}^{m+1}\frac{1}{n}(A_n^\alpha)^{-k}|\varphi_n|^k\sum_{v=1}^{n-1}(A_v^\alpha)^k(\omega_v^\alpha)^k\beta_v^k\times\left\{\frac{1}{n}\sum_{v=1}^{n-1}1\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(\omega_v^\alpha)^k\beta_v^k\\
&= O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{1+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}(\omega_v^\alpha)^k\beta_v^k\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(\omega_v^\alpha)^k\beta_v\beta_v^{k-1}\sum_{n=v+1}^{m+1}\frac{n^{\epsilon-k}|\varphi_n|^k}{n^{1+\alpha k+\epsilon-k}}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(\omega_v^\alpha)^k\beta_v\frac{v^{\epsilon-k}|\varphi_v|^k}{v^{k-1}X_v^{k-1}}\int_v^\infty\frac{dx}{x^{1+\alpha k+\epsilon-k}}\\
&= O(1)\sum_{v=1}^{m}v\beta_v\frac{(\omega_v^\alpha|\varphi_v|)^k}{v^kX_v^{k-1}}=O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{(\omega_r^\alpha|\varphi_r|)^k}{r^kX_r^{k-1}}\\
&+ O(1)m\beta_m\sum_{v=1}^{m}\frac{(\omega_v^\alpha|\varphi_v|)^k}{v^kX_v^{k-1}} \qquad (15)\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v+O(1)m\beta_mX_m\\
&= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by the hypotheses of Theorem 2.1 and Lemma 2.4. Again by using (7), we have that

$$\begin{aligned}
\sum_{n=1}^{m}n^{-k}\mid\varphi_nT_{n,2}^\alpha\mid^k &= \sum_{n=1}^{m}n^{-k}\left|\varphi_n\right|^k\left|\lambda_n\right|\left|\lambda_n\right|^{k-1}(\omega_n^\alpha)^k\\
&= O(1)\sum_{n=1}^{m}\left|\lambda_n\right|\frac{(\mid\varphi_n\mid w_n^\alpha)^k}{n^kX_n{}^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{v=1}^{n}\frac{(\mid\varphi_v\mid w_v^\alpha)^k}{v^kX_v{}^{k-1}} \qquad (16)\\
&+ O(1)\left|\lambda_m\right|\sum_{n=1}^{m}\frac{(\mid\varphi_n\mid w_n^\alpha)^k}{n^kX_n{}^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right|X_n+O(1)\left|\lambda_m\right|X_m
\end{aligned}$$

$$= \quad O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)\left|\lambda_m\right| X_m = O(1) \quad as \quad m \to \infty,$$

by the hypotheses of Theorem 2.1 and Lemma 2.4. □

This completes the proof of Theorem 2.1. Also, if we take $\epsilon = 1$ and $\varphi_n = n^{1-1/k}$ (resp. $\epsilon = 1$ and $\varphi_n = n^{\delta+1-1/k}$), then we obtain two new results dealing with the $\mid C, \alpha \mid_k$ (resp. $\mid C, \alpha; \delta \mid_k$) summability factors. Finally, if we take (X_n) as a positive non-decreasing sequence, then we obtain Theorem 1.1 under weaker conditions. Also, if we take (X_n) as an almost increasing sequence, then we obtain Theorem 1.2.

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