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APPROXIMATE REACHABLE SETS FOR RETARDED SEMILINEAR CONTROL SYSTEMS†

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Abstract. In this paper, we consider a control system for semilinear differential equations in Hilbert spaces with Lipschitz continuous nonlinear term. Our method is to find the equivalence of approximate controllability for the given semilinear system and the linear system excluded the nonlinear term, which is based on results on regularity for the mild solution and estimates of the fundamental solution.

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1. Introduction

Let H and V be complex Hilbert spaces such that V is a dense subspace in H. In this paper, we are concerned with the control results for the following retarded semilinear control system in Hilbert space H:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^{0} a(s)A_1 x(t+s)ds + G(t, x(t)) + Bu(t), \quad t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s \le 0, \end{cases} \tag{1.1}$$

where $h > 0$, B is a bounded linear controller, and $u(t)$ is an appropriate control functions. A is the operator associated with a bounded sesquilinear form defined in $V \times V$ satisfying Gårding inequality. Then A generates an analytic semigroup $S(t)$ in both H and V^* and so the system (1.1) may be considered as an system in both H and V^*. The operator A_1 is bounded linear from V to V^*. The function $a(\cdot)$ is assumed to be real valued and Hölder continuous in the interval $[-h, 0]$, and G is given function satisfying some assumptions. Here, we discuss whether by choosing an u we can steer each initial state to any neighbor hood of

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a given $z \in H$ at a given time T. This is called an approximate controllability problem, and for linear evolution systems in general Banach spaces, there are many papers and monographs, see [1, 2], Triggiani [3], Curtain and Zwart [4] and references therein. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

The controllability for nonlinear control systems has been studied by many authors, for example, Control of nonlinear infinite dimensional systems in [5], controllability for parabolic equations with uniformly bounded nonlinear terms in [6], local controllability of neutral functional differential systems in [7], impulsive functional differential inclusions in [8]. Recently, the approximate controllability for semilinear control systems can be founded in [9, 10, 11], their results give sufficient condition on strict assumptions on the control action operator B.

In this paper we investigate the equivalence of approximate controllability for (1.1) and the linear system excluded the nonlinear term in the time T satisfying $T > 2h$, because that the solution mapping from the initial space to the solution space is Hólder continuous in $(2h, \infty)$, see [12]. We no longer require the strict range condition on B, and the uniform boundedness in [6] but instead we need the regularity and a variation of solutions of the given equations. Based on L^2-regularity properties in Section 2, we will obtain the relations between the reachable set of the semilinear system and that of its corresponding linear system in Section 3.

2. Regularity for retarded semilinear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norms on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. We assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq \omega_1||u||^2 - \omega_2|u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u,v), \quad u,\ v \in V. \tag{2.3}$$

Then A is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. It is well known that A generates an analytic semigroup in both of H and V^*(see [14]). Hence, we assume that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0||u||_{D(A)}^{1/2}|u|^{1/2}, \tag{2.4}$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.5}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.1), (2.4), and (2.5), we have*

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* *(Section 1.3.3 of* [15]*).*

The operator A_1 is a bounded linear operator from V to V^* such that its restrictions to $D(A)$ are bounded linear operators from $D(A)$ equipped with the graph norm of A to H, for example, A_1 is an elliptic differential operator of second order induced by sesquilinear form.

We need to impose the following two conditions:

Assumption (A). The function $a(\cdot)$ is assumed to be real valued and Hölder continuous of order ρ in the interval $[-h, 0]$:

$$|a(s)| \leq H_0, \quad |a(s) - a(\tau)| \leq H_0(s-\tau)^\rho, \quad -h \leq \tau, s \leq 0$$

for a constant H_0.

Assumption (G). Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0, \infty)$ and the Borel σ-field on $[-h, 0]$ respectively. Let μ be a Borel measure on $[-h, 0]$ and $g : [0, \infty) \times [-h, 0] \times V \times V \to H$ be a nonlinear mapping satisfying the following:
(i) For any $x, y \in V$ the mapping $g(\cdot, \cdot, x, y)$ is strongly $\mathcal{L} \times \mathcal{B}$-measurable;
(ii) There exist positive constants L_0, L_1, L_2 such that

$$\begin{aligned} &|g(t, s, 0, 0)| \leq K_0, \\ &|g(t, s, x, y) - g(t, s, \hat{x}, \hat{y})| \leq K_1 ||x - \hat{x}|| + K_2 ||y - \hat{y}|| \end{aligned} \tag{2.6}$$

for all $(t, s) \in [0, \infty) \times [-h, 0]$ and $x, \hat{x}, y, \hat{y} \in V$.

For $x \in L^2(-h, T; V)$, $T > 0$ we set

$$G(t, x) = \int_{-h}^{0} g(t, s, x(t), x(t+s))\mu(ds).$$

Here as in [16] we consider the Borel measurable corrections of $x(\cdot)$.

Remark 2.1. The above operator g satisfying (2.6) is the semilinear case of the nonlinear part of quasilinear equations considered by J. Yong and L. Pan [16].

First, we consider the following linear retarded functional differential equation with forcing term k:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^{0} a(s)A_1 x(t+s)ds + k(t), \quad t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s \leq 0. \end{cases} \tag{2.7}$$

Let $W(\cdot)$ be the fundamental solution of the linear equation (2.7) in the sense of Nakaglri [17], which is the operator valued function satisfying

$$\begin{cases} W(t) & = S(t) + \int_0^t S(t-s)\{\int_{-h}^0 a(\tau)A_1 W(s+\tau)d\tau\}ds, \quad t > 0, \\ W(0) & = I, \qquad W(s) = 0, \quad -h \le s < 0, \end{cases} \tag{2.8}$$

where $S(\cdot)$ is the semigroup generated by A. For each $t > 0$, we introduce the operator valued function $U_t(\cdot)$ defined by

$$U_t(s) = \int_{-h}^s W(t-s+\sigma)a(\sigma)A_1 d\sigma : V \to V, \quad s \in [-h, 0].$$

Then, by using (2.8), (1.1) is represented

$$x(t) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{G(s, x(s)) + k(s)\}ds.$$

From Proposition 4.1 of [12] or Theorem 1 of [13], it follows the following results.

Lemma 2.2. *Under Assumption (A), the fundamental solution $W(t)$ to (2.7) exists uniquely and is bounded. Applying Proposition 4.1 of* [12] *to the equation (2.7) in the space V^*, there exists a constant $C_0 > 0$ such that*

$$||W(t)||_{\mathcal{L}(V^*)} \le C_0, ||W(t^{'}) - W(t)||_{\mathcal{L}(V^*)} \le C_0(t^{'} - t), \tag{2.9}$$

$$||W(t^{'}) - W(t)||_{\mathcal{L}(V^*, V)} \le C_0(t^{'} - t)^{\kappa}(t-h)^{-\kappa}, \tag{2.10}$$

for $h < t < t^{'}$, and $\kappa < \rho$.

Let $T > 0$ be arbitrary fixed. Associated with $U(\cdot)$, we consider the operator $\mathcal{U} : L^2(-h, 0; V) \to L^2(0, T; V)$ defined by

$$(\mathcal{U}\phi^1)(t) = \int_{-h}^0 U_t(s)\phi^1(s)ds, \quad t \in (0, T] \tag{2.11}$$

for $\phi^1 \in L^2(-h, 0; V)$. We can see that $\mathcal{U}$ is into and bounded for each $T > 0$(see [17]).

Lemma 2.3. *Under Assumption (A) and $\kappa < \rho$, the operator $\mathcal{U}$ defined by (2.11) is Hólder continuous of order $(\kappa+1)/2$ in $(2h, \infty)$ in operator norm of $L(H, V)$, i.e., for any $T > 2h$ there exists a constant C_T such that*

$$|(\mathcal{U}\phi^1)(t^{'}) - (\mathcal{U}\phi^1)(t)| \le C_T|t^{'} - t|^{(\kappa+1)/2}. \tag{2.12}$$

Proof. Using $(V, V^*)_{1/2,2} = H$ and the well known interpolation inequality we get from (2.9) and (2.10)

$$||W(t^{'}) - W(t)||_{L(V^*, H)} \le C_0(t^{'} - t)^{(1+\kappa)/2}(t-h)^{-\kappa/2}. \tag{2.13}$$

Hence, with aid of (2.13) we have

$$|(\mathcal{U}\phi^1)(t^{'}) - (\mathcal{U}\phi^1)(t)|$$

$$\le C_0 H_0\big(1-\frac{\kappa}{2}\big)^{-1}\sqrt{h}||A||_{L(V^*,V)}(t^{'}-t)^{(\kappa+1)/2}(t-h)^{1-\kappa/2}||\phi^1||_{L^2(-h,0;V)}.$$

Noting that $t-s+\sigma > h$ since $t > 2h$, we get (2.12). □

Now, consider the following semilinear control system with a forcing term k:

$$\begin{cases} x^{'}(t) = Ax(t) + \int_{-h}^0 a(s)A_1 x(t+s)ds + G(t,x(t)) + k(t), \quad t>0, \\ x(0)=\phi^0, \quad x(s)=\phi^1(s) \quad -h\le s\le 0. \end{cases} \tag{2.14}$$

By virtue of Theorem 3.1 of [11] we have the following result on the semilinear equation (2.7).

Proposition 2.4. *1) Let $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$ and $k\in L^2(0,T;V^*)$, $T>0$. Then there exists a unique solution x of (2.14) belonging to*

$$L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$$

and satisfying

$$\begin{aligned} &||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \\ &\le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}), \end{aligned} \tag{2.15}$$

where C_1 is a constant depending on T.

2) If $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$ and $k\in L^2(0,T;V^*)$, then the mapping

$$H\times L^2(-h,0;V)\times L^2(0,T;V^*)\ni(\phi^0,\phi^1,k)\mapsto x\in L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)$$

is Lipschitz continuous.

Here, we note that by using interpolation theory, we have that for $z\in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$, there exists a constant $C_2>0$ such that

$$||z||_{C([0,T];H)} \le C_2||z||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}. \tag{2.16}$$

3. Approximately reachable sets

Let U be a Banach space and the controller operator B be bounded linear operator from another Banach space $L^2(0,T;U)$ to H. The solution $x(t)=x(t;\phi,G,u)$ of initial value problem (1,1) corresponding to a control $u\in L^2(0,T;U)$ is the following form:

$$\begin{aligned} &x(t;\phi,G,u) \\ &= W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{G(s,x(s)) + (Bu)(s)\}ds. \end{aligned}$$

For $T>0$, $\phi=(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$ and $u\in L^2(0,T;U)$ we define reachable sets as follows.

$$L_T(\phi) = \{x(T;\phi,0,u) : u\in L^2(0,T;U)\},$$
$$R_T(\phi) = \{x(T;\phi,G,u) : u\in L^2(0,T;U)\}.$$

Definition 3.1. (1) System (1.1) is said to be H-approximately controllable for initial value ϕ in time T) if $\overline{R_T(\phi)} = H$.
(2) The linear system corresponding (1.1) is said to be H-approximately controllable for initial value ϕ in time T if $\overline{L_T(\phi)} = H$.

Remark 3.1. Since A generate an analytic semigroup, the following (1)-(2) are equivalent for the linear system (see [2, Theorem 3.10]).

(1) $\overline{L_T(\phi)} = H$
(2) $\overline{L_T(0)} = H$.

Lemma 3.2. *Let $x \in L^2(-h,T;V)$, $T > 0$. Then $G(\cdot,x) \in L^2(0,T;H)$. and*

$$||G(\cdot,x)||_{L^2(0,T;H)} \le \mu([-h,0])\{K_0\sqrt{T} + (K_1+K_2)||x||_{L^2(0,T;V)} + K_2||x||_{L^2(-h,0;V)}\}. \tag{3.1}$$

Moreover if $x_1,\ x_2 \in L^2(-h,T;V)$, then

$$||G(\cdot,x_1) - G(\cdot,x_2)||_{L^2(0,T;H)} \le \mu([-h,0])\{(K_1+K_2)||x_1-x_2||_{L^2(0,T;V)} + K_2||x_1-x_2||_{L^2(-h,0;V)}\}. \tag{3.2}$$

The proof is immediately from Assumption (G).

Theorem 3.3. *For any $T > 0$ we have*

$$\overline{R_T(0)} \subset \overline{L_T(0)}.$$

Proof. Let $z_0 \notin \overline{L_T(0)}$. Since $\overline{L_T(0)}$ is a balanced closed convex subspace, we have $\alpha z_0 \notin \overline{L_T(0)}$ for every $\alpha \in \mathbb{R}$, and

$$\inf\{|z_0 - z| : z \in \overline{L_T(0)}\} = d.$$

By the formula (2.17) we have

$$||x(\cdot;0,G,u)||_{L^2(0,T;V)} \le C_1||B||||u||_{L^2(0,T;U)}, \tag{3.3}$$

where C_1 is the constant in Proposition 2.1. For every $u \in L^2(0,T;U)$, we choose a constant $\alpha > 0$ such that

$$\sqrt{T}C_0\mu([-h,0])\{K_0\sqrt{T} + (K_1+K_2)C_1||B||||u||_{L^2(0,T;U)}\} < \alpha d.$$

Hence form (3.1), (3.3) and by using Hölder inequality, it follows that

$$\begin{aligned} |x(T;0,G,u) - \alpha z_0| &\ge |\int_0^T W(T-s)Bu(s)ds - \alpha z_0| \\ &\quad - |\int_0^T W(T-s)G(s,x(s))ds| \\ &\ge \alpha d - \sqrt{T}C_0\mu([-h,0])\{K_0\sqrt{T} \\ &\quad + (K_1+K_2)C_1||B||||u||_{L^2(0,T;U)}\} \\ &> 0. \end{aligned}$$

Thus, we have $\alpha z_0 \notin \overline{R_T(0)}$. □

We assume the following conditions:

Assumption (B) For $0 \le \tau < t \le T$ and $u \in L^2(0,T;U)$, the $\mathcal{B}(\tau,t)$ from $L^2(0,T;U)$ into H defined by

$$\mathcal{B}(\tau,t)k := \int_\tau^t W(t-s)Bu(s)ds$$

induces an invertible operator $\hat{\mathcal{B}}$ defined on $L^2(0,T;U)/\mathrm{Ker}\mathcal{B}$ and there exists a positive constant L_B such that $||\hat{\mathcal{B}}^{-1}|| \le L_B$, see [18].

Theorem 3.4. *Under Assumptions (A), (B), (G), and $T > 2h$, we have*

$$\overline{L_T(\phi)} \subset \overline{R_T(\phi)}, \quad \phi \in H \times L^2(0,T;V).$$

Proof. Let $T > 2h$, and let $\gamma > 0$ be arbitrary given. We will show that $z \in \overline{L_T(\phi)}$ satisfying $|z| < \gamma$ belongs to $\overline{R_T(\phi)}$. Let $u \in L^2(0,T;U)$ be arbitrary fixed. Then by (2.15) we have

$$||x_u||_{L^2(0,T;V)} \le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||B||||u||_{L^2(0,T;U)}),$$

where x_u is the solution (2.17) corresponding to the control u. For any $\epsilon > 0$, we can choose a constant $\delta > 0$ satisfying

$$\begin{aligned}\max\{\delta,\sqrt{\delta}\}&\\ <\min\big\{&\big(C_0(C_1C_2+C_0)|\phi^0|)\big]^{-1},\\ &\big(C_0||\mathcal{U}||_{L(L^2(-h,0;V),L^2(0,T;V))}||\phi^1||_{L^2(-h,0;V)}\big)^{-1},\\ &\big(C_0C_T||\phi^1||_{L^2(-h,0;V)}\big)^{-1},\\ &\big((C_0+1)C_0\sqrt{T}\mu([-h,0])\big(K_0\sqrt{T}+(K_1+K_2)||x_u||_{L^2(0,T;V)}\\ &+K_2||\phi^1||_{L^2(-h,0;V)}\big)\big)^{-1},\\ &\big(M_0+\frac{\epsilon C_{-\alpha}}{8}\big)^{-1},\ \big((C_0+1)C_0\sqrt{T}||B||||u||_{L^2(0,T;U)}\big)^{-1}\big\}\epsilon/8,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}M_0 =&C_0\mu([-h,0])\big[K_0\sqrt{T}+K_2||\phi^1||_{L^2(-h,0;V)}\\ &+C_1(K_1+K_2)\big\{|\phi^0|+||\phi^1||_{L^2(-h,0;V)})+||u||_{L^2(0,T;U)}\\ &+L_B\big(C_0C_2||x_u||_{L^2(0,T;V)}+\gamma\big)\big\}\big],\end{aligned} \tag{3.5}$$

$$M_1 = C_0C_1L_B\mu([-h,0])(K_1+K_2), \tag{3.6}$$

where c_0, C_1 and C_2 are the constants of (2.11), (2.15) and (2.16), respectively. Set

$$\begin{aligned}x_1 := x(T-\delta;\phi,G,u) &= W(T-\delta)\phi^0 + \int_{-h}^0 U_{T-\delta}(s)\phi^1(s)ds\\ &+\int_0^{T-\delta} W(T-\delta-s)G(s,x_u(s))ds + \int_0^{T-\delta} W(T-\delta-s)Bu(s)ds,\end{aligned}$$

where $x_u(t) = x(t;\phi, G, u)$ for $0 < t \le T$. Consider the following problem:

$$\begin{cases} y'(t) = Ay(t) + \int_{-h}^{0} a(s)A_1 y(t+s)ds + Bu(t), & \delta < t \le T, \\ y(T-\delta) = x_1, \quad y(s) = 0 & -h \le s \le 0. \end{cases} \tag{3.7}$$

The solution of (3.7) with respect to the control $w \in L^2(T-\delta, T; U)$ is denoted by

$$\begin{aligned} y_w(T) &= W(\delta)x_1 + \int_{T-\delta}^{T} W(T-s)Bw(s)ds \\ &= W(\delta)W(T-\delta)\phi^0 + W(\delta)\int_{-h}^{0} U_{T-\delta}(s)\phi^1(s)ds \\ &\quad + W(\delta)\int_0^{T-\delta} W(T-\delta-s)G(s, x_u(s))ds \\ &\quad + W(\delta)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds + \int_{T-\delta}^{T} W(T-s)Bw(s)ds, \end{aligned} \tag{3.8}$$

Then since $z \in \overline{L_T(\phi)}$, and $\overline{L_T(\phi)} = \overline{L(0)}$ is independent of the time T and initial data ϕ(see Remark 2.1), there exists $w_1 \in L^2(T-\delta, T; U)$ such that

$$|y_{w_1}(T) - z| < \frac{\epsilon}{8}. \tag{3.9}$$

and so, by (2.18), (3.8) and (3.9) we have

$$\begin{aligned} |\int_{T-\delta}^{T} W(T-s)Bw_1(s)ds| &\le |y_{w_1}(T) - z| + \gamma + |W(\delta)x_1| \\ &\le C_0C_2||x_u||_{L^2(0,T;V)} + \gamma + \frac{\epsilon}{8}. \end{aligned}$$

Hence, from Assumption (B) it follows that

$$||w_1||_{L^2(0,T;U)} \le L_B\{C_0C_2||x_u||_{L^2(0,T;V)} + \gamma + \frac{\epsilon}{8}\} \tag{3.10}$$

Now we set

$$v(s) = \begin{cases} u & \text{if } 0 \le s \le T-\delta, \\ w_1(s) & \text{if } T-\delta < s < T. \end{cases}$$

Then $v \in L^2(0, T; U)$. Observing that

$$\begin{aligned} &x_v(t;\phi, G, v) \\ &\quad = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-\tau)\{G(\tau, x_v(\tau)) + Bv(\tau)\}d\tau, \end{aligned}$$

from (3.8) and (3.9) we obtain that

$$|x(T;\phi, G, v) - z| \le |y_{w_1}(T) - z| + \left|W(T)\phi^0 - W(\delta)W(T-\delta)\phi^0\right| \tag{3.11}$$

$$+ \big| \int_{-h}^{0} U_T(s)\phi^1(s)ds - W(\delta)\int_{-h}^{0} U_{T-\delta}(s)\phi^1(s)ds\big|$$
$$+ \big| \int_{0}^{T} W(T-s)G(s, x_v(s))ds - W(\delta)\int_{0}^{T-\delta} W(T-\delta-s)G(s, x_u(s))ds\big|$$
$$+ \big| \int_{0}^{T-\delta} W(T-s)Bu(s)ds - W(\delta)\int_{0}^{T-\delta} W(T-\delta-s)Bu(s)ds\big|$$
$$\le \frac{\epsilon}{8} + I + II + III + IV.$$

From (2.17) and (2.18) it follows that

$$\sup_{0\le t\le T} |W(t)\phi^0| \le C_2||W(\cdot)\phi^0||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1C_2|\phi^0|,$$

and so

$$\begin{aligned} I &= \big|W(T)\phi^0 - W(\delta)W(T-\delta)\phi^0\big| \\ &\le ||I - W(\delta)||_{L(H)}|W(T)\phi^0| \\ &\quad + ||W(\delta)||_{L(H)}||W(T) - W(T-\delta)||_{L(H)}|\phi^0| \\ &\le \delta C_0(C_1C_2 + C_0)|\phi^0| < \frac{\epsilon}{8}. \end{aligned} \tag{3.12}$$

With aid of Lemmas 2.2 and 2.3, and (3.4) we have

$$\begin{aligned} II &\le \big|(I - W(\delta))\int_{-h}^{0} U_T(s)\phi^1(s)ds\big| \\ &\quad + \big|W(\delta)\int_{-h}^{0}(U_T(s) - U_{T-\delta}(s))\phi^1(s)ds\big| \\ &\le \delta C_0||\mathcal{U}||_{L(L^2(-h,0;V))}||\phi^1||_{L^2(-h,0;V)} + C_0C_T\delta^{(\kappa+1)/2}||\phi^1||_{L^2(-h,0;V)} < \frac{\epsilon}{4}. \end{aligned} \tag{3.13}$$

By (2.17) and (3.10), we get

$$\begin{aligned} ||x_{w_1}||_{L^2(T-\delta,T;V)} \le& ||x_v||_{L^2(0,T;V)} \\ \le& C_1\big\{|\phi^0| + ||\phi^1||_{L^2(-h,0;V)}) + ||u||_{L^2(0,T;U)} \\ &+ L_B\big(C_0C_2||x_u||_{L^2(0,T;V)} + \gamma + \frac{\epsilon}{8}\big)\big\}. \end{aligned} \tag{3.14}$$

Hence, with aid of (3.1) and by using Hólder inequality, we have

$$\begin{aligned} &\big|\int_{T-\delta}^{T} W(T-s)G(s, x_{w_1}(s))ds\big| \le C_0\sqrt{\delta}||G(\cdot, x_{w_1})||_{L^2(0,T;H)} \\ &\le C_0\sqrt{\delta}\mu([-h,0])\{K_0\sqrt{T} + (K_1+K_2)||x_{w_1}||_{L^2(T-\delta,T;V)} + K_2||\phi^1||_{L^2(-h,0;V)}\} \\ &\le \sqrt{\delta}\big(M_0 + \frac{\epsilon M_1}{8}\big), \end{aligned} \tag{3.15}$$

where M_0 and M_1 are the constants of (3.5) and (3.6), respectively. Thus, from (2.11), (2.12), (3.1), (3.4), and (3.15) it follows that

$$\begin{aligned} III =&\big|W(\delta)\int_0^{T-\delta} W(T-\delta-s)G(s,x_u(s))ds - \int_0^T W(T-s)G(s,x_v(s))ds\big| \\ \le&\big|(W(\delta)-I)\int_0^{T-\delta} W(T-\delta-s)G(s,x_u(s))ds\big| \\ &+\big|\int_0^{T-\delta}\big(W(T-\delta-s)-W(T-s)\big)G(s,x_u(s))ds\big| \\ &+\big|\int_{T-\delta}^T W(T-s)G(s,x_{w_1}(s))ds\big| \\ \le&(C_0+1)C_0\delta\sqrt{T}\mu([-h,0])\{K_0\sqrt{T}+(K_1+K_2)||x_u||_{L^2(0,T;V)}+K_2||\phi^1||\} \\ &+\sqrt{\delta}\big(M_0+\frac{\epsilon M_1}{8}\big) < \frac{\epsilon}{8}+\frac{\epsilon}{8} \le \frac{\epsilon}{4}, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} IV =&\big|W(\delta)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds - \int_0^{T-\delta} W(T-s)Bu(s)ds\big| \\ \le&\big|(W(\delta)-I)\int_0^{T-\delta} W(T-\delta-s)Bu(s)ds\big| \\ &+\big|\int_0^{T-\delta}\big(W(T-\delta-s)-W(T-s)\big)Bu(s)ds\big| \\ \le&(C_0+1)C_0\delta\sqrt{T}||B||||u||_{L^2(0,T;U)} < \frac{\epsilon}{8}. \end{aligned} \tag{3.17}$$

Therefore, by (3.4), and (3.11)-(3.17), we have

$$||x(T;\phi,G,v)-z|| < \epsilon,$$

that is, $z \in \overline{R_T(\phi)}$ and the proof is complete. □

Remark 3.2. Noting that $H([0,T];U)$ is dense in $L^2(0,T;U)$, we can obtain the same results of Theorem 3.4 corresponding to (1.1) with control space

$$H([0,T];U) = \{w:[0,T]\to U : |w(t)-w(s)| \le H_0|t-s|^\theta,\ 0<\theta<1,\ H_0>0\}$$

instead of $L^2(0,T;U)$

From Theorems 3.3-4, we obtain the following control results of (1.1).

Corollary 3.5. *Under Assumptions (A) and (F), for $T > 2h$ we have*

$$\overline{L_T(\phi)} = H \iff \overline{R_T(\phi)} = H.$$

Therefore, the approximate controllability of linear system (1.1) with $G = 0$ is equivalent to the condition for the approximate controllability of the nonlinear system (1.1).

Example 3.6. Example. Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$(Ax)(\xi) = -\frac{d^2x(\xi)}{d\xi^2} \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following control system for a retarded diffusion and reaction process dealt with by [6].

$$\begin{cases} \frac{d}{dt}x(t,\xi) & = Ax(t,\xi) + \int_{-h}^0 a_1(s)Ax(t+s,\xi)ds + G(x(t,\xi)) + B(\xi)u(t), \\ & \quad (t,\xi) \in [0,T]\times[0,\pi], \\ x(t,0) & = \alpha, \quad x(t,\pi) = \beta, \quad t > 0, \\ x(s,\xi) & = \phi^1(s,\xi), \quad -h \le s < 0, \\ x(0,\xi) & = 0, \quad 0 < \xi < \pi, \end{cases} \tag{3.18}$$

where $h > 0$, $\phi^1(\cdot,\xi) \in L^2(0,T;V)$, $B(\cdot) \in L^2(0,\pi)$, and $u \in L^2(0,T;\mathbb{R})$ is a control function. The nonlinear term G is given by

$$G(x) = \frac{\sigma x}{1+|x|}, \quad \sigma \in \mathbb{R}.$$

Setting

$$\phi^0(\xi) = -(1-\xi)\alpha - \xi\beta \ (0 \le \xi \le \pi),$$

and

$$z(t,\xi) = \begin{cases} x(t,\xi) - \phi^0(\xi), & t > 0, \\ \phi^1(t,\xi), & -h \le t < 0. \end{cases}$$

System (3.18) is equivalent to

$$\begin{cases} \frac{d}{dt}z(t,\xi) & = Az(t,\xi) + \int_{-h}^0 a_1(s)Az(t+s,\xi)ds + G(z(t,\xi)) + B(\xi)u(t), \\ & \quad (t,\xi) \in [0,T]\times[0,\pi], \\ z(t,0) & = z(t,\pi) = 0, \quad t > 0, \\ z(0,\xi) & = \phi^0(\xi), \quad 0 < \xi < \pi, \\ z(s,\xi) & = \phi^1(s,\xi), \quad -h \le s < 0, \end{cases} \tag{3.19}$$

where

$$G(y(\xi)) = \frac{\sigma(y(\xi) - \phi^0(\xi))}{(1+|y(\xi) - \phi^0(\xi)|)}, \quad y(\cdot) \in L^2(0,\pi).$$

For $x, y \in L^2(0,\pi)$, we have

$$|G(x(\xi))| \le |\sigma|,$$

$$|G(x(\xi)) - G(y(\xi))| \le \frac{|\sigma|(1+2|y(\xi)-\phi^0(\xi)|\cdot|x(\xi)-y(\xi)|}{(1+|x(\xi)-\phi^0(\xi)|)|1+y(\xi)-\phi^0(\xi)|} \le 2|\sigma||x(\xi)-y(\xi)|$$

for almost all $\xi \in (0,\pi)$. Therefore, we see that G satisfies Assumption (G). Thus by Theorem 3,3, provided the time $T > 2h$, we get that the approximate controllability of linear system (3.18) with $g = 0$ is equivalent to the condition for the approximate controllability of the nonlinear system (3.19).

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