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PRICING STEP-UP OPTIONS USING LAPLACE TRANSFORM†

JERIM KIM, EYUNGHEE KIM, AND CHANGKI KIM*

ABSTRACT. A step-up option is a newly developed financial instrument that simultaneously provides higher security and profitability. This paper introduces two step-up options: step-up type1 and step-up type2 options, and derives the option pricing formulas using the Laplace transform. We assume that the underlying equity price follows a regime-switching model that reflects the long-term maturity of these options. The option prices are calculated for the two types of funds, a pure stock fund composed of risky assets only and a mixed fund composed of stocks and bonds, to reflect possible variety in the fund underlying asset mix. The impact of changes in the model parameters on the option prices is analyzed. This paper provides information crucial to product developments.

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1. Introduction

Investors have experienced dramatic losses as a result of recent financial crises. The credit crunch of 2008 highlighted the need for secure and profitable investments. To meet the changing needs of customers, financial institutions are developing a range of financial instruments that simultaneously provide higher levels of security and profitability. A step-up option is an example of such an instrument.

The pay-off of a step-up option depends on the value of an underlying asset reaching or exceeding a predetermined level or barrier. Once the value exceeds a predetermined level, the minimum guaranteed amount is readjusted so that it is higher than the previous amount.

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However, a step-up option differs from a barrier option. First, a step-up option has multiple stepwise barriers and a long-term maturity. Therefore, an investor is more likely to gain a higher minimum guaranteed amount over the long-term contractual period. Second, to secure benefit, the principal amount is guaranteed even if the underlying asset fails to reach the lowest barrier. Finally, a step-up option is embedded in a portfolio whose underlying assets are purely equities as well as a combination of equities and fixed income securities. Therefore, the pricing of a step-up option should reflect a variety of underlying assets.

The value of a barrier option at any time depends on the underlying asset value at that point and on the path taken by the underlying asset. The classical Black-Scholes approach, therefore, may not be directly applied. A closed-form solution of a down-and-out European call option was derived in Merton [21]. Subsequent articles by Cox and Rubinstein [10] and Rubinstein and Reiner [25, 26] proposed closed-form pricing formulas for single-barrier options.

Double-barrier options have been discussed by various researchers. Kunitomo and Ikeda [18] used the probabilistic approach to a general valuation method for European options whose payoff is restricted by curved boundaries. Moreover, for double-barrier option pricing, expressions for the Laplace transform were derived. Geman and Yor [12] numerically inverted the Laplace transform to obtain option prices and Pelsser [23] inverted the Laplace transform using contour integration to derive the density function of the first-hit times of barriers.

Wang et al. [27] proposed a hybrid method that combines the Laplace transformation and the finite-difference approach. Boyarchenko and Levendorskii [3] derived explicit formulas for the prices of basic types of barrier options and digitals that were touched first using the pseudo differential operators technique and the operator form of the Wiener-Hopf factorization method. Boyarchenko and Levendorskii [5] applied Carr's randomization approximation and the operator form of the Wiener-Hopf method to double barrier options in continuous time. They combined Carr's randomization procedure with the efficient numerical techniques of Boyarchenko and Levendorskii [4].

American barrier option pricing and hedging were studied by Karatzas [16] and Bin et al. [2]. The valuation of barrier options in a binomial model was discussed by Boyle and Lau [6] and a trinomial tree model for barrier options was developed by Cheuk and Vorst [9]. Lindset and Persson [20] and Brown et al. [7] discussed a barrier exchange option that is knocked out when the prices of two underlying assets become equal the first time. Wong and Kwok [28] priced various forms of European-style multi-asset barrier options. The valuation model for time-dependent barrier options is addressed in Hui [14] and Hui and Lo [15]. The closed-form solution for barrier options when a barrier is not active over the entire life of options is covered in Carr [8]. Hong [13] investigated valuation bounds on barrier options under model uncertainty.

Easton et al. [11] empirically compared the pricing of barrier options traded in the Australian Stock Exchange with its theoretical valuation. Easton et al. [11] and Pavel et al. [22] empirically compared the theoretical valuations with the pricing of barrier options traded in the Australian Stock Exchange and of equity-linked structured products in the German market.

This paper suggests a pricing formula for step-up options after considering various attributes. The main contributions of this paper are summarized as follows. First, newly developed step-up options are introduced and option pricing formulas are derived using Laplace transform. Second, considering the long-term maturity of step-up options, the option price is calculated assuming that the equity price movement follows a regime-switching model to adequately capture long-term stock volatility. Third, option prices are computed for various portfolios because funds with step-up options invest in equity as well as fixed income securities. Finally, the effects of parameters of the pricing model on option prices are analyzed and information crucial in product developments is provided.

The remainder of this paper is organized as follows. Section 2 describes the motivation for the development and the payoff structure of step-up options. Section 3 derives the formulas of the step-up options using Laplace transform for two cases: closed-form solutions of a pure equity portfolio and approximate solutions of a mixed fund that is a portfolio composed of equities and fixed income securities. Section 4 analyzes the implications of step-up option pricing under varying parameters of the pricing formula. Finally Section 5 concludes this paper and proposes directions for further research.

2. Step-up Options

2.1. Motivation. Barrier options are used in various types of financial derivatives such as equity-linked notes (ELN) or equity-linked securities (ELS). ELN or ELS are financial instruments that invest in fixed income securities to protect the principal while investing a portion in derivative securities, such as a stock index option, to seek a higher investment return. The instruments can be one of two types: knock-out or early-redemption. The knock-out instrument provides a guaranteed return if the underlying asset reaches a predetermined level at least once during the contractual period, whereas the early-redemption instrument redeems the principal and a guaranteed return once a set of conditions are met on or before maturity.

ELN was first developed in 1992 in the U.S. In South Korea, it was developed in 2003 and, despite a decrease during the financial crisis of 2008, the volume of issuance has increased rapidly. The majority of ELNs were initially simple in structure. Recent ELNs, however, are sophisticated and include step-down options. A step-down ELN is linked to the movement of two underlying assets and possesses a number of early exercise dates. On an early exercise date, a pre-set return is paid upon exercise, provided that the rate of return on the underlying assets is above the pre-set level, and the step-down ELN expires. The multiple barriers of exercise prices are stepwise in descending order; hence, this ELN is called a step-down.

Interesting examples of barrier options are found in the variable insurance (VI) market. VI is a participatory type of insurance and the member premiums are invested in funds. The most remarkable characteristic of VI is the minimum guarantee option.[1] This option obligates the insurer to provide a pre-determined minimum benefit either at maturity or at claim regardless of actual fund value.[2]

In Korea, VI was first introduced in 2001 and has become more popular ever since. As of September 2012, total VI premium income was USD 1.53 billion and accumulated funds amounted to USD 62.5 billion. However, in 2008 and 2012, total premium income and new contract premium showed a significant decline. The slump in 2008 was related to the global financial crisis. In March 2008, the crisis set in and the stock market began to plunge. Hence, the number of new VI contracts decreased and the cancellation rate increased. In 2012, the VI industry experienced another crisis but it was unrelated to stock market performance. A cause of this crisis was reported by the Consumers Union of Korea, which was due to the low investment return on VI. The Consumers Union of Korea asserted in the 2012 K-Consumer Report that although insurers emphasized the profitability of variable annuity, considering the high sales commission in VI, the actual rate of return of VI[3] is lower than the average annual inflation rate over the previous 10 years. This report was distributed to the public through various media. As a result, the average monthly new contracts premium from April 2012 to June 2012 dropped to USD 0.1 billion, a decrease of 50% from USD 0.23 billion from April 2011 to June 2011.

If the decline in premium income during the 2008 financial crisis confirmed the risk adverse behavior of VI consumers, the decline in premium income triggered by the 2012 K-Consumer report alerted insurers that they could no longer emphasize the potential high return of VI to bolster sales. In response, various forms of minimum guarantee options were developed to satisfy customer desire for both stability and profitability. The step-up option has received significant attention.

2.2. The Payoff of Step-up Options. When the fund value reaches a predetermined guarantee amount at each step, the minimum guarantee is readjusted to the next guarantee amount. There are two types of step-up option: step-up type1 and step-up type2. Both types of option provide different benefits depending on the fund value at maturity. Figure 1 illustrates the step-up option (type1 and type2) payoffs. The guarantee amount is divided into five stages from 100% to 200% with a 20% leap between steps.

Figure 1 illustrates the variation of step-up payoffs with two different fund performances from contract signing (A) to maturity (G). At (A), the step-up

[1] Generally, the maturity of a minimum guarantee option is over five years.

[2] The minimum guarantee options are GMIB (guaranteed minimum income benefit), GMAB (guaranteed minimum accumulation benefit), GMWB (guaranteed minimum withdrawal benefit), GLWB (guaranteed minimum lifetime withdrawal), and a step-up option is applied in GMAB and GMWB.

[3] Variable annuity is representative of a saving type VI product.

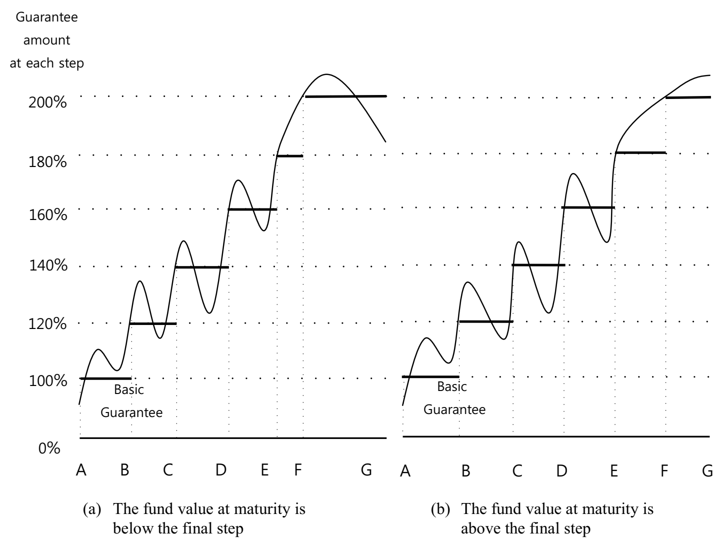

FIGURE 1. Step-up option payoff

minimum guarantee is set at the principal.[4] At (B), the fund value reaches the next step and the minimum guarantee increases to 120% of the initial fund value. As the fund value reaches (C),. . .,(F), the minimum guarantee increases by 20% of the principal at each step and after (F), the highest minimum guarantee (200%) is provided.

The final benefits of step-up type1 and step-up type2 depend on the fund value at maturity (G). The left panel of Figure 1 represents a case where the fund value at maturity (G) is below the final step (200%), whereas the right panel represents a case where the fund value at maturity (G) is above the final step. The final benefit of a step-up type1 is determined irrespective of the fund value at (G). Because the fund has already historically exceeded the final step at (F), the insurer is obligated to provide 200% of the principal invested. Contrastingly, the benefit of a step-up type2 is different based on the fund value at maturity. In the left panel of Figure 1, the fund value at (G) is below the final step and the payoff of the step-up type1 is equivalent to that of the step-up type2 at 200% of the principal. In the right panel of Figure 1, however, the fund value at maturity(G) is above the final step and the step-up type2 payoff is the final fund value, which is higher than the step-up type1 option payoff.

[4]The principle indicates the written premium. In Korea, an insurer deducts the sales commission from the written premium at contract signing and, therefore, the amount of money invested in the fund is less than the principal.

3. Pricing Step-up Options

The pricing formulas for step-up type1 and step-up type2 options are derived in this section using Laplace transform. Moreover, two types of fund are considered, pure stock funds and stock and bond mixed funds. this paper suggests a general pricing formula of step-up options; therefore, we do not consider the sales commission.

The Model. Let S_t be the stock price at time t and B_t be the price of a risk-free asset at time t with B_0= 1. It is assumed that under risk-neutral measure $\mathbb{P}$, S_t, and B_t are given by

$$\begin{aligned} dS_t =& r_{Z(t)} S_t dt + \sigma_{Z(t)} S_t dW_t, \\ dB_t =& r_{Z(t)} B_t dt, \end{aligned}$$

with positive initial values S_0 and $B_0 = 1$. Here, Z is an irreducible continuous time Markov process with a finite state space $\{1, \ldots, m\}$ and an infinitesimal generator G and W_t is the Wiener process. For each $i = 1, 2, \ldots, m$, r_i is the interest rate and $\sigma_i > 0$ is the volatility. For $0 < \alpha < 1$, the value of a stock and bond mixed fund, Y_t, is expressed by

$$Y_t =\alpha S_t + (1-\alpha) S_0 B_t.$$

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a set of barriers with $1 < b_1 < b_2 < \ldots < b_n$ and for notational simplicity, let $b_0 = 1$.

3.1. Pure Stock Funds. In this subsection, the prices of step-up type1 and step-up type2 options are calculated. under the assumption that the underlying fund is composed of risky assets only.

3.1.1. Step-up type1 option. Define

$$\mathcal{T}_b = \inf\{t \geq 0 : \frac{S_t}{S_0} \geq b\}.$$

The payoff of the step-up type1 option on a pure stock fund at maturity T is expressed by

$$\mathcal{X} =S_0 + S_0 \sum_{k=1}^{n} (b_k - b_{k-1}) \mathbb{1}_{\{\mathcal{T}_{b_k} \leq T\}}. \tag{1}$$

The price of the step-up type1 option on a pure stock fund at time 0 with $Z(0) = i$ is given by

$$\begin{aligned} P_{S1}(T,i) =& \mathbb{E}[e^{-\int_0^T r_{Z(u)} du} \mathcal{X} | Z(0) = i] \\ =& S_0 \mathbb{E}[e^{-\int_0^T r_{Z(u)} du} | Z(0) = i] \\ &+ S_0 \sum_{k=1}^{n} (b_k - b_{k-1}) \mathbb{E}[e^{-\int_0^T r_{Z(u)} du} \mathbb{1}_{\{\mathcal{T}_{b_k} \leq T\}} | Z(0) = i]. \end{aligned} \tag{2}$$

We define the Laplace transform of $P_{S1}(T,i)$:

$$\tilde{P}_{S1}(s,i) \equiv \int_0^\infty e^{-st} P_{S1}(t,i)dt.$$

Before stating the next theorem that derives $\tilde{P}_{S1}(s,i)$, we introduce notation that will be used: I is the identity matrix, O is the $m \times m$ zero matrix, $\mathbf{1}$ is the m-dimensional column vector with components all equal to 1, and $\mathbf{1}_i$ is the m-dimensional column vector with an ith component 1 and other components that are 0. For an m-dimensional vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$, $\mathrm{diag}(\boldsymbol{\nu})$ is defined as a diagonal matrix with diagonal entries starting in the upper left corner of $\nu_1, \ldots, \nu_m$. Let $R = \mathrm{diag}(\boldsymbol{r})$, $\Sigma = \mathrm{diag}(\boldsymbol{\sigma})$ and $M = \mathrm{diag}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (r_1 - \frac{1}{2}\sigma_1^2, \ldots, r_m - \frac{1}{2}\sigma_m^2)$. $\mathbb{C}$ denotes the complex number space. For $t \geq 0$, $s, w \in \mathbb{C}$, let

$$\Psi(s,w;\boldsymbol{\nu}) = G - R - sI - w\ \mathrm{diag}(\boldsymbol{\nu}) + \frac{1}{2}w^2\Sigma^2. \tag{3}$$

Theorem 3.1. *Suppose that* $s \in \mathbb{C}$ *satisfies* $\mathrm{Re}(s) > -\max\{r_1, \ldots, r_m\}$. *Then the price of a step-up type1 option with pure stock fund has the following Laplace transform:*

$$\tilde{P}_{S1}(s,i) = S_0 \mathbf{1}_i^\top \Big(I + \sum_{k=1}^n (b_k - b_{k-1}) e^{(\ln b_k) Q(s,0;\boldsymbol{\mu})}\Big)(-G + R + sI)^{-1}\mathbf{1}, \tag{4}$$

where $Q(s,w;\boldsymbol{\nu})$ *is the unique stable matrix which satisfies*

$$\frac{1}{2}\Sigma^2 Q(s,w;\boldsymbol{\nu})^2 + (w\Sigma^2 - \mathrm{diag}(\boldsymbol{\nu}))Q(s,w;\boldsymbol{\nu}) + \Psi(s,w;\boldsymbol{\nu}) = O. \tag{5}$$

Proof. See Appendix A. □

3.2. Step-up type2 option. The payoff of the step-up type2 option on a pure stock fund at maturity T is expressed by

$$\max\{S_T, \mathcal{X}\} = S_T + \max\{\mathcal{X} - S_T, 0\}.$$

The price of the step-up type2 option on a pure stock fund with maturity T and $Z(0) = i$ is given by

$$\begin{aligned} P_{S2}(T,i) =& \mathbb{E}[e^{-\int_0^T r_{Z(u)}du} S_T | Z(0) = i] + \mathbb{E}[e^{-\int_0^T r_{Z(u)}du} \max\{\mathcal{X} - S_T, 0\} | Z(0) = i] \\ =& S_0 + \mathbb{E}[e^{-\int_0^T r_{Z(u)}du} \max\{\mathcal{X} - S_T, 0\} | Z(0) = i]. \end{aligned}$$

We define

$$F(T,x,i) = \mathbb{E}[e^{-\int_0^T r_{Z(u)}du} \max\{\mathcal{X} e^{x - X_p} - S_T, 0\} | Z(0) = i] \tag{6}$$

for an appropriate inversion point; see Petrella [24]. Note that

$$P_{S2}(T,i) = S_0 + F(T, X_p, i).$$

The joint Laplace transform of $F(T,x,i)$, $\tilde{F}(s,w,i)$ is defined as

$$\tilde{F}(s,w,i) \equiv \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{-st-wx}F(t,x,i)dtdx.$$

For $\boldsymbol{\nu} = (\nu_1,\ldots,\nu_m)$, define

$$\mathcal{D}_{\boldsymbol{\nu}} = \Big\{(s,w)\in\mathbb{C}^2 : \mathrm{Re}(w)>0, \mathrm{Re}(s) > \max_{1\le i\le m}\{\frac{\sigma_i^2}{2}(\mathrm{Re}(w))^2 - \nu_i\mathrm{Re}(w) - r_i\}\Big\}.$$

Theorem 3.2. *Suppose that* $(s,w-1)\in\mathcal{D}_{\boldsymbol{\mu}}$. *The joint Laplace transform* $\tilde{F}(s,w,i)$ *is given by*

$$\begin{aligned}\tilde{F}(s,w,i) =& \frac{S_0^{-(w-1)}X_p^w}{w(w-1)}\mathbf{1}_i^\top\Big(I+\sum_{k=1}^{n}(b_k^w - b_{k-1}^w)e^{(\ln b_k)Q(s,w-1;\boldsymbol{\mu})}\Big)\\ &\times(-\Psi(s,w-1;\boldsymbol{\mu}))^{-1}\mathbf{1}.\end{aligned}$$

Proof. See Appendix A. □

3.3. Stock and Bond Mixed Funds. In this subsection, the prices of step-up type1 and step-up type2 options are calculated under the assumption that the underlying fund is composed of stocks and bonds, simply called a mixed fund. It is assumed that the interest rate r is a constant.

3.3.1. Step-up type1 option. Recall $Y_t = \alpha S_t + (1-\alpha)S_0B_t$ and define

$$\mathcal{T}_b^Y = \inf\{t\ge 0 : \frac{Y_t}{Y_0}\ge b\}.$$

The payoff of the step-up type1 option on a stock and bond mixed fund at maturity T is expressed by

$$\mathcal{Y} = S_0 + S_0\sum_{k=1}^{n}(b_k - b_{k-1})\mathbb{1}_{\{\mathcal{T}_{b_k}^Y\le T\}}.$$

The price of the step-up type1 option on a stock and bond mixed fund at time 0 with $Z(0)=i$ is given by

$$\begin{aligned}P_{S1}^Y(T,\alpha,i) =& \mathbb{E}[e^{-rT}\mathcal{Y}|Z(0)=i]\\ =& e^{-rT}S_0 + e^{-rT}S_0\sum_{k=1}^{n}(b_k-b_{k-1})\mathbb{P}(\mathcal{T}_{b_k}^Y\le T|Z(0)=i). \end{aligned}\tag{7}$$

If $b_k-(1-\alpha)e^{rt}\le 0$, then $\mathbb{P}(\mathcal{T}_{b_k}^Y > T|Z(0)=i)=0$. If $b_k-(1-\alpha)e^{rt}>0$, then

$$\begin{aligned}\{\mathcal{T}_{b_k}^Y > T\} =& \{\alpha\frac{S_t}{S_0} + (1-\alpha)B_t < b_k \text{ for all } t\in[0,T]\}\\ =& \{\ln S_t < \beta_k(t) + \ln S_0 \text{ for all } t\in[0,T]\},\end{aligned}\tag{8}$$

where the boundary function $\beta_k(t) = \ln \frac{b_k-(1-\alpha)e^{rt}}{\alpha}$. To approximate (8), we consider a piecewise linear approximation of $\beta_k(t)$. Let $0 = t_0 < t_1 < \cdots < t_l = T$, $\tau_k = t_k = t_{k-1}$ for $k = 0, 1, \ldots, l$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_l)$. Define $\gamma_k(t)$ as

$$\gamma_k(t) = a_{kj}t + c_{kj},\ t_{j-1} \le t \le t_j,\ j = 1, \ldots, l, \tag{9}$$

where

$$a_{kj} = \frac{\beta_k(t_j) - \beta_k(t_{j-1})}{t_j - t_{j-1}},\ c_{kj} = \frac{t_j\beta_k(t_{j-1}) - t_{j-1}\beta_k(t_j)}{t_j - t_{j-1}}.$$

Define

$$\mathcal{T}_{\gamma_k} = \inf\{t : \ln \frac{S_t}{S_0} \ge \gamma_k(t)\}$$

for functions γ_k, $k = 1, \ldots, n$. Then we can write the approximation of $P^Y_{S1}(T, \alpha, i)$ as

$$P^*_{S1}(T, \alpha, i) = e^{-rT}S_0 + e^{-rT}S_0 \sum_{k=1}^{n} (b_k - b_{k-1})\mathbb{P}(\mathcal{T}_{\gamma_k} \le T|Z(0) = i).$$

Let

$$dU_t = \nu_{Z(t)}dt + \sigma_{Z(t)}dWt,$$
$$\mathcal{T}^U_b = \inf\{t \ge 0 : U_t \ge b\}.$$

Define

$$h_{ij}(t, y; x, \boldsymbol{\nu}, b) = \frac{\partial}{\partial y}\mathbb{E}[e^{-rt}\mathbb{1}_{\{\mathcal{T}^U_b > t, U_t \le y, Z(t)=j\}}|U_0 = x, Z(0) = i], \tag{10}$$

and its joint Laplace transform

$$\tilde{h}_{ij}(s, w; x, \boldsymbol{\nu}, b) = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-st-wy}h_{ij}(t, y; x, \boldsymbol{\nu}, b)dtdy. \tag{11}$$

Let $h(t, y; x, \boldsymbol{\nu}, b)$ and $\tilde{h}(s, w; x, \boldsymbol{\nu}, b)$ be $m \times m$ matrices whose (i, j)-th components are $h_{ij}(t, y; x, \boldsymbol{\nu}, b)$ and $\tilde{h}_{ij}(s, w; x, \boldsymbol{\nu}, b)$, respectively.

Theorem 3.3. (a) *The price of a step-up type1 option on a mixed fund is given by*

$$P^*_{S1}(T, \alpha, i) = e^{-rT}S_0 + S_0 \sum_{k=1}^{n} (b_k - b_{k-1})(e^{-rT} - H_k(\boldsymbol{\tau}, i)).$$

where $H_k(\boldsymbol{\tau}, i)$ is given by

$$\begin{aligned} H_k(\boldsymbol{\tau}, i) &= H_k(\tau_1, \tau_2, \ldots, \tau_l, i) = e^{-rT}\mathbb{P}(\mathcal{T}_{\gamma_k} > T|Z(0) = i) \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy_{l-1} \cdots \int_{-\infty}^{\infty} dy_1\ \mathbf{1}_i^\top h(\tau_1, y_1; 0, \bar{\boldsymbol{\mu}}_{k1}, c_{k1})h(\tau_2, y_2; y_1, \bar{\boldsymbol{\mu}}_{k2}, c_{k1}) \cdots \\ &\quad \times h(\tau_{l-1}, y_{l-1}; y_{l-2}, \bar{\boldsymbol{\mu}}_{k,l-1}, c_{k1})h(\tau_l, y; y_{l-1}, \bar{\boldsymbol{\mu}}_{kl}, c_{k1})\mathbf{1}, \end{aligned} \tag{12}$$

with h is given in (10) and $\bar{\boldsymbol{\mu}}_{kj} = \boldsymbol{\mu} - a_{kj}\mathbf{1}$.

(b) *The joint Laplace transform of h, $\tilde{h}$ is given by*

$$\tilde{h}(s,w;x,\boldsymbol{\nu},b) = -\,e^{-wx}(I - e^{(b-x)Q(s,w;\boldsymbol{\nu})})\Psi(s,w;\boldsymbol{\nu})^{-1}.$$

Proof. See Appendix C. □

For $l = 1$, the following proposition is obtained.

Proposition 3.4. *Suppose that $s \in \mathbb{C}$ satisfies $\mathrm{Re}(s) > -\max\{r_1, \dots, r_m\}$. Then the price of the step-up type1 option on a mixed fund has the following Laplace transforms,*

$$\begin{aligned}\tilde{P}^*_{S1}(s,\alpha,i) \equiv & \int_0^\infty e^{-st} P^*_{S1}(t,\alpha,i)dt \\ =& \frac{S_0}{r+s}\Big(1 + \sum_{k=1}^{n}(b_k - b_{k-1})\mathbf{1}_i^\top e^{c_{k,1}Q(s,0;\boldsymbol{\mu} - a_{k,1}\mathbf{1})}\mathbf{1}\Big).\end{aligned}$$

Proof. See Appendix D. □

3.3.2. Step-up type2 option. The payoff of the step-up type2 option on a mixed fund at maturity T is expressed by

$$\max(Y_T, \mathcal{Y}) = Y_T + \max(\mathcal{Y} - Y_T, 0).$$

The price of the step-up type2 option on a mixed fund at time 0 with $Z(0) = i$ is given by

$$\begin{aligned}P^Y_{S2}(T,i) =& \mathbb{E}[e^{-rT}Y_T|Z(0)=i] + \mathbb{E}[e^{-rT}\max(\mathcal{Y} - Y_T, 0)|Z(0)=i] \\ =& Y_0 + \mathbb{E}[e^{-rT}\max(Y_T - \mathcal{Y}, 0)|Z(0)=i]. \qquad (13)\end{aligned}$$

Similar to a pure stock fund, we have

$$\begin{aligned}P^Y_{S2}(T,i) =& S_0 + \sum_{k=1}^{n}\mathbb{E}[e^{-rT}(b_kS_0 - Y_T)^+\mathbb{1}_{\{\mathcal{T}^Y_{b_k} \le T\}}|Z(0)=i] \\ & - \sum_{k=1}^{n}\mathbb{E}[e^{-rT}(b_{k-1}S_0 - Y_T)^+\mathbb{1}_{\{\mathcal{T}^Y_{b_k} \le T\}}|Z(0)=i] \\ & + \mathbb{E}[e^{-rT}(S_0 - Y_T)^+|Z(0)=i]. \qquad (14)\end{aligned}$$

Then (14) can be written as

$$\begin{aligned}P^Y_{S2}(T,i) =& S_0 + \mathbb{E}[e^{-rT}(b_nS_0 - Y_T)^+|Z(0)=i] \\ & - \sum_{k=1}^{n}\mathbb{E}[e^{-rT}(b_kS_0 - Y_T)^+\mathbb{1}_{\{\mathcal{T}^Y_{b_k} > T\}}|Z(0)=i] \\ & + \sum_{k=1}^{n}\mathbb{E}[e^{-rT}(b_{k-1}S_0 - Y_T)^+\mathbb{1}_{\{\mathcal{T}^Y_{b_k} > T\}}|Z(0)=i] \\ \approx& S_0 + \mathbb{E}[e^{-rT}(b_nS_0 - Y_T)^+|Z(0)=i]\end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=1}^{n} \mathbb{E}[e^{-rT}(b_k S_0 - Y_T)^+ \mathbb{1}_{\{\mathcal{T}_{\gamma_k} > T\}} | Z(0) = i] \\
&+ \sum_{k=1}^{n} \mathbb{E}[e^{-rT}(b_{k-1} S_0 - Y_T)^+ \mathbb{1}_{\{\mathcal{T}_{\gamma_k} > T\}} | Z(0) = i] \\
\equiv & P^*_{S2}(T, i). \qquad (15)
\end{aligned}
$$

Theorem 3.5. (a) *The price of the step-up type2 option on a mixed fund is given by*

$$
P^*_{S2}(T, i) = S_0 + G_n(T, c_{nl}, i) + \sum_{k=1}^{n} \Big(G_{k,k-1}(\boldsymbol{\tau}, i) - G_{k,k}(\boldsymbol{\tau}, i) \Big), \qquad (16)
$$

where

$$
\begin{aligned}
G_k(T, x, i) =& \alpha \mathbb{E}[e^{-rT}(S_0 e^{a_{kl}T + x} - S_T)^+ | Z(0) = i], \\
G_{k,k'}(\boldsymbol{\tau}, i) =& \alpha \mathbb{E}[e^{-rT}(S_0 e^{\gamma_{k'}(T)} - S_T)^+ \mathbb{1}_{\{\mathcal{T}_{\gamma_k} > T\}} | Z(0) = i]
\end{aligned}
$$

for $0 \le k' \le k \le n$.

(b) $G_{k,k'}(\boldsymbol{\tau}, i)$ *is given by*

$$
\begin{aligned}
G_{k,k'}(\boldsymbol{\tau}, i) =& S_0 \int_{-\infty}^{c_{k'}} (e^{c_{k'}} - e^y) dy \int_{-\infty}^{\infty} dy_{l-1} \cdots \int_{-\infty}^{\infty} dy_1 \\
& \times \mathbf{1}_i^\top h(\tau_1, y_1; 0, \bar{\boldsymbol{\mu}}_{k1}, c_k) h(\tau_2, y_2; y_1, \bar{\boldsymbol{\mu}}_{k2}, c_k) \cdots \\
& \times h(\tau_{l-1}, y_{l-1}; y_{l-2}, \bar{\boldsymbol{\mu}}_{k,l-1}, c_k) h(\tau_l, y; y_{l-1}, \bar{\boldsymbol{\mu}}_{kl}, c_k) \mathbf{1},
\end{aligned}
$$

where h *is given in (10),* $c_k = \ln \frac{b_k}{\alpha}$ *and* $\bar{\boldsymbol{\mu}}_{kj} = \boldsymbol{\mu} - a_{kj}\mathbf{1}$.

(c) *The Laplace transform of* $G_k(T, x, i)$ *is given as follows:*

$$
\begin{aligned}
\tilde{G}_k(T, w, i) \equiv & \int_0^\infty e^{-wx} G_k(T, x, i) dx \\
=& S_0 \frac{\alpha e^{-(r + a_{kl})T}}{w(w-1)} \mathbf{1}_i^\top e^{T\Psi(0, w-1; \bar{\boldsymbol{\mu}}_{kl})} \mathbf{1}.
\end{aligned}
$$

Proof. See Appendix E. □

Next, consider $l = 1$. Similar to a pure stock fund, we have

$$
P^*_{S2}(T, i) = S_0 + F^*(T, X_p, i)
$$

where

$$
\begin{aligned}
F^*(t, x, i) =& \alpha \mathbb{E}[e^{-rt}(S_0 e^{\gamma_0(T) + (x - X_p)} - S_t)^+ | Z(0) = i] \\
&+ \alpha \sum_{k=1}^{n} \mathbb{E}[e^{-rt}(S_0 e^{\gamma_k(T) - a_{k,1}(T-t) + (x - X_p)} - S_t)^+ \mathbb{1}_{\{\mathcal{T}_{\gamma_k} \le t\}} | Z(0) = i] \\
&- \alpha \sum_{k=1}^{n} \mathbb{E}[e^{-rt}(S_0 e^{\gamma_{k-1}(T) - a_{k,1}(T-t) + (x - X_p)} - S_t)^+ \mathbb{1}_{\{\mathcal{T}_{\gamma_k} \le t\}} | Z(0) = i].
\end{aligned} \qquad (17)
$$

The joint Laplace transform of $F^*(t,x,i)$, $\tilde{F}^*(s,w,i)$, is defined as

$$\tilde{F}^*(s,w,i) \equiv \int_{-\infty}^{\infty}\int_0^{\infty} e^{-st-wx}F^*(t,x,i)dtdx.$$

Proposition 3.6. *For $(s,w-1) \in \mathcal{D}_{\boldsymbol{\mu}}$, $F^*(t,x,i)$ has the following joint Laplace transform:*

$$\tilde{F}^*(s,w,i) = \frac{\alpha(S_0e^{-X})^w}{w(w-1)}\mathbf{1}_i^\top\Bigg[e^{w\gamma_0(T)}(-\Psi(s,w-1;\boldsymbol{\mu}))^{-1} + \sum_{k=1}^{n}(e^{wd_{k,k}} - e^{wd_{k-1,k}})e^{(\ln\frac{b_k}{\alpha})Q(s-a_{k1},w-1;\boldsymbol{\mu}_{k1})}(-\Psi(s-a_{k1},w-1;\bar{\boldsymbol{\mu}}_{k1}))^{-1}\Bigg]\mathbf{1}, \tag{18}$$

where $d_{k,k'} = \gamma_k(T) - a_{k',1}T$.

Proof. See Appendix F. □

4. Numerical Examples

This section provides numerical analyses of step-up type1 and type2 option prices. To compute option prices, one- and two-dimensional Laplace transforms are inverted numerically using the Euler inversion method in Petrella [24]. For a survey of various inversion methods, see Abate and Whitt [1].

Using Theorems 1 and 2, the prices of step-up type1 and type2 options with a pure stock fund are obtained. For a mixed fund, we consider $l = 1$. We obtain the prices of step-up type1 option and type2 option with a mixed fund using Propositions 1 and 2.

Next, consider the model in the case of two regimes. The parameter values in a regime switching model are given as follows: interest rate $r = 0.0313$, state space $\{1,2\}$ and infinitesimal generator $G = \begin{pmatrix} -0.0166 & 0.0166 \\ 0.0393 & -0.0393 \end{pmatrix}$, and volatility $(\sigma_1,\sigma_2) = (0.2092, 0.4430)$. For all cases, we fixed $S_0 = 1$ and $X_p = 3$.

In Tables 1 and 2, we compare our method in Proposition 3.4 with Monte Carlo simulation for step-up type1 option prices $P^*_{S1}(T,i)$ with barriers 120%, 140%, 160%, 180%, 200%. In Tables 3 and 4, we compare our method in Proposition 3.6 with Monte Carlo simulation for step-up type2 option prices $P^*_{S2}(T,i)$ with barriers 120%, 140%, 160%, 180%, 200%. For the simulation values in Tables 1, 2, 3, and 4 we generate 10^5 replications for each simulation. Tables 1, 2, 3, and 4 show that the prices by our methods are at the 95% confidence interval(CI).

Tables 1 and 3 present the numerical results for prices $P^*(T,1)$ of the step-up type1 option and the step-up type2 option with varying T with α=0.3, 0.7.

Tables 2 and 4 present the numerical results for prices $P^*(T,1)$ of the step-up type1 option and the step-up type2 option with varying α with $T = 5, 10$. Because an increase in stock ratio implies an increase in risk, the option price in Tables 2 and 4 increases as the stock ratio rises.

	$\alpha = 0.3$			$\alpha = 0.7$		
	Our method	Simulation		Our method	Simulation	
T	Price	Price	CI	Price	Price	CI
1	0.9788	0.9787	(0.9783, 0.9791)	1.0403	1.0398	(1.0386, 1.0409)
2	0.9843	0.9841	(0.9832, 0.9850)	1.0934	1.0935	(1.0916, 1.0953)
3	0.9949	0.9956	(0.9943, 0.9969)	1.1304	1.1301	(1.1279, 1.1324)
4	1.0042	1.0051	(1.0035, 1.0066)	1.1523	1.1517	(1.1492, 1.1542)
5	1.0133	1.0127	(1.0109, 1.0144)	1.1622	1.1611	(1.1585, 1.1637)
6	1.0171	1.0167	(1.0148, 1.0186)	1.1629	1.1623	(1.1597, 1.1649)
7	1.0180	1.0174	(1.0154, 1.0194)	1.1558	1.1565	(1.1539, 1.1591)
8	1.0160	1.0151	(1.0131, 1.0172)	1.1455	1.1460	(1.1434, 1.1486)
9	1.0115	1.0113	(1.0092, 1.0133)	1.1305	1.1315	(1.1290, 1.1341)
10	1.0049	1.0045	(1.0024, 1.0066)	1.1127	1.1142	(1.1117, 1.1167)

Table 1. Step-up type1 option prices for mixed funds when $\alpha = 0.3$ and $\alpha = 0.7$. (Set of barriers = $\{1.2, 1.4, 1.6, 1.8, 2\}$)

	$T = 5$			$T = 10$		
	Our method	Simulation		Our method	Simulation	
α	Price	Price	CI	Price	Price	CI
0.1	0.9168	0.9169	(0.9161, 0.9178)	0.9423	0.9436	(0.9426, 0.9447)
0.2	0.9631	0.9640	(0.9619, 0.9661)	0.9694	0.9695	(0.9675, 0.9715)
0.3	1.0133	1.0127	(1.0109, 1.0144)	1.0049	1.0045	(1.0024, 1.0066)
0.4	1.0539	1.0523	(1.0502, 1.0543)	1.0323	1.0334	(1.0311, 1.0357)
0.5	1.0937	1.0938	(1.0915, 1.0961)	1.0633	1.0638	(1.0608, 1.0668)
0.6	1.1297	1.1295	(1.1263, 1.1327)	1.0898	1.0914	(1.0891, 1.0937)
0.7	1.1622	1.1611	(1.1585, 1.1637)	1.1127	1.1142	(1.1117, 1.1167)
0.8	1.1915	1.1908	(1.1875, 1.1941)	1.1328	1.1328	(1.1304, 1.1352)
0.9	1.2183	1.2163	(1.2128, 1.2198)	1.1506	1.1493	(1.1470, 1.1516)
1	1.2423	1.2406	(1.2378, 1.2434)	1.1663	1.1654	(1.1629, 1.1679)

Table 2. Step-up type1 option prices for mixed funds when $T = 5$ and $T = 10$. (Set of barriers = $\{1.2, 1.4, 1.6, 1.8, 2\}$)

5. Conclusion

A step-up option is an effective instrument that provides investors security and profitability simultaneously. A step-up option is particularly beneficial in case of extended maturity. This paper introduces two types of step-up option, the step-up type1 option and the step-up type2 option and derives pricing formulas using Laplace transform with the assumption that the underlying equity price series follows a regime-switching model. Moreover, the option prices are calculated using a pure equity portfolio and a portfolio composed of equities and fixed income securities to reflect possible variety in the underlying fund asset mix. The impact of change in the model parameters on the option price are analyzed. This study finds that the option price rises as the stock ratio and the number of steps increase for both types of step-up options. Maturity has a different effect

	$\alpha = 0.3$			$\alpha = 0.7$		
	Our method	Simulation		Our method	Simulation	
T	Price	Price	CI	Price	Price	CI
1	1.0152	1.0147	(1.0142, 1.0153)	1.0647	1.0639	(1.0627, 1.0651)
2	1.0221	1.0215	(1.0206, 1.0224)	1.1096	1.1108	(1.1089, 1.1127)
3	1.0292	1.0291	(1.0278, 1.0304)	1.1516	1.1505	(1.1478, 1.1532)
4	1.0367	1.0366	(1.0350, 1.0382)	1.1831	1.1821	(1.1787, 1.1855)
5	1.0431	1.0431	(1.0412, 1.0450)	1.2067	1.2056	(1.2017, 1.2095)
6	1.0480	1.0478	(1.0456, 1.0499)	1.2261	1.2233	(1.2189, 1.2277)
7	1.0515	1.0512	(1.0488, 1.0536)	1.2381	1.2345	(1.2296, 1.2394)
8	1.0538	1.0535	(1.0509, 1.0561)	1.2437	1.2429	(1.2375, 1.2483)
9	1.0550	1.0550	(1.0522, 1.0578)	1.2472	1.2492	(1.2431, 1.2553)
10	1.0552	1.0555	(1.0524, 1.0587)	1.2484	1.2507	(1.2439, 1.2575)

TABLE 3. Step-up type2 option prices for funds when $\alpha = 0.3$ and $\alpha = 0.7$ (Set of barriers $= \{1.2, 1.4, 1.6, 1.8, 2\}$).

	$T = 5$			$T = 10$		
	Our method	Simulation		Our method	Simulation	
α	Price	Price	CI	Price	Price	CI
0.1	1.0026	1.0028	(1.0021, 1.0034)	1.0055	1.0063	(1.0052, 1.0075)
0.2	1.0154	1.0143	(1.0130, 1.0156)	1.0234	1.0250	(1.0228, 1.0271)
0.3	1.0431	1.0431	(1.0412, 1.0450)	1.0552	1.0555	(1.0524, 1.0587)
0.4	1.0804	1.0806	(1.0782, 1.0830)	1.0982	1.0977	(1.0937, 1.1016)
0.5	1.1212	1.1227	(1.1197, 1.1256)	1.1472	1.1462	(1.1414, 1.1510)
0.6	1.1627	1.1627	(1.1594, 1.1660)	1.1971	1.1981	(1.1924, 1.2037)
0.7	1.2067	1.2056	(1.2017, 1.2095)	1.2484	1.2507	(1.2439, 1.2575)
0.8	1.2430	1.2441	(1.2399, 1.2483)	1.2967	1.2989	(1.2915, 1.3063)
0.9	1.2850	1.2846	(1.2801, 1.2892)	1.3496	1.3484	(1.3401, 1.3567)
1	1.3253	1.3272	(1.3220, 1.3324)	1.4023	1.3934	(1.3838, 1.4030)

TABLE 4. Step-up type2 option prices for mixed funds when $T = 5$ and $T = 10$ (Set of barriers $= \{1.2, 1.4, 1.6, 1.8, 2\}$).

on the step-up type1 option and the step-up type2 option. The price increases as a result of extended maturity was limited for step-up type1 option, whereas the step-up type2 option price shows a consistent increase as a result of extended maturity.

Step-up options might have a broad range applications in developing novel financial products. Therefore, the study of step-up option applications could be a subject for future research. Additionally, hedging strategies to manage the risks involved in providing step-up options could also be a future area of research.

Appendix

Appendix A. Proof of Theorem 3.1

Using (2), $\tilde{P}_{S1}(s,i)$ can be written as

$$\tilde{P}_{S1}(s,i) = S_0 \int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}|Z(0)=i]dt$$
$$+ S_0 \sum_{j=1}^m (b_k - b_{k-1}) \int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]dt. \quad (19)$$

Therefore, we should derive

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}|Z(0)=i]dt \quad (20)$$

and

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]dt. \quad (21)$$

For (20), we know that

$$\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}|Z(0)=i] = \mathbf{1}_i^\top e^{t(G-R)}\mathbf{1}.$$

Then for $\mathrm{Re}(s) > -\max\{r_1,\dots,r_m\}$, we have

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}|Z(0)=i]dt = \int_0^\infty \mathbb{E}[e^{-\int_0^t (r_{Z(u)}+s)du}|Z(0)=i]dt$$
$$= \mathbf{1}_i^\top(-G+R+sI)^{-1}\mathbf{1}. \quad (22)$$

For (21), we can derive

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]dt$$
$$= \mathbb{E}[\int_{\mathcal{T}_{b_k}}^\infty e^{-\int_{\mathcal{T}_{b_k}}^t (r_{Z(u)}+s)du - \int_0^{\mathcal{T}_{b_k}} (r_{Z(u)}+s)du} dt|Z(0)=i].$$

By the Markov property and (22),

$$\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]dt$$
$$= \sum_{j=1}^m \mathbb{E}[e^{-\int_0^{\mathcal{T}_{b_k}} (r_{Z(u)}+s)du}\mathbb{1}_{\{Z(\mathcal{T}_{b_k})=j\}}dt|Z(0)=i]\mathbf{1}_j(-G+R+sI)^{-1}\mathbf{1}. \quad (23)$$

Let $dU_t = \nu_{Z(t)}dt + \sigma_{Z(t)}dW_t$, with $U_0 = 0$ and $\mathcal{T}_y^U = \inf\{t \ge 0 : U_t \ge y\}$. We define

$$f(y,\boldsymbol{\nu},\boldsymbol{\eta}) = \Big(\mathbb{E}[e^{-\int_0^{\mathcal{T}_b^U} \eta_{Z(u)}du}\mathbb{1}_{\{Z(\mathcal{T}_b^U)=j\}}|Z(0)=i]\Big)_{i,j\in E}.$$

It is well known that $f(b,\boldsymbol{\nu},\boldsymbol{\eta}) = e^{b\Phi(\boldsymbol{\nu},\boldsymbol{\eta})}$, where $\Phi(\boldsymbol{\nu},\boldsymbol{\eta})$ satisfies

$$\frac{1}{2}\Sigma^2\Phi^2 - \mathrm{diag}(\boldsymbol{\nu})\Phi + G - \mathrm{diag}(\boldsymbol{\eta}) = O$$

by Feynman-Kac formula. In Kim et al. [17], the iterative algorithm for computing $\Phi(\boldsymbol{\nu},\boldsymbol{\eta})$ is represented. Define $Q(s,w;\boldsymbol{\nu}) = \Phi(\boldsymbol{\nu},\boldsymbol{r}+s) - wI$, then $Q(s,w;\boldsymbol{\nu})$ satisfies (5). Substituting $\eta_i = r_i + s$ and $\nu_i = \mu_i$, we obtain

$$\begin{aligned}\Big(\mathbb{E}[e^{-\int_0^{\mathcal{T}_{b_k}}(r_{Z(u)}+s)du}\mathbb{1}_{\{Z(\mathcal{T}_{b_k})=j\}}dt|Z(0)=i]\Big)_{i,j\in E} =& e^{(\ln b_k)\Phi(\boldsymbol{\mu},\boldsymbol{r}+s)}\\ =& e^{(\ln b_k)Q(s,0;\boldsymbol{\mu})}.\end{aligned}$$

Then, (23) is

$$\begin{aligned}&\int_0^\infty e^{-st}\mathbb{E}[e^{-\int_0^t r_{Z(u)}du}\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]dt\\ &= \mathbf{1}_i^\top e^{(\ln b_k)Q(s,0;\boldsymbol{\mu})}(G-R-sI)^{-1}\mathbf{1}.\end{aligned} \tag{24}$$

Substituting (22) and (24) into (19), we obtain (4).

Appendix B. Proof of Theorem 3.2

Before deriving the pricing formula of the step-up type2 option, consider an up-and-in single-barrier put option with maturity T, strike price K and upper barrier b, that is, the payoff of the single-barrier put option is given by

$$(K - S_T)^+\mathbb{1}_{\{\mathcal{T}_b\le T\}}.$$

Then the price of the up-and-in single-barrier put option is given by

$$\mathbb{E}[e^{-\int_0^T r_{Z(u)}du}(K-S_T)^+\mathbb{1}_{\{\mathcal{T}_b\le T\}}|Z(0)=i].$$

To choose an appropriate inversion point, we use the rescaling parameter $X_p < S_0$. Letting $k = \ln(K/X_p)$, we define

$$P_B(T,k,b,i;\boldsymbol{r},\boldsymbol{\mu},X_p) = \mathbb{E}[e^{-\int_0^T r_{Z(u)}du}X_p(e^k - \frac{S_T}{X_p})^+\mathbb{1}_{\{\mathcal{T}_b\le T\}}|Z(0)=i].$$

The joint Laplace transform of $P_B(t,x,b,\boldsymbol{\mu},i)$ is given by

$$\tilde{P}_B(s,w,b,i;\boldsymbol{r},\boldsymbol{\mu},X_p) \equiv \int_{-\infty}^\infty\int_0^\infty e^{-st-wx}P_B(t,x,b,\boldsymbol{\mu},X_p,i)dtdx.$$

For $\mathrm{Re}(w-1) > 0$, we obtain $\tilde{P}_B(s,w,b,i;\boldsymbol{r},\boldsymbol{\mu},X_p)$ as

$$\begin{aligned}&\tilde{P}_B(s,w,b,i;\boldsymbol{r},\boldsymbol{\mu},X_p)\\ &= \frac{S_0^{-(w-1)}X_p^w}{w(w-1)}\mathbb{E}[\int_{\mathcal{T}_b}^\infty e^{-\int_0^T(r_{Z(u)}+s)du-(w-1)U_t}dt|Z(0)=i],\end{aligned} \tag{25}$$

where $U_t = \ln\frac{S_t}{S_0}$. By the Markov property, we have

$$\mathbb{E}[\int_{\mathcal{T}_b}^\infty e^{-\int_0^T(r_{Z(u)}+s)du-wU_t}dt|Z(0)=i]$$

$$= \sum_{j=1}^{m} \mathbb{E}[e^{-\int_0^{\mathcal{T}_b}(r_{Z(\tau)}+s)d\tau - w\ln b}\mathbb{1}_{\{Z(\mathcal{T}_b^U)=j\}}dt|Z(0)=i]$$
$$\times \mathbb{E}[\int_{\mathcal{T}_b}^{\infty} e^{-\int_{\mathcal{T}_b}^{t}(r_{Z(\tau)}+s)d\tau - w(U_t-\ln b)}dt|Z(\mathcal{T}_b)=j]$$
$$= \sum_{j=1}^{m} \mathbb{E}[e^{-\int_0^{\mathcal{T}_b^U}(r_{Z(\tau)}+s)d\tau - w\ln b}\mathbb{1}_{\{Z(\mathcal{T}_b^U)=j\}}dt|Z(0)=i]$$
$$\times \mathbb{E}[\int_{0}^{\infty} e^{-\int_0^{t}(r_{Z(\tau)}+s)d\tau - wU_t}dt|Z(0)=j]. \tag{26}$$

Given that

$$\mathbb{E}[e^{-\int_0^t(r_{Z(\tau)}+s)d\tau - wU_t}\mathbb{1}_{\{Z(t)=j\}}|Z(0)=i] = (e^{t\Psi(s,w;\boldsymbol{\nu})})_{ij},$$

we have

$$\mathbb{E}[\int_0^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau - wU_t}dt|Z(0)=j] = \mathbf{1}_j^{\top}(-\Psi(s,w;\boldsymbol{\mu})^{-1})\mathbf{1}, \tag{27}$$

for $\mathrm{Re}((\Psi(s,w;\boldsymbol{\nu}))_{ii}) < 0$, $i \in E$, i.e., $\mathrm{Re}(s) > \max_{1\le i\le m}\{\frac{\sigma_i^2}{2}(\mathrm{Re}(w))^2 - \nu_i\mathrm{Re}(w) - r_i\}$. In the proof of Theorem 3.1, we have

$$\mathbb{E}[e^{-\int_0^{\mathcal{T}_b^U}(r_{Z(\tau)}+s)d\tau - w\ln b}\mathbb{1}_{\{Z(\mathcal{T}_b^U)=j\}}dt|Z(0)=i]$$
$$= e^{-wb}\mathbb{E}[e^{-\int_0^{\mathcal{T}_b^U}(r_{Z(\tau)}+s)d\tau}\mathbb{1}_{\{Z(\mathcal{T}_b^U)=j\}}dt|Z(0)=i]$$
$$= (e^{(\ln b)Q(s,w;\boldsymbol{\mu})})_{ij}. \tag{28}$$

Substituting (27) and (28) into (26), we obtain

$$\mathbb{E}[\int_{\mathcal{T}_b}^{\infty} e^{-\int_0^T(r_{Z(u)}+s)du - wU_t}dt|Z(0)=i]$$
$$= \mathbf{1}_i^{\top}e^{(\ln b)Q(s,w;\boldsymbol{\mu})}(-\Psi(s,w;\boldsymbol{\mu})^{-1})\mathbf{1}. \tag{29}$$

Then (25) is derived as

$$\tilde{P}_B(s,w,b,i;\boldsymbol{r},\boldsymbol{\mu},X_p)$$
$$= \frac{S_0^{-(w-1)}X_p^w}{w(w-1)}\mathbf{1}_i^{\top}e^{(\ln b)Q(s,w-1;\boldsymbol{\mu})}(-\Psi(s,w-1;\boldsymbol{\mu})^{-1})\mathbf{1}. \tag{30}$$

Given that

$$F(T,x,i) = \sum_{k=1}^{n}\mathbb{E}[e^{-\int_0^T r_{Z(u)}du}(b_kS_0e^{(x-X)} - S_T)^+\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]$$
$$- \sum_{k=1}^{n}\mathbb{E}[e^{-\int_0^T r_{Z(u)}du}(b_{k-1}S_0e^{(x-X)} - S_T)^+\mathbb{1}_{\{\mathcal{T}_{b_k}\le T\}}|Z(0)=i]$$
$$+ \mathbb{E}[e^{-\int_0^T r_{Z(u)}du}(S_0e^{(x-X)} - S_T)^+|Z(0)=i],$$

it can be written as

$$F(T,x,i) = \sum_{k=1}^{n}\Big(P_B(T,x,b_k,i;\boldsymbol{r},\boldsymbol{\mu},b_kS_0e^{-X}) - P_B(T,x,b_k,i;\boldsymbol{r},\boldsymbol{\mu},b_{k-1}S_0e^{-X})\Big) + P_B(T,x,1,i;\boldsymbol{r},\boldsymbol{\mu},S_0e^{-X}). \tag{31}$$

Substituting (30) into the Laplace transform of (31), the proof is complete.

Appendix C. Proof of Theorem 3.3

(a) We have

$$P(\mathcal{T}_{\gamma_k} > T|Z(0)=i) = \sum_{j=1}^{m}\int_{-\infty}^{\infty}\Big(\frac{\partial}{\partial y}\mathbb{P}(\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} \le y, Z(t_1)=j|Z(0)=i)\Big) \times \mathbb{P}(\mathcal{T}_{\gamma_k} > T|\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} = y, Z(t_1)=j)dy. \tag{32}$$

Let $U_t = \ln\frac{S_t}{S_0} - a_{k1}t$; then, $dU_t = (\bar{\boldsymbol{\mu}}_{k1})_{Z(t)}dt + \sigma_{Z(t)}dW_t$ and $\{\mathcal{T}_{\gamma_k} > t_1\} = \{\mathcal{T}^U_{c_{k1}} > t_1\}$. Therefore the first term on the right-hand side of (32) is

$$\begin{aligned}&\frac{\partial}{\partial y}\mathbb{P}(\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} \le y, Z(t_1)=j|Z(0)=i)\\ =&\frac{\partial}{\partial y}\mathbb{P}(\mathcal{T}^U_{c_{k1}} > t_1, U_{t_1} \le y + a_{c1}t_1, Z(t_1)=j|Z(0)=i)\\ =&h_{ij}(t_1, y+a_{c1}t_1; 0, \bar{\boldsymbol{\mu}}_{k1}, c_{k1}).\end{aligned} \tag{33}$$

Then (32) can be written as

$$\begin{aligned}&P(\mathcal{T}_{\gamma_k} > T|Z(0)=i)\\ =&\sum_{j=1}^{m}\int_{-\infty}^{\infty} h_{ij}(t_1, y+a_{c1}t_1; 0, \bar{\boldsymbol{\mu}}_{k1}, c_{k1})\mathbb{P}(\mathcal{T}_{\gamma_k} > T|\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} = y, Z(t_1)=j)dy\\ =&\sum_{j=1}^{m}\int_{-\infty}^{\infty} h(t_1, y_1; 0, \bar{\boldsymbol{\mu}}_{k1}, c_{k1})\mathbb{P}(\mathcal{T}_{\gamma_k} > T|\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} = y_1 - a_{c1}t_1, Z(t_1)=j)dy_1\end{aligned}$$

Fix $y_1 < a_{k1}t_1 + c_{k1}$ to consider $\mathbb{P}(\mathcal{T}_{\gamma_k} > T|\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} = y_1 - a_{c1}t_1, Z(t_1) = j)$. Let $\gamma_k^{(x,y)}(t) = \gamma_k(t+x) - y$, then

$$\mathbb{P}(\mathcal{T}_{\gamma_k} > T|\mathcal{T}_{\gamma_k} > t_1, \ln\frac{S_{t_1}}{S_0} = y_1 - a_{c1}t_1, Z(t_1)=j) = \mathbb{P}(\mathcal{T}_{\gamma_k^{(t_1,y_1)}} > T - t_1|Z(0)=j).$$

Similar to (33), it can be written as

$$\begin{aligned}&\mathbb{P}(\mathcal{T}_{\gamma_k^{(t_1,y_1)}} > T - t_1|Z(0)=j)\\ =&\sum_{j=1}^{m}\int_{-\infty}^{\infty} h_{ij}(\tau_2, y_2; 0, \bar{\boldsymbol{\mu}}_{k2}, c_{k1})\mathbb{P}(\mathcal{T}_{\gamma_k^{(t_2,y_2)}} > T - t_2|Z(0)=j)dy_2.\end{aligned}$$

By repeating $l-2$ times, we obtain (12).
(b) By definition of h_{ij}, it can be written as

$$\begin{aligned}h_{ij}(t,y;x,\boldsymbol{\nu},b) =& h_{ij}(t,y-x;0,\boldsymbol{\nu},b-x)\\ =& \frac{\partial}{\partial y}\mathbb{E}[e^{-\int_0^t r_{Z(\tau)}d\tau}\mathbb{1}_{\{U_t\le y-x,\mathcal{T}^U_{b-x}>T\}}|Z(0)=i]\\ =& \frac{\partial}{\partial y}\Big(\mathbb{E}[e^{-\int_0^t r_{Z(\tau)}d\tau}\mathbb{1}_{\{U_t\le y-x\}}|Z(0)=i] - f_{ij}(t,y-x;\boldsymbol{\nu},b-x)\Big),\end{aligned}$$

where

$$f_{ij}(t,y;\boldsymbol{\nu},b) = \mathbb{E}[e^{-\int_0^t r_{Z(\tau)}d\tau}\mathbb{1}_{\{\mathcal{T}^U_b\le t,U_t\le y,Z(t)=j\}}|Z(0)=i].$$

The joint Laplace transform is given by

$$\begin{aligned}\tilde{f}_{ij}(s,w;x,\boldsymbol{\nu},b) \equiv& \int_{-\infty}^{\infty}\int_0^{\infty} e^{-st-wy}f_{ij}(t,y;x,\boldsymbol{\nu},b)dtdy\\ =& \mathbb{E}[\int_{U_t}^{\infty}\int_{\mathcal{T}^U_b}^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau-wy}\mathbb{1}_{\{Z(t)=j\}}dtdy|Z(0)=i]\\ =& \frac{1}{w}\mathbb{E}[\int_{\mathcal{T}^U_b}^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau-wU_t}\mathbb{1}_{\{Z(t)=j\}}dt|Z(0)=i].\end{aligned}$$

Using (29), in the proof of Theorem 3.2, we obtain

$$\tilde{f}(s,w;x,\boldsymbol{\nu},b) = \frac{1}{w}e^{bQ(s,w;\boldsymbol{\nu})}(-\Psi(s,w;\boldsymbol{\nu})^{-1}). \tag{34}$$

Then, we derive

$$\begin{aligned}\tilde{h}_{ij}(s,w;x,\boldsymbol{\nu},b) =& w\int_{-\infty}^{\infty}\int_0^{\infty} e^{-st-wy}\mathbb{E}[e^{-\int_0^t r_{Z(\tau)}d\tau}\mathbb{1}_{\{U_t\le y-x\}}|Z(0)=i]dtdy\\ &- w\int_{-\infty}^{\infty}\int_0^{\infty} e^{-st-wy}f_{ij}(t,y-x;\boldsymbol{\nu},b-x)dtdy\\ =& we^{-wx}\mathbb{E}[\int_{-\infty}^{\infty}\int_0^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau-wz}\mathbb{1}_{\{U_t\le z\}}dtdz|Z(0)=i]\\ &- we^{-wx}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-st-wz}f_{ij}(t,z;\boldsymbol{\nu},b-x)dzdt\\ =& we^{-wx}\mathbb{E}[\int_0^{\infty}\int_{U_t}^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau-wz}dzdt|Z(0)=i]\\ &- we^{-wx}\tilde{f}_{ij}(s,w;\boldsymbol{\nu},b-x)\\ =& e^{-wx}\mathbb{E}[\int_0^{\infty} e^{-\int_0^t(r_{Z(\tau)}+s)d\tau-wU_t}dt|Z(0)=i]\\ &- we^{-wx}\tilde{f}_{ij}(s,w;\boldsymbol{\nu},b-x).\end{aligned}$$

Therefore,

$$\tilde{h}(s,w;x,\boldsymbol{\nu},b) = e^{-wx}(I-e^{(b-x)Q(s,w;\boldsymbol{\nu})})(-\Psi(s,w;\bar{\boldsymbol{\mu}}_{k1}))^{-1}.$$

Appendix D. Proof of Proposition 3.4

When $l = 1$, $\gamma_k(t) = a_{k1}t + c_{k1}$. Let $U_t = \ln \frac{S_t}{S_0} - a_{k1}t$ for $k = 1, \dots, n$, i.e., $dU_t = (\bar{\boldsymbol{\mu}}_{k1})_{Z(t)}dt + \sigma_{Z(t)}dWt$ where $\bar{\boldsymbol{\mu}}_{k1} = \boldsymbol{\mu} - a_{k,1}\mathbf{1}$. Since $\mathcal{T}_{\gamma_k} = \mathcal{T}^U_{c_{k1}}$, we have

$$\begin{aligned} e^{-rT}\mathbb{P}(\mathcal{T}_{\gamma_k} \le T|Z(0) = i) =& e^{-rT}\mathbb{P}(\mathcal{T}^U_{c_{k1}} \le T|Z(0) = i) \\ =& \sum_{j=1}^{m} \mathbb{E}[e^{-rT}\mathbb{1}_{\{\mathcal{T}^U_{c_{k1}} \le T, Z(T)=j\}}|Z(0) = i]. \end{aligned} \tag{35}$$

From equation (29), in the proof of Theorem 3.2, we have

$$\begin{aligned} &\int_0^\infty e^{-sT}\mathbb{E}[e^{-rT}\mathbb{1}_{\{\mathcal{T}^U_{c_{k1}} \le T, Z(T)=j\}}|Z(0) = i]dT \\ =&\mathbb{E}[\int_{\mathcal{T}^U_{c_{k1}}}^\infty e^{-(r+s)T}\mathbb{1}_{\{Z(T)=j\}}dT|Z(0) = i] \\ =&\big(e^{c_{k1}Q(s,0;\bar{\boldsymbol{\mu}}_{k1})}(-\Psi(s,0;\bar{\boldsymbol{\mu}}_{k1}))^{-1}\big)_{ij}. \end{aligned}$$

Given that $(\Psi(s,0;\bar{\boldsymbol{\mu}}_{k1}))^{-1}\mathbf{1} = \frac{1}{s+r}\mathbf{1}$, the Laplace transform of (35) can be written as

$$\int_0^\infty e^{-(s+r)T}\mathbb{P}(\mathcal{T}_{\gamma_k} \le T|Z(0) = i)dT = \frac{1}{s+r}\mathbf{1}_i^\top e^{c_{k1}Q(s,0;\bar{\boldsymbol{\mu}}_{k1})}\mathbf{1}.$$

Therefore the Laplace transform of $P^*_{S1}(T,\alpha,i)$, $\tilde{P}^*_{S1}(s,i)$ is given by

$$\tilde{P}^*_{S1}(s,i) = \frac{S_0}{r+s}\Big(1 + \sum_{j=1}^{m}(b_k - b_{k-1})\mathbf{1}_i^\top e^{c_{k1}Q(s,w;\bar{\boldsymbol{\mu}}_{k1})}\mathbf{1}\Big).$$

Appendix E. Proof of Theorem 3.5

(a) We have

$$b_kS_0 - Y_T = \alpha(e^{\gamma_k(T)}S_0 - S_T) = \alpha(e^{a_{kl}T + c_{kl}}S_0 - S_T). \tag{36}$$

Substituting (36) into (15), we have (16).
(b) We omit the proof because it is similar to Theorem 3.3 (a).
(c) By definition of $G_k(T,x,i)$, we have

$$\begin{aligned} \tilde{G}^*_k(T,w,i) \equiv& \int_{-\infty}^\infty e^{-wx}G_k(T,x,i)dx \\ =& \alpha\mathbb{E}[\int_{-\infty}^\infty e^{-wx}e^{-rT}(e^{a_{kl}T+x} - \frac{S_T}{S_0})^+dx|Z(0) = i] \\ =& \alpha S_0 e^{-rT}\mathbb{E}[\int_{\ln\frac{S_T}{S_0} - a_{kl}T}^\infty \big(e^{a_{kl}T-(w-1)x} - \frac{S_T}{S_0}e^{-wx}\big)dx|Z(0) = i] \\ =& S_0\frac{\alpha e^{-(r+a_{kl})T}}{w(w-1)}\mathbb{E}[e^{-(w-1)(\ln\frac{S_T}{S_0} - a_{kl}T)}|Z(0) = i]. \end{aligned} \tag{37}$$

Set $U_t = \ln \frac{S_t}{S_0} - a_{kl}t$; then,

$$\begin{aligned}\mathbb{E}[e^{-w(\ln \frac{S_t}{S_0} - a_{kl}t)} \mathbb{1}_{\{Z(t)=j\}} | Z(0) = i] =& \mathbb{E}[e^{-wU_t} \mathbb{1}_{\{Z(t)=j\}} | Z(0) = i] \\ =& \big(e^{t\Psi(0;w,\bar{\boldsymbol{\mu}}_{kl})}\big)_{ij}\end{aligned}$$

Therefore, (37) can be written as

$$\tilde{G}^*_k(T, w, i) = S_0 \frac{\alpha e^{-(r+a_{kl})T}}{w(w-1)} \mathbf{1}_i^\top e^{T\Psi(0,w-1;\bar{\boldsymbol{\mu}}_{kl})} \mathbf{1}$$

which completes the proof of (c).

Appendix F. Proof of Proposition 3.6

Let $S^k_t = e^{-a_{k1}t} S_t$ and $\mathcal{T}^k_b = \inf\{t \geq 0 : S^k_t \geq b S^k_0\}$. Then $\mathcal{T}_{\gamma_k} = \mathcal{T}^k_{e^{c_{k1}}}$. Using (17), we have

$$\begin{aligned}&F^*(t, x, i) \\ &= \alpha \mathbb{E}[e^{-rt}(S_0 e^{\gamma_0(T)+(x-X)} - S_t)^+ | Z(0) = i] \\ &\quad + \alpha \sum_{k=1}^n \mathbb{E}[e^{-(r-a_{k1})t}(S_0 e^{\gamma_k(T) - a_{k,1}T - X} e^x - S^k_t)^+ \mathbb{1}_{\{\mathcal{T}^k_{e^{c_{k1}}} \leq t\}} | Z(0) = i] \\ &\quad - \alpha \sum_{k=1}^n \mathbb{E}[e^{-(r-a_{k1})t}(S_0 e^{\gamma_{k-1}(T) - a_{k,1}T - X} e^x - S^k_t)^+ \mathbb{1}_{\{\mathcal{T}^k_{e^{c_{k1}}} \leq t\}} | Z(0) = i].\end{aligned}$$

Then $F^*(t, x, i)$ is written as

$$\begin{aligned}F^*(t, x, i) =& \alpha P_B(t, x, 1, i; r, \boldsymbol{\mu}, S_0 e^{\gamma_0(T)-X}) \\ &+ \alpha \sum_{k=1}^n \Big\{ P_B(t, x, e^{c_{k1}}, i; r - a_{k1}, \bar{\boldsymbol{\mu}}_{k1}, S_0 e^{d_{kk}-X}) \\ &- P_B(t, x, e^{c_{k1}}, i; r - a_{k1}, \bar{\boldsymbol{\mu}}_{k1}, S_0 e^{d_{k-1,k}-X}) \Big\}.\end{aligned}$$

The joint Laplace transform of $F^*(t, x, i)$ is given by

$$\begin{aligned}\tilde{F}^*(s, w, i) =& \alpha \tilde{P}_B(s, w, 1, i; r, \boldsymbol{\mu}, S_0 e^{\gamma_0(T)-X}) \\ &+ \alpha \sum_{k=1}^n \Big\{ \tilde{P}_B(s - a_{k1}, w, e^{c_{k1}}, i; r, \bar{\boldsymbol{\mu}}_{k1}, S_0 e^{d_{kk}-X}) \\ &- \tilde{P}_B(s - a_{k1}, w, e^{c_{k1}}, i; r, \bar{\boldsymbol{\mu}}_{k1}, S_0 e^{d_{k-1,k}-X}) \Big\},\end{aligned} \tag{38}$$

since $\tilde{P}_B(s, w, b, i; r - a, \boldsymbol{\mu}, X_p) = \tilde{P}_B(s - a, w, b, i; r, \boldsymbol{\mu}, X_p)$. Substituting (30) in the proof of Theorem 3.2 into (38), we have (18).

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Jerim Kim received M.Sc. and Ph.D. from Korea University. Since 2017 she has been at University of Seoul. Her research interests include probability theory, mathematical finance, insurance models, applied operations research, queueing theory and their applications.

Department of Mathematics, University of Seoul, Seoul 02504, Korea.
e-mail: jerimkim@uos.ac.kr

Eyunghee Kim received M.Sc. from the University of New South Wales, and Ph.D. from Korea University. She is currently at Finance Insurance Research Institute. Her research interests are financial management, insurance, and actuarial science.

Finance Insurance Research Institute, Seoul 07445, Korea.
e-mail: honeyoong@naver.com

Changki Kim received M.Sc. from the University of Iowa, and Ph.D. from the University of Iowa. He is currently a professor at Korea University since 2011. His research interests are insurance, risk management, actuarial science, and financial mathematics.

Korea University Business School, Seoul 02841, Korea.
e-mail: changki@korea.ac.kr