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SOME IDENTITIES INVOLVING THE DEGENERATE BELL-CARLITZ POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATION†

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Abstract. In this paper we define a new degenerate Bell-Carlitz polynomials. It also derives the differential equations that occur in the generating function of the degenerate Bell-Carlitz polynomials. We establish some new identities for the degenerate Bell-Carlitz polynomials. Finally, we perform a survey of the distribution of zeros of the degenerate Bell-Carlitz polynomials.

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1. Introduction

Recently, we have studied in the area of the special numbers and polynomials. Many generalizations of these polynomials have been studied(see [1, 2, 4, 5, 6]). The Bell-Carlitz polynomials $B_n^c(x)(n \geq 0)$, were introduced by Alain M. Robert(see [3]).

The Bell-Carlitz polynomials $B_n^c(x)$ are defined by the generating function

$$\sum_{n=0}^{\infty} B_n^c(x)\frac{t^n}{n!} = e^{(xt+e^t-1)} \text{ (see [3])}. \tag{1.1}$$

As is well known, the Bell numbers B_n are given by the generating function

$$\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = e^{(e^t-1)}. \tag{1.2}$$

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We define the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathbf{B}_n^c(x,\lambda)\frac{t^n}{n!} = (1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)}. \tag{1.4}$$

Since $(1+\lambda)^{\frac{tx}{\lambda}} \to e^{xt}$ as $\lambda \to 0$, it is evident that (1.4) reduces to (1.1). Now, we recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1.5}$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.6}$$

Note that

$$\mathcal{G}(t,x,\lambda) = (1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)}$$

satisfies

$$\frac{\partial \mathcal{G}(t,x,\lambda)}{\partial x} - \frac{\log(1+\lambda)}{\lambda} t(1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)} = 0.$$

Substitute the series in (1.4) for $\mathcal{G}(t,x,\lambda)$ to get

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial x}\mathbf{B}_n^c(x,\lambda)\frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{\log(1+\lambda)}{\lambda} n\mathbf{B}_{n-1}^c(x,\lambda)\frac{t^n}{n!}.$$

Thus we have the following theorem.

Theorem 1.1. *The degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ in generating function (1.4) are the solution of equation*

$$\frac{\partial}{\partial x}\mathbf{B}_n^c(x,\lambda) - \frac{n\log(1+\lambda)}{\lambda}\mathbf{B}_{n-1}^c(x,\lambda) = 0.$$

The generating function (1.4) is useful for deriving several properties of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$. For example, we have the following expression for these polynomials:

Theorem 1.2. *For any positive integer n, we have*

$$\mathbf{B}_n^c(x,\lambda) = \sum_{k=0}^{n}\sum_{l=0}^{k} \binom{n}{k} S_2(k,l) \left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-k} x^{n-k}.$$

Proof. By (1.2) and (1.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{B}_n^c(x,\lambda)\frac{t^n}{n!} &= (1+\lambda)^{\frac{tx}{\lambda}}e^{(e^t-1)}\\ &= \sum_{n=0}^{\infty}\left(\frac{x\log(1+\lambda)}{\lambda}\right)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(e^t-1)^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{x\log(1+\lambda)}{\lambda}\right)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}S_2(n,l)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}S_2(k,l)\left(\frac{x\log(1+\lambda)}{\lambda}\right)^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

On comparing the coefficients of $\frac{t^n}{n!}$, the expected result of Theorem 1.2 is achieved. □

The following basic properties of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ are derived form (1.4), (1.5), and (1.6). We, therefore, choose to omit the details involved.

Theorem 1.3. *For any positive integer n, we have*

(1) $\quad \mathbf{B}_n^c(x,\lambda)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k x^k B_{n-k}.$

(2) $\quad \mathbf{B}_n^c(x_1+x_2,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathbf{B}_l^c(x_1,\lambda)x_2^{n-l}\lambda^{l-n}(\log(1+\lambda))^{n-l}.$

(3) $\quad \mathbf{B}_n^c(x_1+x_2,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathbf{B}_l^c(x_1,\lambda)\sum_{m=n-l}^{\infty}S_1(m,n-l)\frac{\lambda^{m+l-n}(n-l)!}{m!}x_2^{n-l}.$

Recently, in order to give explicit identities for special polynomials, differential equations arising from the generating functions of special polynomials are studied by many authors(see [1, 2, 4, 5, 6]). Inspired by their work, we construct a differential equations by generating function of degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ as follow. Let D denote differentiation with respect to t, D^2 denote differentiation twice with respect to t, and so on; that is, for positive integer N,

$$D^N\mathcal{G}=\left(\frac{\partial}{\partial t}\right)^N\mathcal{G}.$$

We derive a differential equations with coefficients $b_i(N,x,\lambda)$, which is satisfied by

$$D^N\mathcal{G}(t,x,\lambda)-b_0(N,x,\lambda)\mathcal{G}(t,x,\lambda)-\cdots-b_N(N,x,\lambda)e^{Nt}\mathcal{G}(t,x,\lambda)=0.$$

For $0\le i\le N$, by using the coefficients $b_i(N,x,\lambda)$ of this differential equation, we have explicit identities for the degenerate Bell-Carlitz polynomials. This paper is organized as follows. In Sect.2, we construct differential equations arising from the generating functions of degenerate Bell-Carlitz polynomials. We establish some new identities for the degenerate Bell-Carlitz polynomials. In Sect.3, using numerical methods, we investigate the structure of zeros of the degenerate Bell-Carlitz polynomials.

2. Differential equations associated with the degenerate Bell-Carlitz polynomials

In this section, we consider differential equation arising from the generating function of the degenerate Bell-Carlitz polynomials. Let

$$\mathcal{G} = \mathcal{G}(t, x, \lambda) = (1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}\mathcal{G}^{(1)} &= \frac{d}{dt}\mathcal{G}(t,x) = \frac{d}{dt}\left((1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)}\right)\\ &= (1+\lambda)^{\frac{tx}{\lambda}} e^{(e^t-1)}\left(\frac{\log(1+\lambda)}{\lambda}x + e^t\right)\\ &= \left(\frac{\log(1+\lambda)}{\lambda}x + e^t\right)\mathcal{G},\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathcal{G}^{(2)} &= \frac{d}{dt}\mathcal{G}^{(1)} = e^t\mathcal{G} + \left(\frac{\log(1+\lambda)}{\lambda}x + e^t\right)\mathcal{G}^{(1)}\\ &= e^t\mathcal{G} + \left(\frac{\log(1+\lambda)}{\lambda}x + e^t\right)^2\mathcal{G}\\ &= \left(\left(\frac{x\log(1+\lambda)}{\lambda}\right)^2 + \left(2\left(\frac{x\log(1+\lambda)}{\lambda}\right)+1\right)e^t + e^{2t}\right)\mathcal{G},\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathcal{G}^{(3)} = \frac{d}{dt}\mathcal{G}^{(2)} = &\left(\frac{x\log(1+\lambda)}{\lambda}\right)^3\mathcal{G}\\ &+\left(3\left(\frac{x\log(1+\lambda)}{\lambda}\right)^2 + 3\left(\frac{x\log(1+\lambda)}{\lambda}\right)+1\right)e^t\mathcal{G}\\ &+\left(3\left(\frac{x\log(1+\lambda)}{\lambda}\right)+3\right)e^{2t}\mathcal{G}\\ &+e^{3t}\mathcal{G}.\end{aligned}$$

Continuing this process, we can guess that

$$\mathcal{G}^{(N)} = \left(\frac{d}{dt}\right)^N \mathcal{G}(t,x,\lambda) = \left(\sum_{i=0}^{N} b_i(N,x,\lambda)e^{it}\right)\mathcal{G}, \quad (N=0,1,2,\ldots). \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we have

$$\begin{aligned}
\mathcal{G}^{(N+1)} &= \frac{d\mathcal{G}^{(N)}}{dt} \\
&= \left(\sum_{i=0}^{N} ib_i(N,x,\lambda)e^{it}\right)\mathcal{G} + \left(\sum_{i=0}^{N} b_i(N,x,\lambda)e^{it}\right)\mathcal{G}^{(1)} \\
&= \left(\sum_{i=0}^{N} ib_i(N,x,\lambda)e^{it}\right)\mathcal{G} + \left(\sum_{i=0}^{N} b_i(N,x,\lambda)e^{it}\right)\left(\frac{\log(1+\lambda)}{\lambda} + e^t\right)\mathcal{G} \\
&= \left\{\sum_{i=0}^{N}\left(\frac{\log(1+\lambda)}{\lambda}x + i\right) b_i(N,x,\lambda)e^{it} + \sum_{i=0}^{N} b_i(N,x,\lambda)e^{(i+1)t}\right\}\mathcal{G} \\
&= \left\{\sum_{i=0}^{N}\left(\frac{\log(1+\lambda)}{\lambda}x + i\right) b_i(N,x,\lambda)e^{it} + \sum_{i=1}^{N+1} b_{i-1}(N,x,\lambda)e^{it}\right\}\mathcal{G}.
\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$\mathcal{G}^{(N+1)} = \left(\sum_{i=0}^{N+1} b_i(N+1,x,\lambda)e^{it}\right)\mathcal{G}. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$\begin{aligned}
b_0(N+1,x,\lambda) &= \frac{\log(1+\lambda)}{\lambda} x b_0(N,x,\lambda), \\
b_{N+1}(N+1,x,\lambda) &= b_N(N,x,\lambda),
\end{aligned} \tag{2.7}$$

and

$$b_i(N+1,x,\lambda) = b_{i-1}(N,x,\lambda) + \left(\frac{\log(1+\lambda)}{\lambda}x + i\right) b_i(N,x,\lambda), (1 \le i \le N). \tag{2.8}$$

In addition, by (2.4), we get

$$\mathcal{G} = \mathcal{G}^{(0)} = b_0(0,x,\lambda)\mathcal{G}. \tag{2.9}$$

By (2.9), we get

$$b_0(0,x,\lambda) = 1. \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned}
\left(\frac{\log(1+\lambda)}{\lambda} + e^t\right)\mathcal{G} = \mathcal{G}^{(1)} &= \left(\sum_{i=0}^{1} b_i(1,x,\lambda)e^{it}\right)\mathcal{G} \\
&= b_0(1,x,\lambda)\mathcal{G} + b_1(1,x,\lambda)e^t\mathcal{G}.
\end{aligned} \tag{2.11}$$

Thus, by (2.11), we also get

$$b_0(1,x,\lambda)=\frac{\log(1+\lambda)}{\lambda}x,\quad b_1(1,x,\lambda)=1. \tag{2.12}$$

From (2.7), we note that

$$\begin{aligned} b_0(N+1,x,\lambda)&=\frac{\log(1+\lambda)}{\lambda}xb_0(N,x,\lambda)=\left(\frac{\log(1+\lambda)}{\lambda}\right)^2x^2b_0(N-1,x,\lambda)\\ &=\cdots=\left(\frac{\log(1+\lambda)}{\lambda}\right)^N x^N b_0(1,x,\lambda)\\ &=\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1}x^{N+1}, \end{aligned} \tag{2.13}$$

and

$$b_{N+1}(N+1,x)=b_N(N,x,\lambda)=b_{N-1}(N-1,x,\lambda)=\cdots=b_1(1,x,\lambda)=1. \tag{2.14}$$

For $i=1,2,3$ in (2.8), we have

$$b_1(N+1,x,\lambda)=\sum_{k=0}^{N}\left(\frac{\log(1+\lambda)}{\lambda}x+1\right)^k b_0(N-k,x,\lambda), \tag{2.15}$$

$$b_2(N+1,x,\lambda)=\sum_{k=0}^{N-1}\left(\frac{\log(1+\lambda)}{\lambda}x+2\right)b_1(N-k,x,\lambda), \tag{2.16}$$

and

$$b_3(N+1,x,\lambda)=\sum_{k=0}^{N-2}\left(\frac{\log(1+\lambda)}{\lambda}x+3\right)^k b_2(N-k,x,\lambda). \tag{2.17}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$b_i(N+1,x,\lambda)=\sum_{k=0}^{N-i+1}\left(\frac{\log(1+\lambda)}{\lambda}x+i\right)^k b_{i-1}(N-k,x,\lambda). \tag{2.18}$$

Here, we note that the matrix $b_i(j,x,,\lambda)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & \frac{\log(1+\lambda)}{\lambda}x & \left(\frac{\log(1+\lambda)}{\lambda}\right)^2x^2 & \cdot & \cdots & \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N+1}x^{N+1}\\ 0 & 1 & \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & 1 & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & 1 & \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Now, we give explicit expressions for $b_i(N+1,x)$. By (2.15), (2.16), and (2.17), we get

$$
\begin{aligned}
b_1(N+1,x,\lambda) &= \sum_{k_1=0}^{N}\left(\frac{\log(1+\lambda)}{\lambda}x+1\right)^{k_1} b_0(N-k_1,x,\lambda)\\
&= \sum_{k_1=0}^{N}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-k_1}\left(\frac{\log(1+\lambda)}{\lambda}x+1\right)^{k_1} x^{N-k_1},
\end{aligned}
$$

$$
\begin{aligned}
b_2(N+1,x,\lambda) &= \sum_{k_2=0}^{N-1}\left(\frac{\log(1+\lambda)}{\lambda}x+2\right)^{k_2} b_1(N-k_2,x,\lambda)\\
&= \sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-k_2-k_1-1}\left(\frac{\log(1+\lambda)}{\lambda}x+2\right)^{k_2}\\
&\qquad\times\left(\frac{\log(1+\lambda)}{\lambda}x+1\right)^{k_1} x^{N-k_2-k_1-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&b_3(N+1,x,\lambda)\\
&= \sum_{k_3=0}^{N-2}\left(\frac{\log(1+\lambda)}{\lambda}x+3\right)^{k_3} b_2(N-k_3,x,\lambda)\\
&= \sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-k_3-k_2-k_1-2}\left(\frac{\log(1+\lambda)}{\lambda}x+3\right)^{k_3}\\
&\qquad\times\left(\frac{\log(1+\lambda)}{\lambda}x+2\right)^{k_2}\left(\frac{\log(1+\lambda)}{\lambda}x+1\right)^{k_1} x^{N-k_3-k_2-k_1-2}.
\end{aligned}
$$

Continuing this process, we get

$$
\begin{aligned}
&b_i(N+1,x,\lambda)\\
&= \sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-i+1-(k_1-\cdots-k_i)}\\
&\qquad\times\prod_{l=1}^{i}\left(\frac{\log(1+\lambda)}{\lambda}x+l\right)^{k_l} x^{N-i+1-(k_1-\cdots-k_i)}.
\end{aligned}
\tag{2.19}
$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.1. *For $N=0,1,2,\ldots,$ the differential equations*

$$
\mathcal{G}^{(N)} = \left(\sum_{i=0}^{N} b_i(N,x,\lambda)e^{it}\right)\mathcal{G}
$$

have a solution

$$\mathcal{G} = \mathcal{G}(t,x,\lambda) = (1+\lambda)^{\dfrac{tx}{\lambda}} e^{(e^t-1)},$$

where

$$\begin{aligned} b_0(N,x,\lambda) &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^N x^N, \\ b_i(N,x,\lambda) &= \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-i-(k_1-\cdots-k_i)} \\ &\quad \times \prod_{l=1}^{i} \left(\frac{\log(1+\lambda)}{\lambda} x + l\right)^{k_l} x^{N-i-(k_1-\cdots-k_i)}, (1 \le i \le N). \end{aligned}$$

From (2.1), we note that

$$\mathcal{G}^{(N)} = \left(\frac{d}{dt}\right)^N \mathcal{G}(t,x,\lambda) = \sum_{k=0}^{\infty} \mathbf{B}^c_{k+N}(x,\lambda)\frac{t^k}{k!}. \tag{2.20}$$

From Theorem 2.1 and (2.20), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbf{B}^c_{k+N}(x)\frac{t^k}{k!} = \mathcal{G}^{(N)} &= \left(\sum_{i=0}^{N} b_i(N,x,\lambda)e^{it}\right)\mathcal{G} \\ &= \sum_{i=0}^{N} b_i(N,x,\lambda)\left(\sum_{l=0}^{\infty} i^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} \mathbf{B}^c_m(x,\lambda)\frac{t^m}{m!}\right) \\ &= \sum_{i=0}^{N} b_i(N,x,\lambda)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m} i^{k-m}\mathbf{B}^c_m(x,\lambda)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m} i^{k-m} b_i(N,x,\lambda)\mathbf{B}^c_m(x,\lambda)\right)\frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By comparing the coefficients on both sides of (2.21), we obtain the following theorem.

Theorem 2.2. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$\mathbf{B}^c_{k+N}(x,\lambda) = \sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m} i^{k-m} b_i(N,x,\lambda)\mathbf{B}^c_m(x,\lambda), \tag{2.22}$$

where

$$b_0(N,x,\lambda)=\left(\frac{\log(1+\lambda)}{\lambda}\right)^N x^N,$$

$$b_i(N,x,\lambda)=\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-i-k_i}\cdots\sum_{k_1=0}^{N-i-k_i-\cdots-k_2}\left(\frac{\log(1+\lambda)}{\lambda}\right)^{N-i-(k_1-\cdots-k_i)}$$
$$\times\prod_{l=1}^{i}\left(\frac{\log(1+\lambda)}{\lambda}x+l\right)^{k_l}x^{N-i-(k_1-\cdots-k_i)},(1\le i\le N).$$

Let us take $k=0$ in (2.22). Then, we have the following corollary.

Corollary 2.3. *For $N=0,1,2,\dots,$ we have*

$$\mathbf{B}_N^c(x,\lambda)=\sum_{i=0}^{N}b_i(N,x,\lambda).$$

3. Zeros of the degenerate Bell-Carlitz polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$. By using computer, the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ can be determined explicitly.

The first few examples of degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ are

$$\mathbf{B}_0^c(x,\lambda)=1,$$

$$\mathbf{B}_1^c(x,\lambda)=1+x\frac{\log(1+\lambda)}{\lambda},$$

$$\mathbf{B}_2^c(x,\lambda)=2+2x\frac{\log(1+\lambda)}{\lambda}+x^2\left(\frac{\log(1+\lambda)}{\lambda}\right)^2,$$

$$\mathbf{B}_3^c(x,\lambda)=5+6x\frac{\log(1+\lambda)}{\lambda}+3x^2\left(\frac{\log(1+\lambda)}{\lambda}\right)^2+x^3\left(\frac{\log(1+\lambda)}{\lambda}\right)^3,$$

$$\mathbf{B}_4^c(x,\lambda)=15+20x\frac{\log(1+\lambda)}{\lambda}+12x^2\left(\frac{\log(1+\lambda)}{\lambda}\right)^2+4x^3\left(\frac{\log(1+\lambda)}{\lambda}\right)^3$$
$$+x^4\left(\frac{\log(1+\lambda)}{\lambda}\right)^4,$$

$$\mathbf{B}_5^c(x,\lambda)=52+75x\frac{\log(1+\lambda)}{\lambda}+50x^2\left(\frac{\log(1+\lambda)}{\lambda}\right)^2+20x^3\left(\frac{\log(1+\lambda)}{\lambda}\right)^3$$
$$+5x^4\left(\frac{\log(1+\lambda)}{\lambda}\right)^4+x^5\left(\frac{\log(1+\lambda)}{\lambda}\right)^5.$$

We investigate the beautiful zeros of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ by using a computer. We plot the zeros of the $B_n^c(x)$ for $n = 5, 10, 15, 20$ and $x \in \mathbb{C}$(Figure 2). In Figure 1(top-left), we choose $n = 30, \lambda = 1/10$. In

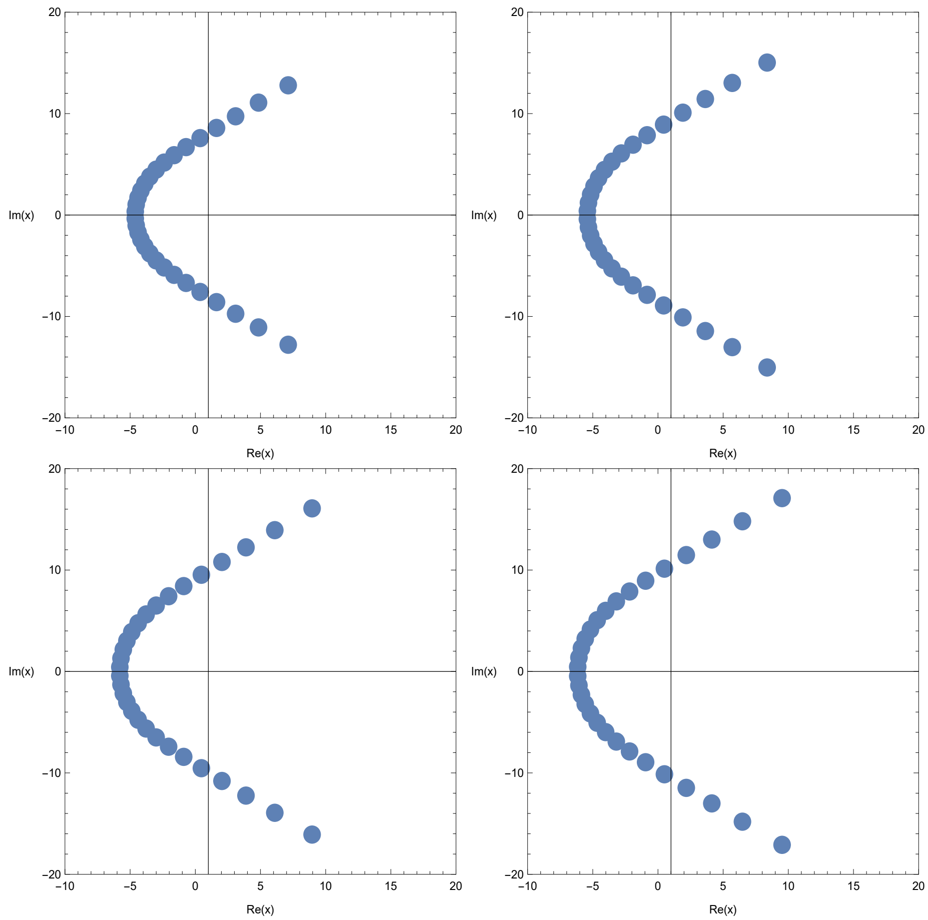

FIGURE 1. Zeros of $\mathbf{B}_n^c(x,\lambda)$

Figure 1(top-right), we choose $n = 30, \lambda = 5/10$. In Figure 1(bottom-left), we choose $n = 30, \lambda = 7/10$. In Figure 1(bottom-right), we choose $n = 30, \lambda = 9/10$. Prove that $\mathbf{B}_n^c(x,\lambda), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions(see Figure 2). Stacks of zeros of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$ for $1 \le n \le 50$ from a 3-D structure are presented(Figure 3). In Figure 2(left), we choose $n = 30, \lambda = 1/10$. In Figure 2(right), we choose $n = 30, \lambda = 9/10$. Plot of real zeros of degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda) = 0$ for $1 \le n \le 50$ structure are presented(Figure 4). In Figure 2(left),

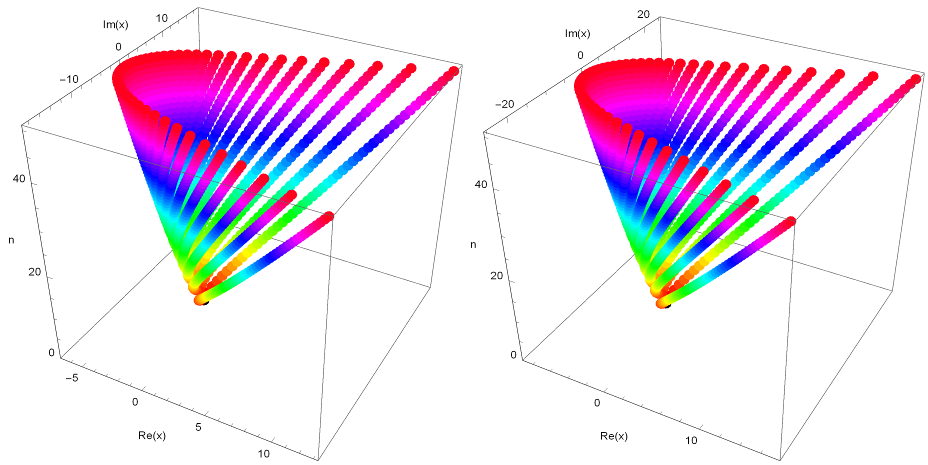

FIGURE 2. Stacks of zeros of $\mathbf{B}_n^c(x,\lambda) = 0, 1 \leq n \leq 50$

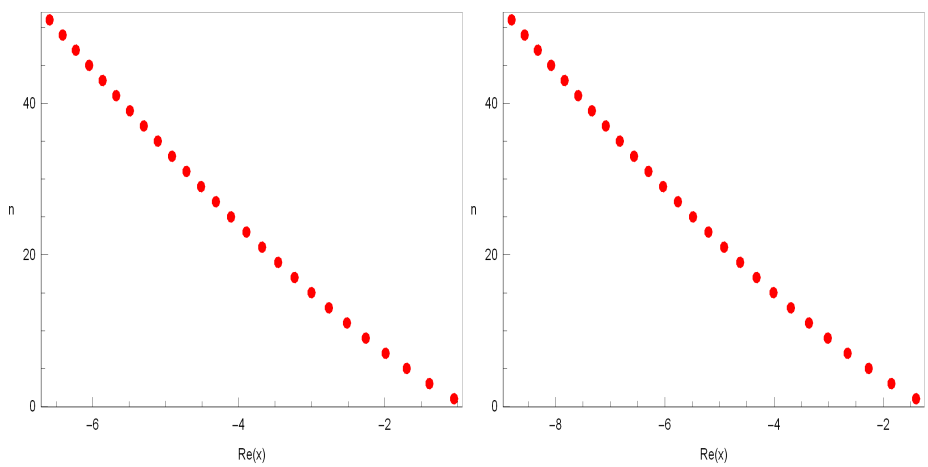

FIGURE 3. Real zeros of $\mathbf{B}_n^c(x,\lambda)$ for $1 \leq n \leq 50$

we choose $\lambda = 1/10$. In Figure 2(right), we choose $\lambda = 9/10$. We observe a remarkably regular structure of the complex roots of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$. We hope to verify a remarkably regular structure of the complex roots of the degenerate Bell-Carlitz polynomials $\mathbf{B}_n^c(x,\lambda)$. Next, we calculated an approximate solution satisfying $\mathbf{B}_n^c(x,\lambda), x \in \mathbb{C}$. The results are given in Table 1.

Table 1. Approximate solutions of $\mathbf{B}_n^c(x, 9/10) = 0, x \in \mathbb{C}$

degree n	x
1	-1.4022
2	$-1.4022 - 1.4022i, \quad -1.4022 + 1.4022i$
3	$-1.8540, \quad -1.1763 - 2.4600i, \quad -1.1763 + 2.4600i$
4	$-1.9383 + 1.0217i, \quad -1.9383 - 1.0217i,$ $-0.8660 - 3.3657i, \quad -0.8660 + 3.3657i$
5	$-2.2694, \quad -1.8554 - 1.8871i, \quad -1.8554 + 1.8871i,$ $-0.5154 + 4.1795i, \quad -0.5154 - 4.1795i$
6	$-2.3810 - 0.8481i, \quad -2.3810 + 0.8481i, \quad -1.6832 + 2.6641i,$ $-1.6832 - 2.6641i, \quad -0.1423 - 4.9295i, \quad -0.1423 + 4.9295i$

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