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UNIQUENESS THEOREM CONCERNING FUNCTIONAL EQUATIONS IN MODULAR SPACES

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ABSTRACT. In this paper, we will prove some uniqueness theorems that can be applied to the generalized Hyers-Ulam stability of some additive-quadratic-cubic functional equation in complete modular spaces without $\triangle_2$-conditions.

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1. Introduction

Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis([6], [7], [9], [10], [11], [12], [13], [16], [19]).

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$.
(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
(M1) $\rho(x) = 0$ if and only if $x = 0$,
(M2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$, and
(M3) $\rho(z) \leq \rho(x) + \rho(y)$ whenever z is a convex combination of x and y.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in V$ and all nonnegative scalars α, β with $\alpha + \beta = 1$, then we say that ρ is *convex*.

For any convex modular ρ on X, the modular space X_ρ is defined by

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

Definition 1.2. Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ. Then
(1) $\{x_n\}$ is called *ρ-convergent* to a point $x \in X_\rho$, denoted by $x =_\rho \lim_{n\to\infty} x_n$, if $\rho(x_n - x) \to 0$ as $n \to \infty$,

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(2) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon > 0$, one has $\rho(x_n - x_m) < \epsilon$ for sufficiently large $m, n \in \mathbb{N}$, and
(3) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence in K is ρ-convergent to a ponit in K.

It is well known that fixed point theories are one of powerful tools in solving mathematical problems. Banach's contraction principle is one of the pivotal results in fixed point theories and they have a board set of applications. Khamsi, Kozowski and Reich [4] investigated the fixed point theorem in modular spaces. In [5], Khamsi proved a series of fixed point theorems in modular spaces.

For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of T at $x \in C$* is the set $\mathbb{O}(x) = \{x, Tx, T^2x, \cdots\}$. If $\delta_\rho(x) = \sup\{\rho(u - v) \mid u, v \in \mathbb{O}(x)\} < \infty$, then one says that T has *a bounded orbit at x*.

Lemma 1.3. [5] *Let X_ρ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a ρ-complete subset. If $T : C \longrightarrow C$ is a ρ-contraction, that is, there is a constant $L \in [0, 1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \ \forall x, y \in C$$

and T has a bounded orbit at a point $x_0 \in C$, then the sequence $\{T^n x_0\}$ is ρ-convergent to a point $w \in C$.

The convergence of a sequence $\{x_n\}$ does not imply that of $\{c \cdot x_n\}$ for a scalar c in modular spaces. In order to avoid such diffculties, some additional conditions are imposed on the modular so that the multiple of $\{x_n\}$ converges naturally. One of such conditions is the so-called $\triangle_2$-condition.

A modular space X_ρ is said to *satisfy the $\triangle_2$-condition* if there exists $k \geq 2$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X_\rho$.

Let X and Y be real vector spaces. For any mapping $f : X \longrightarrow Y$, consider the following functional equations :

$$f(4x + y) + f(4x - y) = 4f(x + y) + 4f(x - y), \tag{1}$$

$$f(4x + y) + f(4x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y), \tag{2}$$

and

$$f(4x + y) + f(4x - y) = 4f(x + y) + 4f(x - y) + 120f(x) \tag{3}$$

for all $x, y \in X$. Then a mapping $f : X \longrightarrow Y$ is called *an additive(quadratic, cubic*, resp.) if f satisfies (1)((2), (3), resp.) and a mapping $f : X \longrightarrow Y$ is called *an additive-quadratic-cubic* if f is represented by the sum of an additive mapping, a quadratic mapping, and a cubic mapping.

The stability problem for functional equations first was planed in 1940 by Ulam [17].

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$

satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. In 1994, P. Găvruta [2] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions.

Recently, Sadeghi [15] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and using the fixed point theorem Lemma 1.3, Wongkum, Chaipunya, and Kumam [18] proved the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

Lee and Jung [8] proved uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation in Banach spaces.

In this paper, using the fixed point theorem, we will prove a general uniqueness theorem that can be applied to the generalized Hyers-Ulam stability of some additive-quadratic-cubic functional equation in modular spaces without $\triangle_2$-conditions.

2. Main results

Throughout this section, we assume that V is a linear space and X_ρ is a ρ-complete modular space whose induced modular is convex lower semi-continuous. In this section, we will prove that, if for given map $f : V \longrightarrow X_\rho$, there is a mapping $F : V \longrightarrow X_\rho$, which is near f in X_ρ, with some properties possessed by additive-quadratic-cubic mappings, then F is uniquely determined.

Define a set $\mathbb{M}$ by

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\tilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\tilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \leq c\phi(x),\ \forall x \in V\},$$

where $\phi : V \longrightarrow [0, \infty)$ is a mapping.

Lemma 2.1. [18] *We have the following :*

(1) $\mathbb{M}$ *is a linear space,*
(2) $\tilde{\rho}$ *is a convex modular on* $\mathbb{M}$*,*
(3) $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$ *and* $\mathbb{M}_{\tilde{\rho}}$ *is* $\tilde{\rho}$*-complete, and*
(4) $\tilde{\rho}$ *is lower semi-continuous.*

Now, with Lemma 1.3 and Lemma 2.1, we will show the following uniquness theorems concerning additive-quadratic-cubic type functional equations.

Theorem 2.2. *Let $\phi : V \longrightarrow [0,\infty)$ be a mapping and L a real number such that $0 \le L < \frac{3}{5}$ and*

$$\phi(2x) \le 2L\phi(x) \tag{4}$$

for all $x \in V$. Let $f, F : V \longrightarrow X_\rho$ be mappings such that

$$\rho(F(x) - f(x)) \le \phi(x) \tag{5}$$

for all $x \in V$ and

$$F(2x) = 2F(x) \tag{6}$$

for all $x \in X$. Then F is determined by

$$\begin{aligned}&\frac{1}{2^3}F(x)\\ &=_\rho \lim_{n\to\infty} \frac{1}{3^n\cdot 2^3}\Big[f(2^{2n}x) + n\Sigma_{k=1}^{n-1}(-1)^k\frac{1}{2^{3k}}f(2^{2n+k}x) + (-1)^n\frac{1}{2^{3n}}f(2^{3n}x)\Big]\end{aligned} \tag{7}$$

for all $x \in V$ and F is the unique mapping with (5) and (6).

Proof. By Lemma 2.1, $\mathbb{M}_{\tilde\rho} = \mathbb{M}$ is $\tilde\rho$-complete and $\tilde\rho$ is lower semi-continuous. Define $T : \mathbb{M}_{\tilde\rho} \longrightarrow \mathbb{M}_{\tilde\rho}$ by $T_a g(x) = \frac{1}{3}g(4x) - \frac{1}{24}g(8x)$ for all $g \in \mathbb{M}_{\tilde\rho}$ and all $x \in V$. By (6), we have $T_aF(x) = F(x)$ for all $x \in V$ and so F is a fixed point of T_a. Suppose that $g, h \in \mathbb{M}_{\tilde\rho}$ and $\tilde\rho(g-h) \le c$ for some positive real number c. By (M3) and (4), we have

$$\begin{aligned}\rho(2T_ag(x) - 2T_ah(x)) &\le \frac{2}{3}\rho(g(4x) - h(4x)) + \frac{1}{12}\rho(g(8x) - h(8x)\\ &\le \frac{2}{3}c\phi(4x) + \frac{1}{12}c\phi(8x) \le \Big(\frac{8}{3} + \frac{2}{3}L\Big)cL^2\phi(x) \le 2cL\phi(x)\end{aligned}$$

for all $x \in V$, because $0 \le L < \frac{3}{5}$. Hence we have

$$\tilde\rho(T_ag - T_ah) \le \frac{1}{2}\tilde\rho(2T_ag - 2T_ah) \le L\tilde\rho(g-h). \tag{8}$$

for all $g, h \in \mathbb{M}_{\tilde\rho}$ and so T is $\tilde\rho$-contractive.

Now, we will show that T_a has a bounded orbit at a point $2^{-3}f$ in $\mathbb{M}_{\tilde\rho}$. Since F is a fixed point of T_a, by (5) and (8), we have

$$\begin{aligned}\tilde\rho(2^{-1}T_af - 2^{-1}f) &\le \frac{1}{2}\tilde\rho(T_af - T_aF) + \frac{1}{2}\tilde\rho(F-f)\\ &\le \frac{1}{2}(1+L)\tilde\rho(F-f)\\ &\le \frac{1}{2}(1+L).\end{aligned}$$

By (M1) and (8), we have

$$\begin{aligned}\tilde\rho(2^{-2}T_a^nf - 2^{-2}f) &\le \frac{1}{2}\tilde\rho(2^{-1}T_a^nf - 2^{-1}T_af) + \frac{1}{2}\tilde\rho(2^{-1}T_af - 2^{-1}f)\\ &\le L\tilde\rho(2^{-2}T_a^{n-1}f - 2^{-2}f) + \frac{1}{2}\tilde\rho(2^{-1}T_af - 2^{-1}f)\end{aligned}$$

for all $n \in \mathbb{N}$. Hence we have

$$\tilde{\rho}(2^{-2}T_a^n f - 2^{-2}f) \leq \frac{1}{2}\Big[\Sigma_{k=0}^{n-1}L^k\Big]\tilde{\rho}(2^{-1}T_a f - 2^{-1}f) \leq \frac{1+L}{2^2(1-L)}$$

for all $n \in \mathbb{N}$. For any non-negative integers m, n with $m > n$,

$$\begin{aligned}\tilde{\rho}(2^{-3}T_a^n f - 2^{-3}T_a^m f) &\leq \frac{1}{2}\tilde{\rho}(2^{-2}T_a^n f - 2^{-2}f) + \frac{1}{2}\tilde{\rho}(2^{-2}T_a^m f - 2^{-2}f)\\ &\leq \frac{1+L}{2^2(1-L)}.\end{aligned} \tag{9}$$

Hence T_a has a bounded orbit at $2^{-3}f$ in $\mathbb{M}_{\tilde{\rho}}$ and thus by Lemma 1.3, there is a $A \in \mathbb{M}_{\tilde{\rho}}$ such that $\{T_a^n 2^{-3}f\}$ is $\tilde{\rho}$-convergent to A in $\mathbb{M}_{\tilde{\rho}}$. That is, $\lim_{n\to\infty} \tilde{\rho}(T_a^n 2^{-3}f - A) = 0$. Since $\tilde{\rho}$ is lower semi-continuous, by (9), we have

$$\tilde{\rho}(A - 2^{-3}f) \leq \frac{1+L}{2^2(1-L)}. \tag{10}$$

Since $\tilde{\rho}(A - T_a A) \leq \liminf_{n\to\infty} \tilde{\rho}(2^{-3}T_a^{n+1}f - T_a A) \leq L \liminf_{n\to\infty}$ A is a fixed pont of T_a. Since F is a fixed point of T_a, by (8),

$$\tilde{\rho}(2^{-1}A - 2^{-4}F) = \tilde{\rho}(2^{-1}T_a A - 2^{-4}T_a F) \leq L\tilde{\rho}(2^{-1}A - 2^{-4}F)].$$

Since $0 \leq L < \frac{3}{5}$, $A = 2^{-3}F$. Moreover, we have

$$T_a^n f(x) = \frac{1}{3^n}\Big[f(2^{2n}x) + n\Sigma_{k=1}^{n-1}(-1)^k\frac{1}{2^{3k}}f(2^{2n+k}x) + (-1)^n\frac{1}{2^{3n}}f(2^{3n}x)\Big]$$

for all $x \in V$ and all $n \in \mathbb{N}$. Thus

$$A(x) =_\rho \lim_{n\to\infty}\frac{1}{3^n\cdot 2^3}\Big[f(2^{2n}x) + n\Sigma_{k=1}^{n-1}(-1)^k\frac{1}{2^{3k}}f(2^{2n+k}x) + (-1)^n\frac{1}{2^{3n}}f(2^{3n}x)\Big]$$

for all $x \in V$. Since $A = 2^{-3}F$, we have (7).

Suppose that G is a mapping with (5) and (6). Then by (5), and (6), we have

$$\rho\Big(\frac{1}{2}F(x) - \frac{1}{2}G(x)\Big) \leq \frac{1}{2^n}\phi(2^n x) \leq L^n\phi(x)$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence $F(x) = G(x)$ for all $x \in V$. □

Similar to the proof of Theorem 2.2, we can show the following two theorems for modular spaces.

Theorem 2.3. *Let $\phi : V \longrightarrow [0, \infty)$ be a mapping and L a real number such that $0 \leq L < 1$ and*

$$\phi(2x) \leq 4L\phi(x) \tag{11}$$

for all $x \in V$. Let $f, F : V \longrightarrow X_\rho$ be mappings satisfying (5) and

$$F(2x) = 4F(x) \tag{12}$$

for all $x \in X$. Then F is determined by

$$\frac{1}{2^3}F(x) =_\rho \lim_{n\to\infty}\frac{1}{2^{2n+3}}f(2^{2n}x) \tag{13}$$

for all $x \in V$ and F is the unique mapping with (5) and (12).

Theorem 2.4. *Let $\phi : V \longrightarrow [0, \infty)$ be a mapping and L a real number such that $0 \le L < \frac{3}{5}$ and*

$$\phi(2x) \le 8L\phi(x) \tag{14}$$

for all $x \in V$. Let $f, F : V \longrightarrow X_\rho$ be mappings satisfying (5) and

$$F(2x) = 8F(x) \tag{15}$$

for all $x \in V$. Then F is determined by

$$\frac{1}{2^3}F(x) =_\rho \lim_{n\to\infty} \frac{1}{3^n \cdot 2^{6n+3}}\Big[(-1)^n f(2^{2n}x) + n\Sigma_{k=1}^{n-1}(-1)^{n+k}\frac{1}{2^k}f(2^{2n+k}x) + \frac{1}{2^n}f(2^{3n}x)\Big] \tag{16}$$

for all $x \in V$ and F is the unique mapping with (5) and (15).

Proof. Define $T : \mathbb{M}_{\tilde\rho} \longrightarrow \mathbb{M}_{\tilde\rho}$ by $T_c g(x) = -\frac{1}{3\cdot 2^6}g(4x) + \frac{1}{3\cdot 2^7}g(8x)$ for all $g \in \mathbb{M}_{\tilde\rho}$ and all $x \in V$. By (15), F is a fixed point of T_c. Suppose that $g, h \in \mathbb{M}_{\tilde\rho}$ and $\tilde\rho(g - h) \le c$ for some positive real number c. By (M3) and (14), we have

$$\begin{aligned}\rho(2T_cg(x) - 2T_ch(x)) &\le \frac{1}{3\cdot 2^6}c\phi(4x) + \frac{1}{3\cdot 2^7}c\phi(8x)\\ &\le \Big(\frac{1}{3} + \frac{4}{3}L\Big)cL^2\phi(x)\\ &\le 2cL\phi(x)\end{aligned}$$

for all $x \in V$, because $0 \le L < \frac{3}{5}$. Hence we have $\tilde\rho(T_cg - T_ch) \le L\tilde\rho(g - h)$. for all $g, h \in \mathbb{M}_{\tilde\rho}$ and so T is $\tilde\rho$-contractive. Similar to Theorem 2.2, we have the results. □

For any map $g : V \longrightarrow X$, let

$$g_o(x) = \frac{g(x) - g(-x)}{2},\ \ g_e(x) = \frac{g(x) + g(-x)}{2},$$
$$g_a(x) = \frac{8g_o(x) - g_o(2x)}{6},\ \ g_c(x) = -\ \frac{2g_o(x) - g_o(2x)}{6}$$

for all $x \in V$. Then g_o, g_a, and g_c are odd mappings, f_e is an even mapping, and

$$g(x) = g_o(x) + g_e(x) = g_a(x) + g_e(x) + g_c(x)$$

for all $x \in V$. Using Theroem 2.2, Theroem 2.3, and Theroem 2.4, we will prove the following theorem which is the main theorem of this paper.

Theorem 2.5. *Let $\phi : V \longrightarrow [0, \infty)$ be a mapping and L a positive real number such that $0 \le L < \frac{3}{5}$ and*

$$\phi(2x) \le 2L\phi(x) \tag{17}$$

for all $x \in V$. Let $f, F : V \longrightarrow X_\rho$ be mappings satisfying (5) and

$$F_o(x) = \frac{5}{8}F_o(2x) - \frac{1}{16}F_o(4x),\ \ F_e(2x) = 4F_e(x) \tag{18}$$

for all $x \in X$. Then F is determined by

$$\begin{aligned}
&\frac{1}{2^6}F(x) =_\rho \lim_{n\to\infty}\Big[\frac{1}{2^6\cdot 3^{n+1}}\Big(4+(-1)^n\frac{1}{2^{6n}}\Big)f_o(2^{2n}x)\\
&-\frac{n+1}{2^7\cdot 3^{n+1}}\Big(1+(-1)^{n+1}\frac{1}{2^{6n}}\Big)f_o(2^{2n+1}x)+\frac{n}{2^6\cdot 3^{n+1}}\Sigma_{k=1}^{n-2}\Big((-1)^{k+1}\cdot\frac{1}{2^{3k}}\\
&+(-1)^{n+k}\frac{1}{2^{6n+k}}\Big)f_o(2^{2n+k+1}x)+\frac{n+1}{2^{3n+4}\cdot 3^{n+1}}\Big((-1)^n-\frac{1}{2^{4n+2}}\Big)f_o(2^{3n}x)\\
&+\frac{1}{6^{n+1}\cdot 2^{3n+1}}\Big((-1)^{n+1}+\frac{1}{2^{4n+5}}\Big)f_o(2^{3n+1}x)+\frac{1}{2^{2n+6}}f_e(2^nx)\Big].
\end{aligned} \tag{19}$$

Moreover, F is the unique mapping with (5) and (18).

Proof. By (18), we get $F_a(2x) = 2F_a(x)$, $F_c(2x) = 8F_c(x)$ for all $x \in V$. By (17) and (5), we have

$$\begin{aligned}
\rho\Big(\frac{1}{2}F_a(x)-\frac{1}{2}f_a(x)\Big) &\le \frac{1}{3}c[\phi(x)+\phi(-x)]+\frac{1}{24}c[\phi(2x)+\phi(-2x)]\\
&\le \Big(\frac{1}{3}+\frac{1}{12}L\Big)c[\phi(x)+\phi(-x)],
\end{aligned} \tag{20}$$

$$\rho(F_e(x)-f_e(x)) \le \frac{1}{2}c[\phi(x)+\phi(-x)], \tag{21}$$

and

$$\begin{aligned}
\rho\Big(\frac{1}{2}F_c(x)-\frac{1}{2}f_c(x)\Big) &\le \frac{1}{6}\rho(F_o(x)-f_o(x)))+\frac{1}{12}\rho(F_o(2x)-f_o(2x))\\
&\le \Big(\frac{1}{12}+\frac{1}{12}L\Big)c[\phi(x)+\phi(-x)]
\end{aligned} \tag{22}$$

for all $x \in V$. By (20), (21), (22), Theorem (2.2), Theorem (2.3), and Theorem (2.4), we have

$$\begin{aligned}
&\frac{1}{2^4}F_a(x)\\
&=_\rho \lim_{n\to\infty}\frac{1}{3^n\cdot 2^4}\Big[f_a(2^{2n}x)+n\Sigma_{k=1}^{n-1}(-1)^k\frac{1}{2^{3k}}f_a(2^{2n+k}x)+(-1)^n\frac{1}{2^{3n}}f_a(2^{3n}x)\Big],
\end{aligned} \tag{23}$$

$$\frac{1}{2^4}F_e(x) =_\rho \lim_{n\to\infty}\frac{1}{2^{2n+4}}f_e(2^{2n}x) \tag{24}$$

and

$$\begin{aligned}
\frac{1}{2^4}F_c(x) =_\rho \lim_{n\to\infty}&\frac{1}{3^n\cdot 2^{6n+4}}\Big[(-1)^nf_c(2^{2n}x)\\
&+n\Sigma_{k=1}^{n-1}(-1)^{n+k}\frac{1}{2^k}f_c(2^{2n+k}x)+\frac{1}{2^n}f_c(2^{3n}x)\Big]
\end{aligned} \tag{25}$$

for all $x \in V$. By the definitions of f_a and f_c, we have

$$\begin{aligned}
&T_a^n f_a(x)\\
&= \frac{1}{3^n}\Big[f_a(2^{2n}x) + n\Sigma_{k=1}^{n-1}(-1)^k \frac{1}{2^{3k}} f_a(2^{2n+k}x) + (-1)^n \frac{1}{2^{3n}} f_a(2^{3n}x)\Big]\\
&= \frac{1}{3^{n+1}}\Big[4f_o(2^{2n}x) - \frac{n+1}{2} f_o(2^{2n+1}x) + n\Sigma_{k=1}^{n-2}(-1)^{k+1}\frac{1}{2^{3k}} f_o(2^{2n+k+1}x)\\
&\quad + (-1)^n \frac{n+1}{2^{3n-2}} f_o(2^{3n}x) + (-1)^{n+1}\frac{1}{2^{3n+1}} f_o(2^{3n+1}x)\Big],
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&T_c^n f_c(x)\\
&= \frac{1}{3^{n+1}\cdot 2^{6n}}\Big[(-1)^{n+1} f_o(2^{2n}x) + (-1)^n \frac{n+1}{2} f_o(2^{2n+1}x)\\
&+ n\Sigma_{k=1}^{n-2}(-1)^{n+k}\frac{1}{2^k} f_o(2^{2n+k+1}x) - \frac{n+1}{2^n} f_o(2^{3n}x) + \frac{1}{2^{n+1}} f_o(2^{3n+1}x)\Big]
\end{aligned} \tag{27}$$

for all $x \in V$ and all $n \in \mathbb{N}$. By (26) and (27), we have

$$\begin{aligned}
&T_a^n f_a(x) + T_c^n f_c(x) + T_e^n f_e(x)\\
&= \frac{1}{3^{n+1}}\Big[4 + (-1)^{n+1}\frac{1}{2^{6n}}\Big] f_o(2^{2n}x) - \frac{n+1}{2\cdot 3^{n+1}}\Big[1 + (-1)^{n+1}\frac{1}{2^{6n}}\Big] f_o(2^{2n+1}x)\\
&+ \frac{n}{3^{n+1}}\Sigma_{k=1}^{n-2}\Big[(-1)^{k+1}\cdot\frac{1}{2^{3k}} + (-1)^{n+k}\frac{1}{2^{6n+k}}\Big] f_o(2^{2n+k+1}x)\\
&+ \frac{n+1}{2^{3n-2}\cdot 3^{n+1}}\Big[(-1)^n - \frac{1}{2^{4n+2}}\Big] f_o(2^{3n}x)\\
&+ \frac{1}{6^{n+1}\cdot 2^{2n}}\Big[(-1)^{n+1} + \frac{1}{2^{4n}}\Big] f_o(2^{3n+1}x) + \frac{1}{2^{2n}} f_e(2^n x)
\end{aligned} \tag{28}$$

for all $x \in V$ and all $n \in \mathbb{N}$. By (23), (24), and (25),

$$\frac{1}{2^6}F(x) =_\rho \lim_{n\to\infty}\frac{1}{2^6}[T_a^n f_a(x) + T_c^n f_c(x) + T_e^n f_e(x)]$$

and by (28), we have (19).

Similar to Theorem 2.2, Theorem 2.3, and Theorem 2.4, F_a, F_e, and F_c are unique mappings with some properties related with (5) and (18) and and hence F is the unique mapping with (5) and (18) □

3. Applications

For any mapping $f : X \longrightarrow Y$, let

$$\begin{aligned}
D_f(x,y) &= f(4x+y) + f(4x-y) - 4f(x+y) - 4f(x-y) - 20f(2x)\\
&\quad + 48f(x) + 8f(-x) + 3f(y) + 3f(-y)
\end{aligned}$$

In this section, we consider the following additive-quadratic-cubic functional equation

$$D_f(x, y) = 0 \tag{29}$$

and using Theorem (2.5), we prove the generalized Hyers-Ulam stability for it in complete modular spaces.

Lemma 3.1. *Let $f : V \longrightarrow X$ be a mapping. Then f satisfies (29) if and only if f is an additive-quadratic-cubic mapping.*

Using Theorem 2.2 - Theorem 2.5 and Lemma 3.1, we can show the generalized Hyers-Ulam stability for (29).

Theorem 3.2. *Let V be a linear space and X_ρ a ρ-complete modular space whose induced modular is convex lower semi-continuous. Suppose that $f : V \longrightarrow X$ is a mapping such that*

$$\rho(D_f(x, y)) \leq \phi(x, y) \tag{30}$$

for all $x, y \in V$ and let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping satisfying

$$\phi(2x, 2y) \leq 2L\phi(x, y),\ \forall x, y \in V \tag{31}$$

for some real number L with $0 \leq L < 5$. Then there is a unique additive-quadratic-cubic mapping $F : V \longrightarrow X_\rho$ such that

$$\begin{aligned}&\rho(2^{-2}F(x) - 2^{-5}f(x))\\ &\leq \Big(\frac{M_a}{2^4(1-M_a)} + \frac{M_e}{2^4(1-M_e)} + \frac{M_c}{2^4(1-M_c)}\Big)[\psi(x, 0) + \psi(x, x) + \psi(x, 4x)]\end{aligned} \tag{32}$$

for all $x \in V$, where $M_a = \frac{1}{24}\Big(1 + L + \frac{L^2}{3}\Big)$, $M_e = \frac{13}{80}$, $M_c = \frac{1}{96}\Big(1 + \frac{1}{4}L + \frac{1}{12}L^2\Big)$ and $\psi(x, y) = \frac{1}{2}[\phi(x, y) + \phi(-x, -y)]$ for all $x, y \in V$.

Proof. By (30), we have

$$\rho(D_{f_o}(x, y)) \leq \frac{1}{2}\rho(D_f(x, y)) + \frac{1}{2}\rho(D_f(-x, -y)) \leq \psi(x, y) \tag{33}$$

for all $x, y \in V$. Letting $y = 0$ in (33), we get

$$\rho(2f_o(4x) - 20f_o(2x) + 32f_o(x)) \leq \psi(x, 0) \tag{34}$$

for all $x, y \in X$ and by (34), we have

$$\rho(12[f_a(2x) - 2f_a(x)]) \leq \psi(x, 0) \tag{35}$$

for all $x \in X$. By (35), we have

$$\begin{aligned}&\rho(T_a f_a(x) - f_a(x))\\ &\leq \frac{1}{2}\rho(2f_a(x) - f_a(2x)) + \frac{1}{4}\rho(2f_a(2x) - f_a(4x)) + \frac{1}{24}\rho(2f_a(4x) - f_a(8x))\\ &\leq \frac{1}{24}\Big(1 + L + \frac{L^2}{3}\Big)\psi(x, 0)\end{aligned} \tag{36}$$

for all $x \in X$. By (34), we have

$$\rho(12[f_c(2x) - 8f_c(x)]) \le \psi(x, 0) \tag{37}$$

for all $x \in X$ and

$$\rho(T_c f_c(x) - f_c(x)) \le \frac{1}{3 \cdot 2^5}\Big(1 + \frac{1}{4}L + \frac{1}{12}L^2\Big)\psi(x, 0). \tag{38}$$

By (30), we have

$$\rho(D_{f_e}(x, y)) \le \frac{1}{2}\rho(D_f(x, y)) + \frac{1}{2}\rho(D_f(-x, -y)) \le \psi(x, y) \tag{39}$$

for all $x, y \in V$. Letting $y = 0$ in (39), we get

$$\rho(2f_e(4x) - 20f_e(2x) + 48f_e(x)) \le \psi(x, 0) \tag{40}$$

for all $x \in X$ and letting $y = x$ in (39), we get

$$\rho(f_e(5x) + f_e(3x) - 24f_e(2x) + 62f_e(x)) \le \psi(x, x) \tag{41}$$

for all $x \in X$. Letting $y = 4x$ in (39), we get

$$\rho(f_e(8x) - 4f_e(5x) - 4f_e(3x) - 20f_e(2x) + 56f_e(x) + 6f_e(4x)) \le \psi(x, 4x) \tag{42}$$

for all $x \in X$ and letting $x = 2x$ in (40), we get

$$\rho(2f_e(8x) - 20f_e(4x) + 48f_e(2x)) \le \psi(2x, 0) \tag{43}$$

for all $x \in X$. By (40)-(43), we get

$$\rho(f_e(2x) - 4f_e(x)) \le \Big(\frac{2}{5} + \frac{1}{20}L\Big)\psi(x, 0) + \frac{1}{5}\psi(x, x) + \frac{1}{20}\psi(x, 4x) \tag{44}$$

for all $x \in X$. By (45), we have

$$\begin{aligned} \rho(T_e f_e(x) - f_e(x)) &\le \frac{1}{80}(8 + L)\psi(x, 0) + \frac{1}{20}\psi(x, x) + \frac{1}{80}\psi(x, 4x) \\ &\le \frac{13}{80}[\psi(x, 0) + \psi(x, x) + \psi(x, 4x)] \end{aligned} \tag{45}$$

for all $x \in X$. Since $0 \le L < 5$, $\frac{1}{24} \le M_a < \frac{2}{3} < \frac{3}{5}$ and for any t $g, h \in \mathbb{M}_{\tilde\rho}$, we have

$$\tilde\rho(T_a g - T_a h) \le \frac{1}{2}\tilde\rho(2T_a g - 2T_a h) \le M_a \tilde\rho(g - h).$$

for all $g, h \in \mathbb{M}_{\tilde\rho}$. Hence T_a is $\tilde\rho$-contractive and similar to the proof of Theorem 2.2, there is a fixed point $A \in \mathbb{M}_{\tilde\rho}$ of T_a such that

$$\rho(A(x) - 2^{-3}f_a(x)) \le \frac{M_a}{2^2(1 - M_a)}\psi(x, 0). \tag{46}$$

for all $x \in V$. Since $0 \le L < 5$, $\frac{1}{96} \le M_c < \frac{3}{5}$ and similarly, there is a fixed points C of T_c such that

$$\rho(C(x) - 2^{-3}f_c(x)) \le \frac{M_c}{2^2(1 - M_c)}\psi(x, 0). \tag{47}$$

for all $x \in V$. Since $0 < M_e < 1$, there is a fixed points Q of T_e such that

$$\rho(Q(x) - 2^{-3} f_e(x)) \leq \frac{M_e}{2^2(1 - M_e)}[\psi(x, 0) + \psi(x, x) + \psi(x, 4x)]. \tag{48}$$

for all $x \in V$. Since A, Q, and C are fixed points of T_a, T_e and T_c, respectively,

$$A(2x) = 2A(x),\ Q(2x) = 4Q(x),\ C(2x) = 8C(x) \tag{49}$$

for all $x \in V$. Let $F = A + Q + C$. By (49), $F_a = A$, $F_e = Q$, and $F_c = C$. Since $\psi(x, 0) \leq \psi(x, 0) + \psi(x, x) + \psi(x, 4x)$, by (46), (47), and (48), we get (32) and by Theorem 2.5, F is the unique mapping with (49) and (32).

Since T_a is contractive, $\lim_{n\to\infty} \rho(T_a^n D_{f_a}(x, y)) = 0$ and so

$$\begin{aligned}\rho\Big(\frac{1}{2^4} D_{F_a}(x, y)\Big) \leq{}& \frac{1}{2^4}\rho(F_a(4x + y) - T_a^n 2^{-3} f_a(4x + y)) + \frac{1}{2^4}\rho(F_a(4x - y) \\ &- T_a^n 2^{-3} f_a(4x - y)) + \frac{1}{2^4}\rho(F_a(x + y) - T_a^n f_a 2^{-3}(x + y)) \\ &+ \frac{1}{2^4}\rho(F_a(x - y) - T_a^n 2^{-3} f_a(x - y)) + \frac{1}{2^7}\rho(T_a^n D_{f_a}(x, y))\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Hence we have $D_{F_a}(x, y) = 0$ for all $x \in V$ and by Lemma 3.1, F_a is an additive mapping. Similarly, F_e is a quadratic mapping and F_c is a cubic mapping. Thus F is an additive-quadratic-cubic mapping. □

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