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THREE SOLUTIONS FOR A SECOND-ORDER STURM-LIOUVILLE EQUATION WITH IMPULSIVE EFFECTS

HADI HAGHSHENAS*, GHASEM A. AFROUZI

Abstract. In this article, a second-order Sturm-Liouville problem with impulsive effects and involving the one-dimensional p-Laplacian is considered. The existence of at least three weak solutions via variational methods and critical point theory is obtained.

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1. Introduction

In 1836-1837, the French mathematicians Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1882) published several papers that initiated a new subtopic of mathematical analysis: the Sturm-Liouville theory. Sturm and Liouville were concerned with the general linear, homogeneous second-order differential equation of the form

$$(q(x)u')' + s(x)u = \lambda w(x)u, \qquad x \in [a, b]. \tag{1}$$

Under various boundary conditions, Sturm and Liouville established that solutions of problem (1) can exist only for particular values of the real parameter λ. In these latest years, problems of Sturm-Liouville type have been investigated by using topological degree theory, the supersolution and subsolution method, or critical point theory. For the background and results, we refer the reader to some recent contributions such as [1, 2, 3, 6, 9, 12] and the references therein. For instance, Afrouzi et al. in [2], by using critical point theorems due to Ricceri, considered multiplicity of the classical solutions for the Dirichlet problem

$$\begin{cases} -(p-1)|u'(x)|^{p-2}u''(x) = \lambda f(x,u)h(x,u'), \ x \in]a,b[, \\ u(a) = u(b) = 0. \end{cases} \tag{2}$$

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However, existence of multiple solutions for p-Laplacian Sturm-Liouville problems with impulsive effects whose right-hand side nonlinear term is depending on two control parameters λ and μ has attracted less attention.
In this paper, using a three critical point theorem (Theorem 2.1) established in [7], we obtain the existence of three solutions to the following Sturm-Liouville problem with impulsive effects and involving the ordinary p-Laplacian

$$\begin{cases} -\big(q(x)\phi_p(u'(x))\big)' + s(x)\phi_p(u(x)) \\ = \lambda f(x,u(x)) + \mu g(x,u(x)),\ x \neq x_j,\ x \in [a,b], \\ \triangle\big(q(x_j)\phi_p(u'(x_j))\big) = I_j(u(x_j)), \qquad j = 1,2,...,l, \\ u'(a) = u'(b) = 0, \end{cases} \tag{3}$$

where $\phi_p(x) = |x|^{p-2}x$, $p > 1$, $a < b$, $q, s \in L^\infty\big([a,b]\big)$ with $q_0 = \operatorname{ess\,inf}_{[a,b]} q > 0$ and $s_0 = \operatorname{ess\,inf}_{[a,b]} s \geq 0$, $q(a^+) = q(a) > 0$, $q(b^-) = q(b) > 0$. Here, $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $x_0 = a < x_1 < x_2 < ... < x_l < x_{l+1} = b$, $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,...,l$ are continuous, $\lambda \in]0,+\infty[$ and $\mu \in [0,+\infty[$ are two real parameters and

$$\triangle\big(q(x_j)\phi_p(u'(x_j))\big) = q(x_j^+)\phi_p(u'(x_j^+)) - q(x_j^-)\phi_p(u'(x_j^-)),$$

where $z(y^+)$ and $z(y^-)$ denote the right and left limits of $z(y)$ at y, respectively. Differential equations with impulsive effects arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the interest readers to [4, 8, 10, 13, 14] and the references therein. In [4], Bai and Dai obtained sufficient conditions that guarantee the existence of at least three solutions for the problem

$$\begin{cases} -\big(q(x)\phi_p(u'(x))\big)' + s(x)\phi_p(u(x)) = \lambda f(x,u(x)),\ x \neq x_j,\ x \in [a,b], \\ \triangle\big(q(x_j)\phi_p(u'(x_j))\big) = I_j(u(x_j)), \qquad j = 1,2,...,l, \\ \alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0. \end{cases}$$

Their technical approach is based on critical points theorems obtained by Ricceri [11]. In [14], the authors investigated the existence of multiple solutions of the following Neumann boundary value problem with impulsive Sturm-Liouville type equation

$$\begin{cases} -\big(q(x)u'(x)\big)' + s(x)u(x) = \lambda f(x,u(x)),\ x \neq x_j,\ x \in [a,b], \\ \triangle\big(q(x_j)u'(x_j)\big) = I_j(u(x_j)), \qquad j = 1,2,...,l, \\ u'(a) = u'(b) = 0. \end{cases}$$

The present paper is arranged as follows. Some fundamental facts will be given in Section 2 and the main result of this paper will be presented in Section 3.

2. Preliminaries.

Let $X = W^{1,p}([a,b])$ be the Sobolev space endowed with the norm

$$\|u\| = \left(\int_a^b q(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx\right)^{\frac{1}{p}},$$

which is equivalent to the usual one. It is well known that $(X, \|.\|)$ is compactly embedded in $\left(C^0([a,b]), \|.\|_\infty\right)$ and one has

$$\|u\|_\infty < \frac{(b-a)^{\frac{p-1}{p}}}{(q_0)^{\frac{1}{p}}}\|u\| \qquad \text{for all } u \in X. \tag{4}$$

Formula (4), can be obtained observing that, for all $x \in [a,b]$

$$|u(x)| \leq \int_a^b |u'(x)|dx,$$

by Hölder inequality one has

$$\int_a^b |u'(x)|dx \leq (b-a)^{\left(\frac{p-1}{p}\right)}\|u'\|_{L^p([a,b])} \leq \frac{(b-a)^{\frac{p-1}{p}}}{(q_0)^{\frac{1}{p}}}\|u\|.$$

Given the continuous functions $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$, put

$$F(x,\xi) = \int_0^\xi f(x,t)dt, \qquad G(x,\xi) = \int_0^\xi g(x,t)dt,$$

for all $(x,\xi) \in [a,b] \times \mathbb{R}$. Moreover, set $G^c = \int_a^b \max_{|\xi|\leq c} G(x,\xi)dx$ for all $c > 0$ and $G_e = \inf_{[a,b]\times[0,e]} G$ for all $e > 0$. If g is sign-changing, then $G^c \geq 0$ and $G_e \leq 0$.

A weak solution of problem (3) is a function $u : [a,b] \to \mathbb{R}$ in X such that the equality

$$\int_a^b q(x)\phi_p(u'(x))v'(x)dx + \int_a^b s(x)\phi_p(u(x))v(x)dx + \sum_{j=1}^{l} I_j(u(x_j))v(x_j)$$
$$-\lambda \int_a^b f(x,u(x))v(x)dx - \mu \int_a^b g(x,u(x))v(x)dx = 0,$$

holds for all $v \in X$.

Our main tool is the following theorem of existence of three critical points, obtained in [7].

Theorem 2.1. [7, Theorem 2.6] *Let X be a reflexive real Banach space; $\phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gateaux differentiable functional whose Gateaux derivative admits a continuous inverse on X^* and $\psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable functional whose Gateaux derivative is compact, such that*

$$\phi(0) = \psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{u} \in X$, with $r < \phi(\bar{u})$ such that

- (H1) $\sup_{\phi(u)\leq r} \psi(u) < r\frac{\psi(\bar{u})}{\phi(\bar{u})}$;
- (H2) *for each $\lambda \in \Lambda_r := \left]\frac{\phi(\bar{u})}{\psi(\bar{u})}, \frac{r}{\sup_{\phi(u)\leq r}\psi(u)}\right[$ the functional $I_\lambda = \phi - \lambda\psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional I_λ has at least three distinct critical points in X.

3. Main results.

In this section we derive conditions under which problem (3) admits at least three solutions. For this purpose, we introduce the following assumptions.

(M1) Assume that there exists a positive constant k_1 such that for each $u \in X$

$$0 \leq \sum_{j=1}^{l} \int_0^{u(x_j)} I_j(x)dx \leq k_1 \max_{j\in\{1,2,\ldots,l\}} |u(x_j)|^p.$$

(M2) Assume that there exist positive constants α, β, and $\gamma \in [0,1[$, such that

$$f(x,t) \leq \alpha + \beta|t|^\gamma, \quad \text{for } (x,t) \in [a,b]\times\mathbb{R}.$$

Let $k_0 q_0^{-1}(b-a)^{p-1} < 1$ where $k_0 = \int_a^b s(x)dx$ and let $k_2 = \frac{1}{p}k_0 + k_1$. For constants c_1, c, we define

$$\lambda_1 = p\frac{\int_a^b \max_{|\xi|\leq c_1} F(x,\xi)dx}{k_0 c_1^p}, \quad \lambda_2 = \frac{\int_a^b F(x,c)dx}{k_2 c^p}.$$

Theorem 3.1. *Assume that (M1), (M2) are satisfied, and there exist two positive constants c_1, c satisfying $c_1 < c$ such that $\lambda_1 < \lambda_2$. Then, for each $\lambda \in \left]\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right[$, for every continuous function $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ satisfying the condition*

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[a,b]} G(x,\xi)}{\xi^2} < +\infty, \tag{5}$$

and for every $\mu \in [0,\delta[$ where

$$\delta = \min\left\{\frac{k_0 c_1^p - p\lambda\int_a^b \max_{|\xi|\leq c_1} F(x,\xi)dx}{pG^{c_1}}, \frac{k_2 c^p - \lambda\int_a^b F(x,c)dx}{G_d}\right\},$$

for some constant $d > c$, problem (3) *has at least three weak solutions.*

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, fix $\lambda \in \left]\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right[$. Since $\frac{1}{\lambda_2} < \lambda < \frac{1}{\lambda_1}$ we have $\delta > 0$. Now fix $\mu \in [0,\delta[$ and set

$$J(x,\xi) = F(x,\xi) + \frac{\mu}{\lambda}G(x,\xi),$$

for all $(x,\xi) \in [a,b]\times\mathbb{R}$. For each $u \in X$, we let the functionals $\phi, \psi : X \to \mathbb{R}$ be defined by

$$\phi(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{l}\int_0^{u(x_j)} I_j(x)dx,$$

$$\psi(u) = \int_a^b J(x,u(x))dx,$$

and put

$$I_\lambda(u) = \phi(u) - \lambda\psi(u).$$

Note that the weak solutions of problem (3) are exactly the critical points of I_λ. The functionals ϕ and ψ satisfy the regularity assumptions of Theorem 2.1 and we have $\phi(0) = \psi(0) = 0$. Indeed, by standard arguments, ϕ is a sequentially weakly lower semi-continuous functional. Also, ϕ is coercive and continuously Gateaux differentiable and its Gateaux derivative at point $u \in X$ is defined by

$$\phi'(u)(v) = \int_a^b q(x)\phi_p(u'(x))v'(x)dx + \int_a^b s(x)\phi_p(u(x))v(x)dx + \sum_{j=1}^{l} I_j(u(x_j))v(x_j),$$

for every $v \in X$.
Moreover, $\phi' : X \to X^*$ admits a continuous inverse on X^* (see [12, Lemma 2.3]).
On the other hand, ψ is continuously Gateaux differentiable and its Gateaux derivative at point $u \in X$ is defined by

$$\psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx + \frac{\mu}{\lambda}\int_a^b g(x, u(x))v(x)dx, \qquad (v \in X),$$

and $\psi' : X \to X^*$ is a compact operator. For this, let $u_n \rightharpoonup u$ as $n \to \infty$ on X; by compactness of embedding $X \hookrightarrow C([a, b])$ we have u_n converges uniformly to u on $[a, b]$ as $n \to \infty$. Since f and g are two continuous functions, one has $f(x, u_n) \to f(x, u)$ and $g(x, u_n) \to g(x, u)$ as $n \to \infty$. So, $\psi'(u_n) \to \psi'(u)$ as $n \to \infty$. Thus, we have showed that ψ' is strongly continuous on X, which implies that ψ' is a compact operator by [15, Proposition 26.2]. Set $r = \frac{1}{p}k_0c_1^p$. Note that $\bar{u}(x) = c \in X$. It then follows from (M1) that

$$\begin{aligned}
\phi(\bar{u}) &= \frac{1}{p}\|\bar{u}\|^p + \sum_{j=1}^{l}\int_0^{\bar{u}(x_j)} I_j(x)dx \\
&= \frac{1}{p}c^p\int_a^b s(x)dx + \sum_{j=1}^{l}\int_0^{c} I_j(x)dx \\
&\le \frac{1}{p}k_0c^p + k_1c^p \\
&= k_2c^p,
\end{aligned}$$

and we have $\phi(\bar{u}) > \frac{1}{p}\|\bar{u}\|^p > r$. For $u \in X$ satisfying $\phi(u) \le r$, one has $\|u\|_\infty \le c_1$, since

$$\begin{aligned}
\|u\|_\infty^p &\le q_0^{-1}(b-a)^{p-1}\|u\|^p \\
&\le pq_0^{-1}(b-a)^{p-1}\phi(u) \\
&\le k_0^{-1}pr \\
&= c_1^p.
\end{aligned}$$

Therefore

$$\frac{\sup_{u\in\phi^{-1}((-\infty,r])}\psi(u)}{r} \leq \frac{\int_a^b \max_{|\xi|\leq c_1} F(x,\xi)dx + \frac{\mu}{\lambda}\int_a^b \max_{|\xi|\leq c_1} G(x,\xi)dx}{\frac{1}{p}k_0 c_1^p}$$
$$= p\frac{\int_a^b \max_{|\xi|\leq c_1} F(x,\xi)dx}{k_0 c_1^p} + p\frac{\mu}{\lambda}\frac{G^{c_1}}{k_0 c_1^p}.$$

From this, if $G^{c_1} = 0$, we deduce that

$$\frac{\sup_{u\in\phi^{-1}((-\infty,r])}\psi(u)}{r} < \frac{1}{\lambda}, \tag{6}$$

while, if $G^{c_1} > 0$, it turns out to be true bearing in mind that

$$\mu < \frac{k_0 c_1^p - p\lambda\int_a^b \max_{|\xi|\leq c_1} F(x,\xi)dx}{pG^{c_1}}.$$

On the other hand

$$\psi(\bar{u}) = \int_a^b F(x,\bar{u})dx + \frac{\mu}{\lambda}\int_a^b G(x,\bar{u})dx$$
$$\geq \int_a^b F(x,c)dx + \frac{\mu}{\lambda}G_d,$$

and so

$$\frac{\psi(\bar{u})}{\phi(\bar{u})} \geq \frac{\int_a^b F(x,c)dx}{k_2 c^p} + \frac{\mu}{\lambda}\frac{G_d}{k_2 c^p}.$$

Hence, if $G_d = 0$, we find

$$\frac{\psi(\bar{u})}{\phi(\bar{u})} > \frac{1}{\lambda}, \tag{7}$$

while, if $G_d < 0$, the same relation holds since

$$\mu < \frac{k_2 c^p - \lambda\int_a^b F(x,c)dx}{G_d}.$$

Therefore, from (6) and (7), condition (H1) of Theorem 2.1 is verified. By (M1), (M2) and condition (5), when $\lambda \in \left]\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right[$, we easily obtain that the functional I_λ is coercive and hence the condition (H2) of Theorem 2.1 is verified, too. Since, from (6) and (7)

$$\lambda \in \Lambda_r = \left]\frac{\phi(\bar{u})}{\psi(\bar{u})}, \frac{r}{\sup_{\phi(u)\leq r}\psi(u)}\right[,$$

Theorem 2.1 ensures the existence of the least three critical points for the functional I_λ and the proof is complete. □

The technical approach used to prove the previous result uses some ideas from [5]. In the cited work, the existence of at least three classical solutions for a perturbed two-point boundary value problem has been investigated under suitable conditions on the potentials F and G.

Now, we present the following example to illustrate the result.

Example 3.2. Take $p = 2$ and consider the problem

$$\begin{cases} -(e^x u'(x))' + \frac{1}{2}u(x) = \lambda f(x, u(x)) + 2\mu u(x), \ x \neq x_1, \ x \in [0,1], \\ \triangle\big(e^{x_1} u'(x_1)\big) = u(x_1), \quad x_1 = \frac{1}{2}, \\ u'(0) = u'(1) = 0, \end{cases} \tag{8}$$

where

$$f(x, u(x)) = \begin{cases} e^{u(x)}, & u(x) \leq 16, \ x \in [0,1], \\ u^{\frac{1}{2}}(x) + e^{16} - 4, & u(x) > 16, \ x \in [0,1]. \end{cases}$$

Here, $I_1(s) = s$, $q(x) = e^x$, $s(x) = \frac{1}{2}$ and $l = 1$. Note that (M1), (M2) are satisfied. Moreover, we have $k_0 = \frac{1}{2}$, $k_1 = \frac{1}{2}$, $k_2 = \frac{3}{4}$, and

$$F(x, u(x)) = \begin{cases} e^{u(x)} - 1, & u(x) \leq 16, \ x \in [0,1], \\ \frac{2}{3}u^{\frac{3}{2}}(x) + (e^{16} - 4)u(x) & \\ \qquad + \frac{61}{3} - 15e^{16}, & u(x) > 16, \ x \in [0,1]. \end{cases}$$

Choose $c_1 = 1$, $c = 16$. Direct calculations give

$$\lambda_1 = 4(e-1), \qquad \lambda_2 = \frac{1}{192}(e^{16} - 1).$$

Since, $G(\xi) = \xi^2$, we have $\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{\xi^2} < \infty$ and by choosing $d = 2$, we obtain that

$$G_d = \inf_{\xi \in [0,2]} G(\xi) = 0, \qquad G^{c_1} = \max_{|\xi| \leq 1} G(\xi) = 1.$$

Therefore, it follows from Theorem 3.1 that problem (8) admits at least three solutions in $W^{1,2}([0,1])$ provided that $\lambda \in \left]\frac{1}{\lambda_2}, \frac{1}{\lambda_1}\right[$ and $\mu \in [0, 0.25 - \lambda(e-1)[$.

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Hadi Haghshenas received Ph.D. from University of Mazandaran. He works on nonlinear analysis, nonlinear functional analysis theory of differential equations and applied functional analysis.

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
e-mail: haghshenas60@gmail.com

Ghasem A. Afrouzi is a member in Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran. Since 1989 His current research interests are nonlinear analysis, theory of Differential equations, Applied functional Analysis, nonlinear functional Analysis, and calculus of Variations.

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
e-mail: afrouzi@umz.ac.ir