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# EFFECT OF FEAR ON A MODIFIED LESLI-GOWER PREDATOR-PREY ECO-EPIDEMIOLOGICAL MODEL WITH DISEASE IN PREDATOR

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Abstract. The anti-predator factor due to fear of predator in eco- epidemiological models has a great importance and cannot be evaded. The present paper consists of a modified Lesli-Gower predator-prey model with contagious disease in the predator population only and also consider the fear effect in the prey population. Boundedness and positivity have been studied to ensure the eco-epidemiological model is well-behaved. The existence and stability conditions of all possible equilibria of the model have been studied thoroughly. Considering the fear constant as bifurcating parameter, the conditions for the existence of limit cycle under which the system admits a Hopf bifurcation are investigated. The detailed study for direction of Hopf bifurcation have been derived with the use of both the normal form and the central manifold theory. We observe that the increasing fear constant, not only reduce the prey density, but also stabilize the system from unstable to stable focus by excluding the existence of periodic solutions.

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## 1. Introduction

Ecology and epidemiology are leading area of research in their individual pertinent but there are some typical aspects among these two systems. At present many researchers are involved to review the ecological systems based on epidemiological components. To the best of our knowledge, after the pioneering work of Anderson and May [1], most of the precedent reviews such as Hadeler and Freedman [14], Venturino [37], Chattopadhyay and Arino [5], Han et. al. [15], Hethcote et. al. [17], Xiao and Chen [44], Greenhalgh and Haque [11], Pal and Samanta ([27], [28]), investigated the effect of the predation on epidemics.

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Most of these works have discussed the dynamics of predator-prey models with disease in prey. But the study about the predator control, the dynamics of predator-prey system with an infected predator has also ample influence. Some studies considered the transmission of disease in predator population in Lotka-Volterra type predator-prey models (Venturino [38], Haque et. al. [16]). Das [7] considered a predator-prey model with disease in predator and also derived the ecological as well as disease basic reproduction numbers and studied the community structure of model system by these numbers. Mondal et. al. [23] also studied dynamical behaviors of a Lesli-Gower predator-prey eco-epidemiological model with disease in predator.

Most of the predator-prey models are formulated on typical Lotka-Volterra formalism, where the prey consumption rate of predators is the growth rate of predators with a conversion factor. Lesli ([20], [21]) modified the Lotka-Volterra predator-prey model into Lesli-Gower model. The Lesli-Gower predator-prey model formulation is based on the assumption that reduction in a predator population is proportional to the reciprocal of per capita availability of its most preferred food. Here, the environmental carrying capacity of the predators is proportional to the biomass of prey, indicating the fact that there are upper limits to the rates of increase in both prey and predator, which are not focused in the Lotka-Volterra model. Many researchers studied the Lesli-Gower type predator-prey model and its modified version. Aziz-Aloui and Okiye [2] demonstrated the modified Lesli-Gower model by adding a positive constant with the carrying capacity of the predator and also its global stability. Zhu and Wang [46] studied the global attractive behavior of positive periodic solution for the modified Lesli-Gower model, where predator prey interaction follow Holling-type II scheme. Gupta and Chandra [13] showed the bifurcation analysis of a modified Lesli-Gower predator-prey model with Michaelis-Menten type prey harvesting. The modified Lesli-Gower model with time delay has discussed by Nindjin et. al. [24]. Sarwardi et. al. [32] proposed a modified Lesli-Gower predator-prey model which is speculated in the eco-epidemiological situation, with disease spreading among the prey species only. But, there is lack of research for the modified Lesli-Gower model with Holling-type II response function for predation with disease in predator.

Several experimental analysis showed that the population dynamics of ecological systems can be affected by the control of fear. In the natural world, an array of anti-predator responses such as variation in foraging behaviors, habitat usage and physiology can arouse prey to direct killing by the fear of predators, which may induce a long-term decrease in prey population. In 2011, Zanette et. al. [45] experimentally showed that the song sparrows, *Melospiza Melodia*, surrounded by predators sound produce 40% less offspring than the others hearing no predators sounds, in the absence of direct killing. Hua et. al. [18]

manipulated that realization of predation risk adversely disturb blue bird's reproduction by employing the vocal cues of their predators. Also, there are plenty information demonstrate that fear can disturb populations such as in Elk (Creel et. al. (2007) [6]) or in birds (Orrock & Fletcher (2014) [25]). Wang et. al. [39] first proposed a predator-prey model by incorporating fear of the predator on prey, where the cost of fear reduce the growth of prey. They explored that predator-prey dynamics can be stabilize by excluding the existence of limit cycle oscillation. Based on these facts, many researchers recommended several mathematical modeling approach to study the impact of fear in the predator-prey systems ([29], [26], [33], [34], [40], [41], ).

The main aim of this work is to study the impact of the fear effect on the mutual relations occurring in a modified Lesli-Gower type ecosystem where a microparasite affects a predator populations feeding on a prey, the latter being unaffected by the parasite. The remainder of this work is organized as follows. In Section 2, we illustrate the mathematical model with basic assumptions. The analysis made in Section 3 assures that the model is biologically well-posed. Some conditions are derived in Section 4 for which the prey, susceptible and infected predator will become extinct in the long-time run. Existence criteria, local and global stability analysis of the equilibria and permanence of the system are discussed in Section 5. Switching of stability of equilibrium points have been verified by Sotomayor's Theorem and also the criteria for Hopf bifurcation and the stability of periodic oscillation using the center manifold and normal form theory have been studied in Section 6. The dynamics of the system without fear factor have been observed in Section 7. Numerical simulations are performed to substantiate our analytical results in Section 8. Finally, a brief discussion is given in Section 9.

## 2. The Mathematical Model

The model we analyze in this paper has three populations:
(1) the prey, whose population biomass is denoted by $N(T)$,
(2) the susceptible predator, whose population biomass is denoted by $P_S(T)$,
(3) the infected predator, whose population biomass is denoted by $P_I(T)$.

In construction of the model the following assumptions are made:

$(A_1)$ In the absence of predators, the prey population grows according to logistic fashion with carrying capacity $K_1$ ($K_1 > 0$) and with an intrinsic growth rate constant $R_1$ ($R_1 > 0$).

$(A_2)$ In predator-prey ecological modeling, Holling Type-II or Michaelis-Menten functional response has drawn the attention of the researchers for its vivid application ([8], [31], [22]). Here we consider that predator-prey interaction is guided by a Holling Type-II functional response. It is assumed that only susceptible predator consumes prey population.

$(A_3)$ The growth of the predator is assumed to be of the modified Leslie-Gower type which was also studied by Aziz-Alaoui and Daher Okiye (2003) [2]. The growth of the total predator population follows logistic curve where the carrying capacity of the predator environment is proportional to the number of prey (i.e., prey dependent carrying capacity for the predator) ([20], [21], [36]). It is assumed that in case of severe scarcity, the predator can switch over the other population.

$(A_4)$ The parasite attacks the predator population only and the infected population does not recover or become immune. In general, most of the epidemic models deals with a mass action incidence rate $C_2P_SP_I$, $C_2$ being the infection rate. So $C_2$ is the maximum number of infections an infective predator can cause in a unit of time. It may be noted that if the degree of infectivity increases, many other components often come into the consideration which tend to saturate the effect that a large numbers of parasites may have. Therefore, it is more reasonable to replace the simple mass action term by Holling type-II term $\frac{C_2P_SP_I}{B+P_S}$ in order to have a clear insight of the microparasite infection. Many studies in the epidemiological literature have considered the Holling type-II function to describe the infection mechanism.

$(A_5)$ Now we incorporate the cost of fear in our model. We consider that due to fear of predator, the growth rate of the prey population reduces. We assume that the modified growth rate of the prey species is $\frac{R_1}{1+KP_S}$ in the presence of predator, which are monotonic decreasing function of both $K$ and $P_S$. Here $K$ is the fear parameter of the prey species. Hence, by considering $\psi(K,P_S)=\frac{1}{1+KP_S}$ and by the biological meaning of $K,\ P_S,$ and $\psi(K,P_S)$, it is reasonable to assume that

$$\psi(0,P_S)=1, \qquad \psi(K,0)=1, \quad \lim_{K\to\infty}\psi(K,P_S)=0,$$

$$\lim_{P_S\to\infty}\psi(K,P_S)=0, \quad \frac{\partial\psi(K,P_S)}{\partial K}<0, \quad \frac{\partial\psi(K,P_S)}{\partial P_S}<0.$$

On the above considerations, we introduced an eco-epidemic model under the groundwork of the following set of ordinary differential equations:

$$\begin{aligned}
\frac{dN}{dT} &= \frac{R_1N}{1+KP_S}\left(1-\frac{N}{K_1}\right)-\frac{C_1NP_S}{A+N}\\
\frac{dP_S}{dT} &= R_2P_S\left\{1-\frac{h(P_S+\eta P_I)}{N+M}\right\}-\frac{C_2P_SP_I}{B+P_S}-D_1P_S \qquad (2.1)\\
\frac{dP_I}{dT} &= \frac{C_2P_SP_I}{B+P_S}-(D_1+\nu)P_I
\end{aligned}$$

with initial biomasses $N(0)>0,\ P_S(0)>0,\ P_I(0)>0$.

All the model parameters are assumed to be positive constants with following interpretation:

$C_1$ : Predation rate (search efficiency of susceptible predator for prey).

$A,\ B$ : half saturation constants.
$R_2$ : intrinsic growth rate of susceptible predator population.
$h$ : measure of the food quantity that the prey provides and converted to predator birth.
$\eta$ : relative fecundity of an infected predator.
$M$ : measure the extent to which environment provides protection to predator $P_S$ and $P_I$.
$C_2$ : transmission coefficient from susceptible predator to infected predator.
$D_1$ : parasite-independent death rate of predator.
$\nu$ : parasite-induced excess death rate.

To reduce the number of parameters, we use the following scaling:

$$n = \frac{N}{K_1},\ p_s = \frac{P_S}{K_1},\ p_i = \frac{P_I}{K_1} \text{ and } t = R_1 T.$$

Then the system (2.1) takes the form (after some simplification):

$$\begin{aligned}
\frac{dn}{dt} &= \frac{n(1-n)}{1+kp_s} - \frac{\alpha n p_s}{a+n} = f^{(1)}(n, p_s, p_i) \\
\frac{dp_s}{dt} &= rp_s\left\{1 - \frac{h(p_s+\eta p_i)}{n+m}\right\} - \frac{\beta p_s p_i}{b+p_s} - d_1 p_s = f^{(2)}(n, p_s, p_i) \\
\frac{dp_i}{dt} &= \frac{\beta p_s p_i}{b+p_s} - d_2 p_i = f^{(3)}(n, p_s, p_i)
\end{aligned} \tag{2.2}$$

together with $n(0) > 0,\ p_s(0) > 0,\ p_i(0) > 0$, where

$$k = KK_1,\ r = \tfrac{R_2}{R_1},\ \alpha = \tfrac{C_1}{R_1},\ a = \tfrac{A}{K_1},\ \beta = \tfrac{C_2}{R_1},$$
$$m = \tfrac{M}{K_1},\ b = \tfrac{B}{K_1},\ d_1 = \tfrac{D_1}{R_1},\ d_2 = \tfrac{D_1+\nu}{R_1}.$$

## 3. Positivity and Boundedness

We shall discuss positivity and boundedness of the system (2.2) to ensure that the model is well behaved.

**Theorem 3.1.** *All the solutions of system* (2.2) *that start in* $\mathbb{R}^3_+$ *remain positive for all time.*

*Proof.* From the first equation of system (2.2), we get

$$n(t) = n(0)\exp\left[\int_0^t \left\{\frac{1-n(\theta)}{1+kp_s(\theta)} - \frac{\alpha p_s(\theta)}{a+n(\theta)}\right\} d\theta\right] \Rightarrow n(t) > 0.$$

From the second equation of system (2.2), we get

$$p_s(t) = p_s(0)\exp\left[\int_0^t \left\{r\left(1 - \tfrac{h(p_s(\theta)+\eta p_i(\theta))}{x(\theta)}\right) - \tfrac{\beta p_i(\theta)}{b+p_s(\theta)} - d_1\right\} d\theta\right] \Rightarrow p_s(t) > 0.$$

From the third equation of system (2.2), we get

$$p_i(t) = p_i(0)\exp\left[\int_0^t \left\{\frac{\beta p_s(\theta)}{b+p_s(\theta)} - d_2\right\} d\theta\right] \Rightarrow p_i(t) > 0.$$

Hence the theorem is proved.

□

**Theorem 3.2.** *All solutions of the system* (2.2) *that start in* $\mathbb{R}^3_+$ *are uniformly bounded.*

*Proof.* Let $(n(t), p_s(t), p_i(t))$ be any solution of the system (2.2). Since

$$\frac{dn}{dt} \le n(1-n)$$

we have

$$\lim_{t\to\infty} \sup n(t) \le 1.$$

Let us define a function $W = n + p_s + p_i$
Along solutions of (2.2) :

$$\begin{aligned}\frac{dW}{dt} &= \tfrac{dn}{dt} + \tfrac{dp_s}{dt} + \tfrac{dp_i}{dt}\\ &\le \tfrac{n(1-n)}{1+kp_s} + rp_s - d_1p_s - d_2p_i\\ &\le n - (d_1 - r)p_s - d_2p_i\\ &\le 2n - \gamma W, \text{ where } \gamma = \min\{1, d_1 - r, d_2\}, \text{ assuming } d_1 > r.\end{aligned}$$

Hence $\frac{dW}{dt} + \gamma W \le 2$, for large $t$, since $\lim_{t\to\infty} \sup n(t) \le 1$.
Applying a theorem on differential inequalities [3], we obtain

$$0 < W(n, p_s, p_i) \le W(n(0), p_s(0), p_i(0))e^{-\gamma t} + \frac{2}{\gamma}(1 - e^{-\gamma t}),$$

and for $t \to \infty$, we get $0 < W \le \frac{2}{\gamma}$.
Thus, all solutions of the system (2.2) enters into the region

$$B = \left\{(n, p_s, p_i) : 0 < W \le \frac{2}{\gamma} + \epsilon, \text{ for any } \epsilon > 0\right\}.$$

This proves the theorem. □

## 4. Extinction Scenarios

This section contains the conditions for which prey and both predator species will get extinct in the long run. Suppose:

$$\bar{p_s} = \limsup_{t\to\infty} p_s(t) \text{ and } \underline{p_s} = \liminf_{t\to\infty} p_s(t),$$

Here we shall use the following fact: $\lim\sup_{t\to\infty} n(t) \le 1$

The first theorem will show the extinction criteria of prey species and the second and last theorem will show the extinction criteria of the infected and susceptible predator species respectively.

**Theorem 4.1.** *If* $\underline{p_s} > \frac{a+2}{\alpha}$, *then* $\limsup_{t\to\infty} n(t) = 0$.

*Proof.* If possible, let $\lim_{t\to\infty} n(t) = \zeta > 0$. Since $\bar{n} \leq 1$, then for any $0 < \epsilon < 1$, there exists $t_\epsilon > 0$ such that $n(t) < 1 + \epsilon$ for $t > t_\epsilon$.
From the definition of $\underline{p_s}$, it follows that for any $0 < \epsilon_1 < \underline{p_s} - \frac{a+2}{\alpha}$, there exists $t_{\epsilon_1} > 0$ such that $p_s(t) > \underline{p_s} - \epsilon_1$ for $t > t_{\epsilon_1}$.
Then, for $t > \max\{t_\epsilon, t_{\epsilon_1}\}$, the first equation of the system (2.2) can be written as

$$\begin{aligned}\frac{dn}{dt} &\leq n - \tfrac{\alpha n p_s}{a+n}, \\ &< n - \tfrac{\alpha n p_s}{a+1+\epsilon}, \\ &< n\{1 - \tfrac{\alpha(\underline{p_s}-\epsilon_1)}{a+2}\}, \\ &= -\tfrac{\alpha n}{a+2}(\underline{p_s} - \epsilon_1 - \tfrac{a+2}{\alpha}) < 0,\end{aligned}$$

a contradiction and hence $\limsup_{t\to\infty} n(t) = 0$. □

**Theorem 4.2.** *If* $d_2 > \beta\bar{p_s}$, *then* $\limsup_{t\to\infty} p_i(t) = 0$.

*Proof.* Choose $\epsilon$ such that $0 < \epsilon < \frac{d_2}{\beta} - \bar{p_s}$. By definition: there exists $t'$ such that $p_s(t) < \bar{p_s} + \epsilon$, $\forall\ t > t'$.
For $t > t'$:

$$\begin{aligned}\frac{dp_i}{dt} &= p_i(\tfrac{\beta p_s}{b+p_s} - d_2), \\ &< p_i(-d_2 + \beta p_s), \\ &< p_i\{-d_2 + \beta(\bar{p_s} + \epsilon)\}, \\ &< 0.\end{aligned}$$

Hence, $\limsup_{t\to\infty} p_i(t) = 0$. □

**Theorem 4.3.** *If* $r < d_1$, *then* $\lim_{t\to\infty} p_s(t) = 0$.

*Proof.* From second and third equations of the system (2.2), we have

$$\frac{dp_s}{dt} + \frac{dp_i}{dt} = p_s\{r(1 - \frac{h(p_s + \eta p_i)}{n+m})\} - d_1 p_s - d_2 p_i,$$

$$\begin{aligned}\frac{d}{dt}(p_s + p_i) &< rp_s - d_1 p_s - d_2 p_i, \\ &< r(p_s + p_i) - d_1(p_s + p_i), \text{ since } d_1 < d_2 \\ &< (r - d_1)(p_s + p_i).\end{aligned}$$

Therefore, $p_s(t) + p_i(t) = [p_s(t_0) + p_i(t_0)]\exp\{\int_{t_0}^{t}(r - d_1)\}d\xi$
$\leq [p_s(t_0) + p_i(t_0)]\exp\{(r - d_1)(t - t_0)\}$.
Thus $\lim_{t\to\infty}\{p_s(t) + p_i(t)\} = 0$ provided $r < d_1$.
Again $\lim_{t\to\infty} p_i(t) = 0$, hence $\lim_{t\to\infty} p_s(t) = 0$. □

## 5. Model Analysis

The objective of this section is to study the existence and stability of the equilibrium points of the system (2.2).

### 5.1. Existence of Equilibrium points

The system of equation has the following equilibrium points:
1. The trivial equilibrium: $E_0(0,0,0)$.

2. The predator-free axial equilibrium: $E_1(1,0,0)$.

3. The prey extinction axial equilibrium: $E_2(0,\tilde{p_s},0)$, where $\tilde{p_s}=\frac{m(r-d_1)}{rh}$, provided $r>d_1$, i.e., in original parameter, intrinsic growth rate of susceptible predator population is greater than parasite-independent death rate of predator, which is biologically meaningful.

4. Infection free boundary equilibrium: $E_3(\hat{n},\hat{p_s},0)$, where $\hat{n}=\frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}$, $\hat{p_s}=\frac{(r-d_1)(\hat{n}+m)}{rh}$ and $A_1=1+\frac{k\alpha(r-d_1)^2}{r^2h^2}$, $A_2=\frac{\alpha(r-d_1)}{rh}+\frac{2km\alpha(r-d_1)^2}{r^2h^2}+a-1$, $A_3=\frac{k\alpha m^2(r-d_1)^2}{r^2h^2}+\frac{\alpha m(r-d_1)}{rh}-a$, provided $r>d_1$ and $\frac{\alpha m(r-d_1)}{rh}\{\frac{km(r-d_1)}{rh}+1\}<a$.

5. Interior equilibrium: $E^*(n^*,p_s^*,p_i^*)$, where $n^*$ is the positive root of the quadratic $(n^*)^2+A_4n^*+A_5=0$, such that $A_4=a-1$, $A_5=\frac{\alpha bd_2}{(\beta-d_2)^2}(\beta-d_2+kbd_2)-a$ and $p_s^*=\frac{bd_2}{(\beta-d_2)}$, and $p_i^*=(r-\frac{rhp_s^*}{n^*+m}-d_1)/(\frac{rh\eta}{n^*+m}+\frac{\beta}{b+p_s^*})$, provided $\beta>d_2$, $k<\frac{(\beta-d_2)}{bd_2}\{\frac{a(\beta-d_2)}{\alpha bd_2}-1\}$ and $r(1-\frac{hp_s^*}{n^*+m})>d_1$ holds.

### 5.2. Stability analysis of equilibrium points

The variational matrix $V$ of the system (2.2) at any arbitrary point $(n,p_s,p_i)$ is given by $V=(v_{ij})_{3\times3}$ where

$$v_{11}=\frac{1-2n}{1+kp_s}-\frac{a\alpha p_s}{(a+n)^2},\ v_{12}=-\frac{kn(1-n)}{(1+kp_s)^2}-\frac{\alpha n}{a+n},\ v_{13}=0,$$

$$v_{21}=\frac{rhp_s(p_s+\eta p_i)}{(n+m)^2},\ v_{22}=r-\frac{rh(2p_s+\eta p_i)}{n+m}-\frac{b\beta p_i}{(b+s)^2}-d_1,\ v_{23}=-\frac{rh\eta p_s}{n+m}-\frac{\beta p_s}{b+p_s},$$

$$v_{31}=0,\ v_{32}=\frac{b\beta p_i}{b+p_s},\ v_{33}=\frac{\beta p_s}{b+p_s}-d_2.$$

**5.2.1.** The behavior of the system around $E_0(0,0,0)$

At $E_0$, the variational matrix $V(E_0)$ is given by

$$V(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

Then the corresponding eigenvalues are $1, r - d_1$ and $-d_2$.

**Theorem 5.1.** *The trivial equilibrium point $E_0$ is always unstable.*

**5.2.2.** The behavior of the system around $E_1(1, 0, 0)$

**Theorem 5.2.** $(a)$ *The predator-free axial equilibrium point $E_1(1, 0, 0)$ of the system* (2.2) *is locally asymptotically stable if $R_{01} < 1$, where $R_{01} = \frac{r}{d_1}$.*
$(b)$ *$E_1$ is globally asymptotically stable if $d_1 \geq r + \alpha$.*

*Proof.* $(a)$ At $E_1$, the variational matrix $V(E_1)$ is given by

$$V(E_1) = \begin{bmatrix} -1 & -\frac{\alpha}{a+1} & 0 \\ 0 & r - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}.$$

Then the corresponding eigenvalues are $-1,\ r - d_1,\ -d_2$. Therefore, $E_1$ is locally asymptotically stable if $R_{01} < 1$. □

**Remark 5.1.** $(i)$ In terms of original parameters of the system, the condition $R_{01} < 1$ becomes $R_2 < D_1$. This means that when growth rate of predator population $(R_2)$ is less than parasite independent death rate $(D_1)$ then the system becomes predator free, which is biologically meaningful.

$(ii)$ The existence of $E_3$ destabilizes $E_1$.

$(iii)$ The system (2.2) is stable about the predator-free equilibrium $(E_1)$ if $R_{01} < 1$ and becomes unstable if $R_{01} > 1$. Therefore, the predator-free equilibrium undergoes a transcritical bifurcation at $R_{01} = 1$, i.e. $r = d_1 = r_{[TC_1]}$.

*Proof.* $(b)$ Let $\Gamma^3_{+n} = \{(n, p_s, p_i) \in \mathbb{R}^3_+ : n > 0,\ p_s \geq 0,\ p_i \geq 0\}$ and consider a positive definite function $L_1 : \Gamma^3_{+n} \to \Re$ about $E_1(1, 0, 0)$ given by

$$L_1(n, p_s, p_i) = \frac{1}{2}(n-1)^2 + p_s + p_i$$

The derivative of $L_1$ w.r.t time $t$ along the solution of the system (2.2), we get

$$\begin{aligned} &\frac{dL_1}{dt} \\ &= (n-1)\{\tfrac{n(1-n)}{1+kp_s} - \tfrac{\alpha n p_s}{a+n}\} + rp_s(1 - \tfrac{h(p_s+\eta p_i)}{n+m}) - \tfrac{\beta p_s p_i}{b+p_s} - d_1 p_s + \tfrac{\beta p_s p_i}{b+p_s} - d_2 p_i \\ &\leq -\tfrac{n(n-1)^2}{1+kp_s} - \tfrac{\alpha n^2 p_s}{a+n} + \alpha p_s + rp_s - \tfrac{rhp_s(p_s+\eta p_i)}{n+m} - d_1 p_s - d_2 p_i \ [\because \tfrac{n}{a+n} < 1] \\ &= -\tfrac{n(n-1)^2}{1+kp_s} - \tfrac{\alpha n^2 p_s}{a+n} - (d_1 - r - \alpha)p_s - \tfrac{rhp_s(p_s+\eta p_i)}{n+m} - d_2 p_i \end{aligned}$$

$\leq 0$, if $d_1 \geq r+\alpha$; and $\dot{L_1} = 0$ at $(n, p_s, p_i) = (1, 0, 0)$. Hence the equilibrium point $E_1$ is globally asymptotically stable for $d_1 \geq r+\alpha$. □

**5.2.3.** The behavior of the system around $E_2(0, \tilde{p_s}, 0)$

**Theorem 5.3.** *The prey extinction axial equilibrium point $E_2(0, \tilde{p_s}, 0)$ of the system* (2.2) *is locally asymptotically stable if $\beta\tilde{p_s} < d_2(b+\tilde{p_s})$ and $a < \alpha\tilde{p_s}(1+k\tilde{p_s})$.*

*Proof.* At $E_2$, the variational matrix $V(E_2)$ is given by

$$V(E_2) = \begin{bmatrix} \frac{1}{1+k\tilde{p_s}} - \frac{\alpha\tilde{p_s}}{a} & 0 & 0 \\ \frac{rh\tilde{p_s}^2}{m^2} & -\frac{rh\tilde{p_s}}{m} & -(\frac{rh\eta}{m} + \frac{\beta}{b+\tilde{p_s}})\tilde{p_s} \\ 0 & 0 & \frac{\beta\tilde{p_s}}{b+\tilde{p_s}} - d_2 \end{bmatrix}.$$

Then the corresponding eigenvalues are $\frac{1}{1+k\tilde{p_s}} - \frac{\alpha\tilde{p_s}}{a}$, $-\frac{rh\tilde{p_s}}{m}$, and $\frac{\beta\tilde{p_s}}{b+\tilde{p_s}} - d_2$. Therefore, $E_2$ is locally asymptotically stable if $\frac{\beta\tilde{p_s}}{b+\tilde{p_s}} - d_2 < 0$ and $\frac{1}{1+k\tilde{p_s}} - \frac{\alpha\tilde{p_s}}{a} < 0$ which implies the conditions. □

**5.2.4.** The behavior of the system around $E_3(\hat{n}, \hat{p_s}, 0)$

**Theorem 5.4.** $(a)$ *The infection free boundary equilibrium point $E_3(\hat{n}, \hat{p_s}, 0)$ of the system* (2.2) *is locally asymptotically stable if $R_{02} < 1$ and $\alpha\hat{p_s}(1+k\hat{p_s}) < (a+\hat{n})^2$ where $R_{02} = \frac{\beta\hat{p_s}}{d_2(b+\hat{p_s})}$.*
$(b)$ *If the equilibrium $E_3(\hat{n}, \hat{p_s}, 0)$ exists and is locally asymptotically stable in the interior of positive quadrant of $n-p_s$ plane, then it will be globally asymptotically stable.*

*Proof.* $(a)$ At $E_3$, the variational matrix $V(E_3)$ is given by

$$V(E_3) = \begin{bmatrix} v_{11}^{[3]} & v_{12}^{[3]} & 0 \\ v_{21}^{[3]} & v_{22}^{[3]} & v_{23}^{[3]} \\ 0 & 0 & v_{33}^{[3]} \end{bmatrix}.$$

where

$$v_{11}^{[3]} = -\frac{\hat{n}}{1+k\hat{p_s}} + \frac{\alpha\hat{n}\hat{p_s}}{(a+\hat{n})^2},\ v_{12}^{[3]} = -\frac{k\hat{n}(1-\hat{n})}{(1+k\hat{p_s})^2} - \frac{\alpha\hat{n}}{a+\hat{n}},\ v_{21}^{[3]} = \frac{rh\hat{p_s}^2}{(\hat{n}+m)^2},$$

$$v_{22}^{[3]} = -\frac{rh\hat{p_s}}{\hat{n}+m},\ v_{23}^{[3]} = -\frac{rh\eta\hat{p_s}}{\hat{n}+m} - \frac{\beta\hat{p_s}}{b+\hat{p_s}},\ v_{33}^{[3]} = \frac{\beta\hat{p_s}}{b+\hat{p_s}} - d_2$$

The corresponding eigenvalues are $v_{33}^{[3]}$ and the roots of the quadratic $\lambda^2 + B_1\lambda + B_2 = 0$, where $B_1 = -(v_{11}^{[3]} + v_{22}^{[3]})$ and $B_2 = v_{11}^{[3]}v_{22}^{[3]} - v_{12}^{[3]}v_{21}^{[3]}$. Now the quadratic has negative real part if $B_1 > 0$ and $B_2 > 0$. So, if only $v_{11}^{[3]} < 0$ then both the eigenvalues have negative real parts. Therefore, $E_3$ is locally asymptotically stable if $\frac{\beta\hat{p_s}}{b+\hat{p_s}} - d_2 < 0$ and $-\frac{\hat{n}}{1+k\hat{p_s}} + \frac{\alpha\hat{n}\hat{p_s}}{(a+\hat{n})^2} < 0$ which implies the conditions $R_{02} < 1$ and $\alpha\hat{p_s}(1+k\hat{p_s}) < (a+\hat{n})^2$. □

**Remark 5.2.** $(i)$ The system (2.2) is stable about the equilibrium ($E_2$) if $R_{02} < 1$ and becomes unstable if $R_{02} > 1$. Therefore, the infection free equilibrium undergoes a transcritical bifurcation at $R_{02} = 1$, i.e. $\beta = \frac{d_2(b+\hat{p_s})}{\hat{p_s}} = \beta_{[TC_2]}$.

$(ii)$ Here $\frac{\beta\hat{p_s}}{b+\hat{p_s}}$ is the infection rate of a newly infected predator appearing in a totally susceptible predator and $\frac{1}{d_2}$ is the duration of infectivity of an infective predator, the product of which is the disease basic reproduction number. $R_{02} < 1$ implies that the infected predator will dies out from the system and only the prey and the susceptible predator survive.

*Proof.* $(b)$ For $E_3(\hat{n}, \hat{p_s}, 0)$: let

$$U(n,p_s) = \frac{1}{np_s},\ u_1(n,p_s) = \frac{n(1-n)}{1+kp_s} - \frac{\alpha n p_s}{a+n},\ \text{and } u_2(n,p_s) = p_s\Big(r - \frac{rhp_s}{n+m} - d_1\Big).$$

So, $U(n,p_s) > 0$ in the interior of positive quadrant of $n - p_s$ plane.
Hence

$$\begin{aligned}
\Delta(n,p_s) &= \tfrac{\partial}{\partial n}(Uu_1) + \tfrac{\partial}{\partial p_s}(Uu_2),\\
&= -\frac{1}{p_s(1+kp_s)} + \frac{\alpha}{(a+n)^2} - \frac{rh}{n(n+m)},\\
&= -\frac{\{(a+n)^2 - \alpha p_s(1+kp_s)\}}{p_s(1+kp_s)(a+n)^2} - \frac{rh}{n(n+m)},\\
&< 0.\ (\because\ E_3\ is\ \text{LAS})
\end{aligned}$$

So, there exists no limit cycle in the positive quadrant of $n - p_s$ plane, by Bendixson-Dulac criterion. Hence, it will be globally asymptotically stable in the positive quadrant of $n - p_s$ plane, if $E_3(\hat{n}, \hat{p_s}, 0)$ is locally asymptotically stable. □

**5.2.5.** The behavior of the system around $E^*(n^*, p_s^*, p_i^*)$

**Theorem 5.5.** $(a)$ *The interior equilibrium point* $E^*(n^*, p_s^*, p_i^*)$ *of the system* (2.2) *is locally asymptotically stable if the following conditions hold:*
*(i)* $\frac{\alpha p_s^*}{(a+n^*)^2} < \frac{1}{1+kp_s^*}$
*(ii)* $\frac{\beta p_i^*}{(b+p_s^*)^2} < \frac{rh}{n^*+m}$.
$(b)$ *The interior equilibrium point* $E^*$ *is globally asymptotically stable if* $\Pi_1 > 0,\ \Pi_2 > 0,\ \Pi_3 > 0,$ *where* $\Pi_1,\ \Pi_2,\ \Pi_3$ *are defined later.*

*Proof.* $(a)$ At $E^*$, the variational matrix $V(E^*)$ is given by

$$V(E^*) = \begin{bmatrix} V_{11} & V_{12} & 0 \\ V_{21} & V_{22} & V_{23} \\ 0 & V_{32} & 0 \end{bmatrix}.$$

where

$$V_{11} = -\frac{n^*}{1+kp_s^*} + \frac{\alpha n^* p_s^*}{(a+n^*)^2},\ V_{12} = -\frac{kn^*(1-n^*)}{(1+kp_s^*)^2} - \frac{\alpha n^*}{a+n^*},\ V_{21} = \frac{rhp_s^*(p_s^* + \eta p_i^*)}{(n^*+m)^2},$$

$$V_{22} = -\frac{rhp_s^*}{n^*+m} + \frac{\beta p_s^* p_i^*}{(b+p_s^*)^2},\ V_{23} = -\frac{rh\eta p_s^*}{n^*+m} - \frac{\beta p_s^*}{b+p_s^*},\ V_{32} = \frac{\beta b p_i^*}{(b+p_s^*)^2}$$

The characteristic equation of the variational matrix is given by $\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3 = 0$
where $Q_1 = -(V_{11}+V_{22}),\ Q_2 = V_{11}V_{22} - V_{12}V_{21} - V_{23}V_{32},\ Q_3 = V_{11}V_{23}V_{32} = -\det[V(E^*)]$
Now, $\Delta = Q_1Q_2 - Q_3 = -(V_{11}+V_{22})(V_{11}V_{22} - V_{12}V_{21}) + V_{22}V_{23}V_{32}$.
If $V_{11} < 0,\ V_{22} < 0$ then $Q_1 > 0,\ Q_3 > 0,$ and $\Delta > 0$. Using the Routh-Hurwitz criteria we observe that the system (2.2) is locally asymptotically stable around the interior equilibrium point $E^*$ if the conditions stated in the theorem hold.

($b$) Let us define the function $L_2(n,p_s,p_i) = L_{21}(n,p_s,p_i) + L_{22}(n,p_s,p_i) + L_{23}(n,p_s,p_i)$. where $L_{21} = n - n^* - n^* ln\frac{n}{n^*},\ L_{22} = p_s - p_s^* - p_s^* ln\frac{p_s}{p_s^*},\ L_{23} = p_i - p_i^* - p_i^* ln\frac{p_i}{p_i^*}$.
It is to be shown that $L_2$ is Lyapunov function and $L_2$ vanishes at $E^*$ and it is positive for all $n,\ p_s,\ p_i\ >\ 0$. Hence $E^*$ represents its global minimum. Let us calculate the time derivative of $L_{2i}$ $(i = 1,2,3)$ along the solution of the system (2.2).

$$\frac{dL_{21}}{dt} = (n-n^*)\left(\frac{1-n}{1+kp_s} - \frac{\alpha p_s}{a+n}\right),$$
$$= (n-n^*)\left[-\frac{(n-n^*)}{(1+kp_s)} + \frac{\alpha p_s^*(n-n^*)}{(a+n)(a+n^*)} - \frac{k(1-n^*)(p_s-p_s^*)}{(1+kp_s)(1+kp_s^*)} - \frac{\alpha(p_s-p_s^*)}{a+n}\right],$$

Similarly

$$\begin{aligned}\frac{dL_{22}}{dt} &= (p_s-p_s^*)\left(r\{1-\frac{h(p_s+\eta p_i)}{n+m}\} - \frac{\beta p_i}{b+p_s} - d_1\right),\\ &= (p_s-p_s^*)\Big[\frac{rh(p_s^*+\eta p_i^*)(n-n^*)}{(n+m)(n^*+m)} - \frac{rh(p_s-p_s^*)}{(n+m)} + \frac{\beta p_i^*(p_s-p_s^*)}{(b+p_s)(b+p_s^*)}\\ &- \frac{rh\eta(p_i-p_i^*)}{(n+m)} - \frac{\beta(p_i-p_i^*)}{(b+p_s)}\Big],\end{aligned}$$

And

$$\begin{aligned}\frac{dL_{23}}{dt} &= (p_i-p_i^*)\left(\frac{\beta p_s}{b+p_s} - d_2\right),\\ &\le (p_i-p_i^*)\left[\frac{b\beta(p_s-p_s^*)}{(b+p_s)(b+p_s^*)} + (p_i-p_i^*)\right],\end{aligned}$$

Now we consider

$$\frac{dL_2}{dt} = -[A_{11}(n-n^*)^2 + A_{22}(p_s-p_s^*)^2 + A_{33}(p_i-p_i^*)^2 + 2A_{12}(n-n^*)(p_s-p_s^*)$$
$$+2A_{13}(n-n^*)(p_i-p_i^*) + 2A_{23}(p_s-p_s^*)(p_i-p_i^*)] = -V^TQV,$$

where $V = ((n-n^*), (p_s - p_s^*), (p_i - p_i^*))^T$ and $Q$ is quadratic form given by

$$Q = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

with the entries that are functions only of the variable $n$ as follows

$$A_{11} = \frac{1}{1+kp_s} - \frac{\alpha p_s^*}{(a+n)(a+n^*)},\ A_{22} = \frac{rh}{(n+m)} - \frac{\beta p_i^*}{(b+p_s)(b+p_s^*)},\ A_{33} = -1,$$
$$A_{13} = 0,\ A_{12} = \frac{1}{2}\left[\frac{k(1-n^*)}{(1+kp_s)(1+p_s^*)} + \frac{\alpha}{a+n} - \frac{rh(p_s^* + \eta p_i^*)}{(n+m)(n^*+m)}\right],$$
$$A_{23} = \frac{1}{2}\left[\frac{rh\eta}{n+m} + \frac{\beta}{b+p_s} - \frac{b\beta}{(b+p_s)(b+p_s^*)}\right].$$

If the matrix $Q$ is positive definite, then $\frac{dL_2}{dt} < 0$. So, all the principal minors of $Q$, namely, $\Pi_1 \equiv A_{11},\ \Pi_2 \equiv A_{11}A_{22} - A_{12}^2,\ \Pi_3 \equiv A_{11}A_{22}A_{33} + 2A_{12}A_{13}A_{23} - A_{11}A_{23}^2 - A_{22}A_{13}^2 - A_{33}A_{12}^2$, to be positive, i.e., $\Pi_1 > 0,\ \Pi_2 > 0,\ \Pi_3 > 0$. □

### 5.3. Permanence of the System

To establish the permanence of the system (2.2), we shall apply the Average Lyapunov functions (Gard and Hallam (1979), Th.4 [10] and Freedman and Ruan [9]).

**Theorem 5.6.** *Suppose that the system* (2.2) *satisfies the following conditions:*

(i) $r > d_1$;

(ii) $\frac{1}{1+k\tilde{p_s}} > \frac{\alpha\tilde{p_s}}{a}$ *and/or* $\frac{\beta\tilde{p_s}}{b+\tilde{p_s}} > d_2$;

(iii) $\frac{\beta\hat{p_s}}{b+\hat{p_s}} > d_2$,

*then system* (2.2) *is permanence.*

*Proof.* Let us consider the average Lyapunov function in the form $L(n, p_s, p_i) = n^{\mu_1} p_s^{\mu_2} p_i^{\mu_3}$ where each $\mu_i$ $(i = 1, 2, 3)$ is assumed to be positive. In the interior of $\mathbb{R}_+^3$, we have

$$\begin{aligned} \frac{\dot{L}}{L} &= \phi(n, p_s, p_i) = \mu_1\left[\tfrac{1-n}{1+kp_s} - \tfrac{\alpha p_s}{a+n}\right] \\ &+ \mu_2\left[r\left\{1 - \tfrac{h(p_s+\eta p_i)}{n+m}\right\} - \tfrac{\beta p_i}{b+p_s} - d_1\right] + \mu_3\left[\tfrac{\beta p_s}{b+p_s} - d_2\right]. \end{aligned}$$

To prove permanence of the system we shall have to show that $\phi(n, p_s, p_i) > 0$ for all boundary equilibria of the system. The values of $\phi(n, p_s, p_i)$ at the boundary equilibrium $E_0,\ E_1,\ E_2,\ E_3$ are the following:

$$\begin{aligned} E_0 &: \mu_1 + \mu_2(r - d_1) - \mu_3 d_2, \\ E_1 &: \mu_2(r - d_1) - \mu_3 d_2, \\ E_2 &: \mu_1\left(\tfrac{1}{1+k\tilde{p_s}} - \tfrac{\alpha\tilde{p_s}}{a}\right) + \mu_3\left(\tfrac{\beta\tilde{p_s}}{b+\tilde{p_s}} - d_2\right), \\ E_3 &: \mu_3\left\{\tfrac{\beta\hat{p_s}}{b+\hat{p_s}} - d_2\right\}. \end{aligned}$$

Now, $\phi(0,0,0) > 0$ is always true for some suitable $\mu_i > 0$ $(i = 1,2,3)$. Also, if the inequalities $(i) - (iii)$ hold, $\phi$ is positive at $E_1,\ E_2$, and $E_3$. Therefore, the system (2.2) is permanent. Hence the theorem. □

**Remark 5.3.** The conditions $(i)-(iii)$ guarantee that the boundary equilibrium points $E_1,\ E_2,$ and $E_3$ are unstable.

## 6. Bifurcation Analysis

In this section we study local bifurcation and Hopf bifurcation of the system (2.2).

### 6.1. Local bifurcation of the system

The variational matrices of the system (2.2) corresponding to $E_1$ and $E_3$ has a zero eigenvalue if $r = d_1$ and $\beta = \frac{d_2(b+\hat{p_s})}{\hat{p_s}}$ respectively. Therefore the non-hyperbolic equilibrium point $E_1$ and $E_3$ may have a bifurcation for the parameter $r$ and $\beta$ respectively.

**Theorem 6.1.** *The system* (2.2) *undergoes a transcritical bifurcation with respect to the bifurcation parameter $r$ around $E_1(1,0,0)$ if $r = d_1 = r_{[TC_1]}$.*

*Proof.* To prove the theorem, we apply Sotomayor's theorem [30], by considering $r$ as bifurcation parameter. In order to apply Sotomayor's theorem exactly one of the eigenvalues of the variational matrix at the bifurcation point must be zero and other eigenvalues have negative real parts.

One of the eigenvalue of $V(E_1)$ will be zero if the eigenvalue $\lambda = r - d_1 = 0$, i.e., $r = d_1 = r_{[TC_1]}$. Now when $r = r_{[TC_1]}$, the other two eigenvalues are given by $\lambda_1 = -1 < 0$ and $\lambda_2 = -d_2 < 0$. We have obtained that $W = (\frac{\alpha}{a+1}, -1, 0)^T,\ \tilde{W} = (0,1,0)^T$, where $W,\ \tilde{W}$ are the eigenvectors corresponding to the eigenvalue $\lambda = 0$ of the matrices $V(E_1)$ and $[V(E_1)]^T$ respectively. Now,

$$\tilde{W}^T.[f_r(E_1, r_{[TC_1]})] = 0.$$

Again,

$$\tilde{W}^T.[Df_r(E_1, r_{[TC_1]})W] = -1 \neq 0,$$

$$\text{and } \tilde{W}.[D^2 f(E_1, r_{[TC_1]}).(W,W)] = -\frac{2rh}{1+m} \neq 0,$$

where $[Df_r(E_1, r_{[TC_1]})] = (\alpha_{ij})_{3\times 3}$ of which $\alpha_{22} = 1$, and all other $\alpha_{ij} = 0$ for $i,j = 1,2,3,$; and $[D^2 f(X,r)] = \begin{bmatrix} \nabla\frac{\partial f^{(1)}}{\partial n} & \nabla\frac{\partial f^{(2)}}{\partial n} & \nabla\frac{\partial f^{(3)}}{\partial n} \\ \nabla\frac{\partial f^{(1)}}{\partial p_s} & \nabla\frac{\partial f^{(2)}}{\partial p_s} & \nabla\frac{\partial f^{(3)}}{\partial p_s} \\ \nabla\frac{\partial f^{(1)}}{\partial p_i} & \nabla\frac{\partial f^{(2)}}{\partial p_i} & \nabla\frac{\partial f^{(3)}}{\partial p_i} \end{bmatrix} \in \mathbb{R}^{3\times 3\times 3}$, $\nabla\frac{\partial f^i}{\partial n} = (\frac{\partial^2 f^i}{\partial n^2}, \frac{\partial^2 f^i}{\partial p_s \partial n}, \frac{\partial^2 f^i}{\partial p_i \partial n}),\ \nabla\frac{\partial f^i}{\partial p_s} = (\frac{\partial^2 f^i}{\partial n \partial p_s}, \frac{\partial^2 f^i}{\partial^2 p_s}, \frac{\partial^2 f^i}{\partial p_i \partial p_s}),$

$\nabla \frac{\partial f^i}{\partial p_i} = (\frac{\partial^2 f^i}{\partial n \partial p_i}, \frac{\partial^2 f^i}{\partial p_s \partial p_i}, \frac{\partial^2 f^i}{\partial^2 p_i})$, for $i = 1, 2, 3$. Thus the system (2.2) possesses a transcritical bifurcation around $E_1$ at $r = r_{[TC_1]}$, by Sotomayor's theorem. □

**Theorem 6.2.** *The system* (2.2) *undergoes a transcritical bifurcation with respect to the bifurcation parameter* $\beta$ *around* $E_3(\hat{n}, \hat{p_s}, 0)$ *if* $\beta = \frac{d_2(b+\hat{p_s})}{\hat{p_s}} = \beta_{[TC_2]}$ *together with* $a < \alpha \hat{p_s}(1 + k\hat{p_s})$

*Proof.* Here also, we apply Sotomayor's theorem, by considering $\beta$ as bifurcation parameter.
One of the eigenvalue of $V(E_3)$ will be zero if the eigenvalue $\lambda = \frac{\beta \hat{p_s}}{b+\hat{p_s}} - d_2 = 0$, i.e., $\beta = \frac{d_2(b+\hat{p_s})}{\hat{p_s}} = \beta_{[TC_2]}$. Now when $\beta = \beta_{[TC_1]}$, the other two eigenvalues have negative real parts if this condition $a < \alpha \hat{p_s}(1 + k\hat{p_s})$ is satisfied. We have obtained that $W = (v_{12}^{[3]}, -v_{11}^{[3]}, \frac{v_{21}^{[3]} v_{12}^{[3]} - v_{11}^{[3]} v_{22}^{[3]}}{v_{23}^{[3]})^T$, $\tilde{W} = (0, 0, 1)^T$, where $W$, $\tilde{W}$ are the eigenvectors corresponding to the eigenvalue $\lambda = 0$ of the matrices $V(E_3)$ and $[V(E_3)]^T$ respectively and $v_{ij}^{[3]}$'s are given by Theorem 5.4($a$). Now,

$$\tilde{W}^T.[f_\beta(E_3, \beta_{[TC_2]})] = 0.$$

Again,

$$\tilde{W}^T.[Df_\beta(E_3, \beta_{[TC_2]})W] = \frac{\hat{p_s}}{b + \hat{p_s}} \left( \frac{v_{21}^{[3]} v_{12}^{[3]} - v_{11}^{[3]} v_{22}^{[3]}}{v_{23}^{[3]}} \right) \neq 0,$$

$$\text{and } \tilde{W}.[D^2 f(E_3, \beta_{[TC_2]}).(W, W)] = -\frac{b\beta}{(b + \hat{p_s})^2} \left( \frac{v_{11}^{[3]}(v_{21}^{[3]} v_{12}^{[3]} - v_{11}^{[3]} v_{22}^{[3]})}{v_{23}^{[3]}} \right) \neq 0,$$

where $[Df_\beta(E_3, \beta_{[TC_2]})] = (\alpha_{ij})_{3\times 3}$ of which $\alpha_{23} = -\frac{\hat{p_s}}{b+\hat{p_s}}$, $\alpha_{32} = \frac{\hat{p_s}}{b+\hat{p_s}}$ and all other $\alpha_{ij} = 0$ for $i, j = 1, 2, 3,$; and $[D^2 f(X, \beta)] = \begin{bmatrix} \nabla \frac{\partial f^{(1)}}{\partial n} & \nabla \frac{\partial f^{(2)}}{\partial n} & \nabla \frac{\partial f^{(3)}}{\partial n} \\ \nabla \frac{\partial f^{(1)}}{\partial p_s} & \nabla \frac{\partial f^{(2)}}{\partial p_s} & \nabla \frac{\partial f^{(3)}}{\partial p_s} \\ \nabla \frac{\partial f^{(1)}}{\partial p_i} & \nabla \frac{\partial f^{(2)}}{\partial p_i} & \nabla \frac{\partial f^{(3)}}{\partial p_i} \end{bmatrix}$
$\in \mathbb{R}^{3\times 3\times 3}$, $\nabla \frac{\partial f^i}{\partial n} = (\frac{\partial^2 f^i}{\partial n^2}, \frac{\partial^2 f^i}{\partial p_s \partial n}, \frac{\partial^2 f^i}{\partial p_i \partial n})$, $\nabla \frac{\partial f^i}{\partial p_s} = (\frac{\partial^2 f^i}{\partial n \partial p_s}, \frac{\partial^2 f^i}{\partial^2 p_s}, \frac{\partial^2 f^i}{\partial p_i \partial p_s})$,
$\nabla \frac{\partial f^i}{\partial p_i} = (\frac{\partial^2 f^i}{\partial n \partial p_i}, \frac{\partial^2 f^i}{\partial p_s \partial p_i}, \frac{\partial^2 f^i}{\partial^2 p_i})$, for $i = 1, 2, 3$. Thus the system (2.2) possesses a transcritical bifurcation around $E_3$ at $\beta = \beta_{[TC_2]}$, by Sotomayor's theorem. □

### 6.2. Hopf bifurcation at $E^*(n^*, p_s^*, p_i^*)$

We now establish the conditions that guarantee the occurrence of an Hopf bifurcation near the positive equilibrium point $E^*$ for the fear parameter $k$.

The characteristic equation of the system (2.2) at $E^*(n^*, p_s^*, p_i^*)$ is given by

$$\lambda^3 + Q_1(k)\lambda^2 + Q_2(k)\lambda + Q_3(k) = 0,$$

where

$$Q_1(k) = -\Big\{ - \frac{n^*}{1+kp_s^*} + \frac{\alpha n^* p_s^*}{(a+n^*)^2} - \frac{rhp_s^*}{n^*+m} + \frac{\beta p_s^* p_i^*}{(b+p_s^*)^2} \Big\},$$

$$Q_2(k) = \Big\{ - \tfrac{n^*}{1+kp_s^*} + \tfrac{\alpha n^* p_s^*}{(a+n^*)^2} \Big\}\Big\{ - \tfrac{rhp_s^*}{n^*+m} + \tfrac{\beta p_s^* p_i^*}{(b+p_s^*)^2} \Big\} - \Big\{ - \tfrac{kn^*(1-n^*)}{(1+kp_s^*)^2} - \tfrac{\alpha n^*}{a+n^*} \Big\} \Big\{ \frac{rhp_s^*(p_s^* + \eta p_i^*)}{(n^*+m)^2} \Big\} - \Big\{ - \frac{rh\eta p_s^*}{n^*+m} - \frac{\beta p_s^*}{b+p_s^*} \Big\}\Big\{ \frac{\beta b p_i^*}{(b+p_s^*)^2} \Big\},$$

$$Q_3(k) = \Big\{ - \frac{n^*}{1+kp_s^*} + \frac{\alpha n^* p_s^*}{(a+n^*)^2} \Big\}\Big\{ - \frac{rh\eta p_s^*}{n^*+m} - \frac{\beta p_s^*}{b+p_s^*} \Big\}\Big\{ \frac{\beta b p_i^*}{(b+p_s^*)^2} \Big\}.$$

**Theorem 6.3.** *When the fear constant $k$ of prey population $n$ crosses a critical value $k^*$, the model system (2.2) undergoes an Hopf bifurcation around the positive equilibrium $E^*$ if the following conditions are satisfied:*
*(i)* $Q_1(k^*) > 0,\ Q_3(k^*) > 0,$
*(ii)* $\triangle(k^*) = Q_1(k^*)Q_2(k^*) - Q_3(k^*) = 0,$
*(iii)* $\frac{d\triangle(k^*)}{dk}\Big|_{k=k^*} \neq 0.$

*Proof.* We choose $k$ as the bifurcation parameter and investigate if their exists a critical value $k^*$ such that $Q_1(k^*) > 0,\ Q_3(k^*) > 0,\ \triangle(k^*) = Q_1(k^*)Q_2(k^*) - Q_3(k^*) = 0,$ and $\frac{d\triangle(k^*)}{dk}\Big|_{k=k^*} \neq 0$. For the occurrence of Hopf bifurcation at $k = k^*$, the characteristic equation must be of the form

$$(\lambda^2(k^*) + Q_2(k^*))(\lambda(k^*) + Q_1(k^*)) = 0, \tag{6.1}$$

which has roots $\lambda_1(k^*) = i\sqrt{Q_2(k^*)},\ \lambda_2(k^*) = -i\sqrt{Q_2(k^*)},\ \lambda_3(k^*) = -Q_1(k^*) < 0$. Now, we need to validate the transversality condition:

$$\left[\frac{d(Re\lambda_j(k))}{dk}\right]_{k=k^*} \neq 0,\ j = 1, 2.$$

Substituting $\lambda_j(k) = p_1(k) \pm ip_2(k)$ into (6.1) and calculating derivative, we have

$$G(k)p_1'(k) - H(k)p_2'(k) + I(k) = 0, \tag{6.2}$$

$$G(k)p_2'(k) + H(k)p_1'(k) + J(k) = 0, \tag{6.3}$$

where

$$G(k) = (3p_1^2(k) - 3p_2^2(k) + 2p_1(k)Q_1(k) + Q_2(k)),$$
$$H(k) = (6p_1(k)p_2(k) + 2Q_1(k)p_2(k)),$$
$$I(k) = Q_2'(k)p_1(k) - Q_1'(k)p_2^2(k) + p_1^2(k)Q_1'(k) + Q_3'(k),$$
$$J(k) = 2p_1(k)p_2(k)Q_1'(k) + Q_2'(k)p_2(k).$$

Here, $p_1(k^*) = 0,\ p_2(k^*) = \sqrt{Q_2(k^*)}$; hence, we have
$G(k^*) = -2Q_2(k^*),\ H(k^*) = 2A_1(k^*)\sqrt{Q_2(k^*)},\ I(k^*) = Q_3'(k^*) - Q_1'(k^*)Q_2(k^*),$

$J(k^*) = Q_2'(k^*)\sqrt{Q_2(k^*)}$.
Solving for $p_1'(k)$ from (6.2) and (6.3), we have

$$\begin{aligned}\left[\frac{dRe(\lambda_j(k))}{dk}\right]_{k=k^*} &= p_1'(k^*) = -\tfrac{H(k^*)J(k^*)+G(k^*)I(k^*)}{G^2(k^*)+H^2(k^*)} \\ &= \frac{Q_3'(k^*) - Q_1(k^*)Q_2'(k^*) - Q_1'(k^*)Q_2(k^*)}{2(Q_1^2(k^*) + Q_2^2(k^*))} \neq 0.\end{aligned}$$

If $\frac{d}{dk}[Q_1(k)Q_2(k) - Q_3(k)]_{k=k^*} \neq 0$ and $\lambda_3(k^*) = -Q_1(k^*) < 0$. Hence the transversality condition $\left.\frac{d\triangle(k)}{dk}\right|_{k=k^*} \neq 0$ holds. This implies that an Hopf bifurcation occurs at $k = k^*$. □

**Remark 6.1.** Similar bifurcation study can be done with another important parameter $\beta$, i.e., transmission coefficient from susceptible predator to infected predator.

### 6.2.1. Direction of Hopf bifurcation

In this section, we explain the direction and stability properties of the bifurcating periodic solutions commencing from the interior equilibrium point $E^*(n^*, p_s^*, p_i^*)$ via Hopf-bifurcation. To explore the stability and direction of Hopf bifurcation, we determine the first Lyapunov coefficient [42].
So, we transfer the origin at the equilibrium point $E^*(n^*, p_s^*, p_i^*)$ by considering $x_1 = n - n^*,\ x_2 = p_s - p_s^*,\ x_3 = p_i - p_i^*$. Then the system (2.2) becomes

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{(x_1+n^*)(1-x_1-n^*)}{1+k(x_2+p_s^*)} - \frac{\alpha(x_1+n^*)(x_2+p_s^*)}{a+x_1+n^*} \\ \frac{dx_2}{dt} &= r(x_2+p_s^*)\{1 - \frac{h((x_2+p_s^*)+\eta(x_3+p_i^*))}{x_1+n^*+m}\} - \frac{\beta(x_2+p_s^*)(x_3+p_i^*)}{b+x_2+p_s^*} \\ &\quad -d_1(x_2+p_s^*) \\ \frac{dx_3}{dt} &= \frac{\beta(x_2+p_s^*)(x_3+p_i^*)}{b+x_2+p_s^*} - d_2(x_3+p_i^*)\end{aligned}$$

Now expanding the above system by Taylor's series at $(x, y, z) = (0, 0, 0)$ up to 3rdorder terms, we get:

$$\begin{aligned}
\dot{x_1} &= e_{100}x_1 + e_{010}x_2 + e_{200}x_1^2 + e_{110}x_1x_2 + e_{020}x_2^2 + e_{300}x_1^3 \\
&+ e_{210}x_1^2x_2 + e_{120}x_1x_2^2 + e_{030}x_2^3 + O(|x|^4), \\
\dot{x_2} &= f_{100}x_1 + f_{010}x_2 + f_{001}x_3 + f_{200}x_1^2 + f_{020}x_2^2 + f_{110}x_1x_2 + f_{101}x_1x_3 \\
&+ f_{011}x_2x_3 + f_{300}x_1^3 + f_{210}x_1^2x_2 + f_{201}x_1^2x_3 + f_{021}x_2^2x_3 + f_{030}x_2^3 + O(|x|^4), \\
\dot{x_3} &= g_{010}x_2 + g_{020}x_2^2 + g_{011}x_2x_3 + g_{003}x_2^3 + g_{021}x_2^2x_3 + O(|x|^4),
\end{aligned} \tag{6.4}$$

where

$$\begin{bmatrix} e_{100} & e_{010} & 0 \\ f_{100} & f_{010} & f_{001} \\ 0 & g_{010} & 0 \end{bmatrix} = V(E^*), \text{ where } V(E^*) \text{ is given in Theorem 5.5(a) and}$$

$$\begin{aligned}
&e_{200} = -\frac{1}{1+kp_s^*} + \frac{\alpha a p_s^*}{(a+n^*)^3},\ e_{110} = -\frac{k(1-2n^*)}{(1+kp_s^*)^2} - \frac{\alpha a}{(a+n^*)^2}, \\
&e_{020} = \frac{kn^*(1-n^*)}{(1+kp_s^*)^3},\ e_{300} = \frac{\alpha n^* p_s^*}{(a+n^*)^4},\ e_{210} = \frac{k}{(1+kp_s^*)^2} + \frac{\alpha a}{(a+n^*)^3}, \\
&e_{120} = \frac{k^2(1-2n^*)}{(1+kp_s^*)^3},\ e_{030} = -\frac{k^3n^*(1-n^*)}{(1+kp_s^*)^4},\ f_{200} = -\frac{rh(p_s^*+\eta p_i^*)}{(n^*+m)^3}, \\
&f_{110} = \frac{rh}{(n^*+m)^2},\ f_{020} = \frac{\beta b p_i^*}{(b+p_s^*)^3},\ f_{101} = \frac{rh\eta}{(n^*+m)^2},\ f_{011} = -\frac{\beta b}{(b+p_s^*)^2}, \\
&f_{300} = \frac{rh(p_s^*+\eta p_i^*)}{(n^*+m)^4},\ f_{210} = -\frac{rh}{(n^*+m)^3},\ f_{201} = -\frac{rh\eta}{(n^*+m)^3},\ f_{021} = \frac{\beta b}{(b+p_s^*)^3}, \\
&f_{300} = \frac{\beta p_s^* p_i^*}{(b+p_s^*)^4},\ g_{020} = -\frac{\beta b p_i^*}{(b+p_s^*)^3},\ g_{011} = \frac{\beta b}{(b+p_s^*)^2},\ g_{300} = \frac{\beta b p_i^*}{(b+p_s^*)^4}, \\
&g_{021} = -\frac{\beta b}{(b+p_s^*)^3}.
\end{aligned}$$

By neglecting the higher order terms of degree 4 and above, the system (6.4) can be written as

$$\dot{X} = V(E^*)X + J(X), \tag{6.5}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and}$$

$$J = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix}$$

$$= \begin{pmatrix} e_{200}x_1^2 + e_{110}x_1x_2 + e_{020}x_2^2 + e_{300}x_1^3 + e_{210}x_1^2x_2 + e_{120}x_1x_2^2 + e_{030}x_2^3 \\ \\ f_{200}x_1^2 + f_{020}x_2^2 + f_{110}x_1x_2 + f_{101}x_1x_3 + f_{011}x_2x_3 + f_{300}x_1^3 + f_{210}x_1^2x_2 \\ +f_{201}x_1^2x_3 + f_{021}x_2^2x_3 + f_{030}x_2^3 \\ \\ g_{020}x_2^2 + g_{011}x_2x_3 + g_{003}x_2^3 + g_{021}x_2^2x_3 \end{pmatrix}. \tag{6.6}$$

At the Hopf bifurcation, characteristic equation holds and eigenvalues of $V(E^*)$ are $\lambda_1 = -Q_1,\ \lambda_2 = +i\sqrt{Q_2},\ \lambda_3 = -i\sqrt{Q_2}$. If the eigenvectors corresponding to the above eigenvalues are respectively $P_1,\ P_2 \pm iP_3$ (where $P_1,\ P_2,\ P_3$ are real) then a non-singular matrix $P = (P_3,\ P_2,\ P_1)$ can be formed which satisfying

$$P^{-1}V(E^*)P = \begin{pmatrix} 0 & -\sqrt{Q_2} & 0 \\ \sqrt{Q_2} & 0 & 0 \\ 0 & 0 & Q_1 \end{pmatrix}, \text{ where}$$

$$P = \begin{pmatrix} e_{010}\sqrt{Q_2} & 0 & e_{010}Q_1 \\ -e_{100}\sqrt{Q_2} & -Q_2 & -(e_{100}+Q_1)Q_1 \\ g_{010}\sqrt{Q_2} & -e_{100}g_{010} & (e_{100}+Q_1)g_{010} \end{pmatrix} = (p_{ij})_{3\times 3}.$$

and $R = P^{-1} = (r_{ij})_{3\times 3}$, then the matrix $R$ is as follows:

$$R = \frac{1}{\det P^{-1}} \begin{pmatrix} p_{22}p_{33} - p_{23}p_{32} & p_{13}p_{32} & -p_{13}p_{22} \\ p_{23}p_{31} - p_{21}p_{33} & p_{11}p_{33} - p_{13}p_{31} & p_{11}p_{23} - p_{13}p_{21} \\ p_{21}p_{32} - p_{22}p_{31} & -p_{11}p_{32} & p_{11}p_{22} \end{pmatrix},$$

where $\det P^{-1} = p_{11}(p_{22}p_{33} - p_{23}p_{32}) + p_{13}(p_{21}p_{32} - p_{22}p_{31})$.

Let us apply linear transformation $X = PS \ \Rightarrow\ S = P^{-1}X$, where $S = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$,

then equation (6.5) reduces to

$$\frac{dS}{dt} = (P^{-1}V(E^*)P)S + P^{-1}J. \tag{6.7}$$

Now the system of equations (6.7) can be written as

$$\frac{d}{dt}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} R^1(s_1, s_2, s_3; k = k^*) \\ R^2(s_1, s_2, s_3; k = k^*) \end{pmatrix}, \tag{6.8}$$

$$\frac{ds_3}{dt} = -Q_1 s_3 + R^3(s_1, s_2, s_3; k = k^*), \tag{6.9}$$

where $C = \begin{pmatrix} 0 & -\sqrt{Q_2} \\ \sqrt{Q_2} & 0 \end{pmatrix}$, and $R^1,\ R^2,$and $R^3$ are functions of $s_1,\ s_2,\ s_3$. The center manifold up to a quadratic approximation can be described by ([35],

[4], [19])

$$s_3 = \psi(s_1, s_2) = \frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2), \tag{6.10}$$

which leads to (by using (6.8))

$$\begin{aligned} \frac{ds_3}{dt} &= (c_{11}s_1 + c_{12}s_2 \quad c_{12}s_1 + c_{22}s_2)\begin{pmatrix} 0 & -\sqrt{Q_2} \\ \sqrt{Q_2} & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\ &= \sqrt{Q_2}c_{12}s_1^2 - \sqrt{Q_2}(c_{11} - c_{22})s_1s_2 - \sqrt{Q_2}c_{12}s_2^2. \end{aligned} \tag{6.11}$$

Again, from (6.7) and (6.9), we have

$$\frac{ds_3}{dt} = -Q_1s_3 + r_{31}J_1 + r_{32}J_2 + r_{33}J_3. \tag{6.12}$$

From the above two equations, we have

$$\sqrt{Q_2}c_{12}s_1^2 + \sqrt{Q_2}(c_{22} - c_{11})s_1s_2 - \sqrt{Q_2}c_{12}s_2^2$$

$$= -\frac{1}{2}Q_1(c_{11}s_1^2+2c_{12}s_1s_2+c_{22}s_2^2)+r_{31}[e_{200}\{p_{11}s_1+p_{12}s_2+p_{13}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+ c_{22}s_2^2)\}^2 + e_{110}\{p_{11}s_1 + p_{12}s_2 + p_{13}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}\{p_{21}s_1 + p_{22}s_2 + p_{23}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\} + e_{020}\{p_{21}s_1 + p_{22}s_2 + p_{23}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}^2+r_{32}[f_{200}\{p_{11}s_1+p_{12}s_2+p_{13}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+c_{22}s_2^2)\}^2+f_{020}\{p_{21}s_1+ p_{22}s_2 + p_{23}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}^2 + f_{110}\{p_{11}s_1 + p_{12}s_2 + p_{13}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2+c_{22}s_2^2)\}\{p_{21}s_1+p_{22}s_2+p_{23}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+c_{22}s_2^2)\}+f_{101}\{p_{11}s_1+ p_{12}s_2+p_{13}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+c_{22}s_2^2)\}\{p_{31}s_1+p_{32}s_2+p_{33}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+ c_{22}s_2^2)\} + f_{011}\{p_{21}s_1 + p_{22}s_2 + p_{23}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}\{p_{31}s_1 + p_{32}s_2 + p_{33}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+c_{22}s_2^2)\}]+r_{33}[g_{020}\{p_{21}s_1+p_{22}s_2+p_{23}\frac{1}{2}(c_{11}s_1^2+2c_{12}s_1s_2+ c_{22}s_2^2)\}^2 + g_{011}\{p_{21}s_1 + p_{22}s_2 + p_{23}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}\{p_{31}s_1 + p_{32}s_2 + p_{33}\frac{1}{2}(c_{11}s_1^2 + 2c_{12}s_1s_2 + c_{22}s_2^2)\}]$$

Computing the coefficients of $s_1^2,\ s_1s_2,$ and $s_2^2$ from both sides, we have

$$\begin{aligned} &\tfrac{Q_1}{2}c_{11} + \sqrt{Q_2}c_{12} \\ &\quad= r_{31}[e_{200}p_{11}^2 + e_{110}p_{11}p_{21} + e_{020}p_{21}^2] + r_{32}[f_{200}p_{11}^2 + f_{020}p_{21}^2 \\ &\qquad +f_{110}p_{11}p_{21} + f_{101}p_{11}p_{31} + f_{011}p_{21}p_{31}] + r_{33}[g_{020}p_{21}^2 + g_{011}p_{21}p_{31}] \\ &\quad= \Omega_1, \end{aligned} \tag{6.13}$$

$$
\begin{aligned}
& Q_1c_{12}+\sqrt{Q_2}(c_{22}-c_{11}) \\
&\quad = r_{31}[2e_{200}p_{11}p_{12}+e_{110}(p_{11}p_{22}+p_{12}p_{21})+2e_{020}p_{21}p_{22}] \\
&\quad\quad +r_{32}[2f_{200}p_{11}p_{12}+2f_{020}p_{21}p_{22}+f_{110}(p_{11}p_{22}+p_{12}p_{21}) \\
&\quad\quad +f_{101}(p_{11}p_{32}+p_{12}p_{31})+f_{011}(p_{21}p_{32}+p_{22}p_{31})] \qquad (6.14)\\
&\quad = +r_{33}[2g_{020}p_{21}p_{22}+g_{011}(p_{21}p_{32}+p_{22}p_{31})] \\
&\quad = \Omega_2,
\end{aligned}
$$

$$
\begin{aligned}
& \tfrac{Q_1}{2}c_{22}-\sqrt{Q_2}c_{12} \\
&\quad = r_{31}[e_{200}p_{12}^2+e_{110}p_{12}p_{22}+e_{020}p_{22}^2]+r_{32}[f_{200}p_{12}^2+f_{020}p_{22}^2+f_{110}p_{12}p_{22} \\
&\quad\quad +f_{101}p_{12}p_{32}+f_{011}p_{22}p_{32}]+r_{33}[g_{020}p_{22}^2+g_{011}p_{22}p_{32}] \\
&\quad = \Omega_3,
\end{aligned} \qquad (6.15)
$$

The above three equations can be expressed as:

$$
\begin{pmatrix} \frac{1}{2}Q_1 & \sqrt{Q_1} & 0 \\ -\sqrt{Q_2} & Q_1 & \sqrt{Q_2} \\ 0 & -\sqrt{Q_2} & \frac{1}{2}Q_1 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{22} \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}. \qquad (6.16)
$$

Solving the above equation we have the coefficients $c_{11}$, $c_{12}$, and $c_{22}$ as $c_{11}=\frac{4\{\Omega_1(\frac{Q_1^2}{2}+Q_2)-\Omega_2\frac{Q_1}{2}\sqrt{Q_2}+\Omega_3 Q_2\}}{Q_1^3}$, $c_{12}=\frac{4(\Omega_1\frac{Q_1}{2}\sqrt{Q_2}+\Omega_2\frac{Q_1^2}{4}-\Omega_3\frac{Q_1}{2}\sqrt{Q_2})}{Q_1^3}$, $c_{22}=\frac{4\{\Omega_1 Q_2+\Omega_2(\frac{Q_1}{2}\sqrt{Q_2}-Q_2)+\Omega_3\frac{Q_1^2}{3}\}}{Q_1^3}$. Therefore, the flow on the central manifold is characterized by the simplified system

$$
\frac{d}{dt}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{Q_2} \\ \sqrt{Q_2} & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} R^1 \\ R^2 \end{pmatrix} \qquad (6.17)
$$

where $R^1=r_{11}J_1+r_{12}J_2+r_{13}J_3+\text{h.o.t.}$, $R^2=r_{21}J_1+r_{22}J_2+r_{23}J_3+\text{h.o.t.}$. Here h.o.t. stands for higher order terms.

The stability of the bifurcating limit cycle can be determined by the sign of the parametric expression

$$
\begin{aligned}
R = \; & R_{111}^1+R_{112}^1+R_{222}^2 \\
& +\tfrac{1}{\sqrt{Q_2}}\{R_{12}^1(R_{11}^1-R_{22}^1)-R_{12}^2(R_{11}^2+R_{22}^2)-R_{11}^1R_{11}^2+R_{22}^1R_{22}^2\},
\end{aligned} \qquad (6.18)
$$

where $R_{ijk}^l$ denotes the partial derivative $\frac{\partial^3 R^l}{\partial s_1\partial s_2\partial s_3}$, $l=1,2$ at the origin. The partial derivatives are given as follows:

$$
\begin{aligned}
R_{11}^1 = \; & 2r_{11}[e_{200}p_{11}^2+e_{020}p_{21}^2+e_{110}p_{11}p_{21}]+2r_{12}[f_{200}p_{11}^2+f_{020}p_{21}^2 \\
& +f_{110}p_{11}p_{21}+f_{101}p_{11}p_{31}+f_{011}p_{21}p_{31}]+2r_{13}[g_{020}p_{21}^2+g_{011}p_{21}p_{31}],
\end{aligned}
$$

$$
\begin{aligned}
R^1_{12} = {} & r_{11}[2e_{200}p_{11}p_{12} + 2e_{020}p_{21}p_{22} + e_{110}(p_{11}p_{22} + p_{12}p_{21})] + r_{12}[2f_{200}p_{11}p_{12} \\
& + 2f_{020}p_{21}p_{22} + f_{110}(p_{11}p_{22} + p_{12}p_{21}) + f_{101}(p_{12}p_{31} + p_{11}p_{32}) \\
& + f_{011}(p_{22}p_{31} + p_{21}p_{32})] + r_{13}[2g_{020}p_{21}p_{22} + g_{011}(p_{22}p_{31} + p_{21}p_{32})], \\
R^1_{22} = {} & 2r_{11}[e_{200}p^2_{12} + e_{020}p^2_{22} + e_{110}p_{12}p_{22}] + 2r_{12}[f_{200}p^2_{12} + f_{020}p^2_{22} \\
& + f_{110}p_{12}p_{22} + f_{101}p_{12}p_{32} + f_{011}p_{22}p_{32}] + 2r_{13}[g_{020}p^2_{22} + g_{011}p_{22}p_{32}], \\
R^1_{111} = {} & 3r_{11}c_{11}[2e_{200}p_{11}p_{13} + 2e_{020}p_{21}p_{23} + e_{110}(p_{13}p_{21} + p_{11}p_{23})] + 3r_{12}c_{11} \\
& [2f_{200}p_{11}p_{13} + 2f_{020}p_{21}p_{23} + f_{110}(p_{13}p_{21} + p_{11}p_{23}) + f_{101}(p_{13}p_{31} + p_{11}p_{33}) \\
& + f_{011}(p_{23}p_{31} + p_{21}p_{33})] + 3r_{13}c_{11}[2g_{020}p_{21}p_{23} + g_{011}(p_{23}p_{31} + p_{21}p_{33})], \\
R^1_{122} = {} & r_{11}[2e_{200}p_{13}(p_{11}c_{22} + 2p_{12}c_{12}) + 2e_{020}p_{23}(p_{21}c_{22} + 2p_{22}c_{12}) + e_{110} \\
& \{p_{13}(p_{21}c_{22} + 2p_{22}c_{12}) + p_{23}(p_{11}c_{22} + 2p_{12}c_{12})\}] + r_{12}[2f_{200}p_{13} \\
& (p_{11}c_{22} + 2p_{12}c_{12}) + 2f_{020}p_{23}(p_{21}c_{22} + 2p_{22}c_{12}) + f_{110}\{p_{13}(p_{21}c_{22} \\
& + 2p_{22}c_{12}) + f_{101}\{p_{13}(p_{31}c_{22} + 2p_{32}c_{12}) + p_{33}(p_{11}c_{22} + 2p_{12}c_{12})\} + f_{011} \\
& \{p_{23}(p_{31}c_{22} + 2p_{32}c_{12}) + p_{33}(p_{21}c_{22} + 2p_{22}c_{12})\}] + r_{13}[2g_{020}p_{23} \\
& (p_{21}c_{22} + 2p_{22}c_{12}) + g_{011}\{p_{23}(p_{31}c_{22} + 2p_{32}c_{12}) + p_{33}(p_{21}c_{22} + 2p_{22}c_{12})\}], \\
R^2_{11} = {} & 2r_{21}[e_{200}p^2_{11} + e_{020}p^2_{21} + e_{110}p_{11}p_{21}] + 2r_{22}[f_{200}p^2_{11} + f_{020}p^2_{21} \\
& + f_{110}p_{11}p_{21} + f_{101}p_{11}p_{31} + f_{011}p_{21}p_{31}] + 2r_{23}[g_{020}p^2_{21} + g_{011}p_{21}p_{31}], \\
R^2_{12} = {} & r_{21}[2e_{200}p_{11}p_{12} + 2e_{020}p_{21}p_{22} + e_{110}(p_{11}p_{22} + p_{12}p_{21})] + r_{22}[2f_{200}p_{11}p_{12} \\
& + 2f_{020}p_{21}p_{22} + f_{110}(p_{11}p_{22} + p_{12}p_{21}) + f_{101}(p_{12}p_{31} + p_{11}p_{32}) \\
& + f_{011}(p_{22}p_{31} + p_{21}p_{32})] + r_{23}[2g_{020}p_{21}p_{22} + g_{011}(p_{22}p_{31} + p_{21}p_{32})], \\
R^2_{22} = {} & 2r_{21}[e_{200}p^2_{12} + e_{020}p^2_{22} + e_{110}p_{12}p_{22}] + 2r_{22}[f_{200}p^2_{12} + f_{020}p^2_{22} + f_{110}p_{12}p_{22} \\
& + f_{101}p_{12}p_{32} + f_{011}p_{22}p_{32}] + 2r_{23}[g_{020}p^2_{22} + g_{011}p_{22}p_{32}], \\
R^2_{222} = {} & 3r_{21}c_{22}[2e_{200}p_{12}p_{13} + 2e_{020}p_{22}p_{23} + e_{110}(p_{13}p_{22} + p_{12}p_{23})] + 3r_{22}c_{22} \\
& [2f_{200}p_{12}p_{23} + 2f_{020}p_{22}p_{23} + f_{110}(p_{13}p_{22} + p_{12}p_{23}) + f_{101}(p_{13}p_{32} + p_{12}p_{33}) \\
& + f_{011}(p_{23}p_{32} + p_{22}p_{33})] + 3r_{23}c_{22}[2g_{020}p_{22}p_{23} + g_{011}(p_{23}p_{32} + p_{22}p_{33})],
\end{aligned}
$$

$$\begin{aligned} R_{112}^2 = \quad & r_{21}[2e_{200}p_{13}(p_{12}c_{11}+2p_{11}c_{12})+2e_{020}p_{23}(p_{22}c_{11}2+p_{21}c_{12})+e_{110}\{p_{13} \\ & (p_{22}c_{11}+2p_{21}c_{12})+p_{23}(p_{12}c_{11}+2p_{11}c_{12})\}]+r_{22}[2f_{200}p_{13}(p_{12}c_{11} \\ & +2p_{11}c_{12})+2f_{020}p_{23}(p_{22}c_{11}2+p_{21}c_{12})+f_{110}\{p_{13}(p_{22}c_{11}+2p_{21}c_{12}) \\ & +p_{23}(p_{12}c_{11}+2p_{11}c_{12})\}+f_{101}\{p_{13}(p_{32}c_{11}+2p_{31}c_{12})+p_{33}(p_{12}c_{11} \\ & +2p_{11}c_{12})\}+f_{011}\{p_{23}(p_{32}c_{11}+2p_{31}c_{12})+p_{33}(p_{22}c_{11}+2p_{21}c_{12})\}] \\ & +r_{13}[2g_{020}p_{23}(p_{22}c_{11}2+p_{21}c_{12})+g_{011}\{p_{23}(p_{32}c_{11}+2p_{31}c_{12}) \\ & +p_{33}(p_{22}c_{11}+2p_{21}c_{12})\}], \end{aligned}$$

Now we have the following theorem:

**Theorem 6.4.** *If $R < 0$, the bifurcating limit cycle is stable and the Hopf bifurcation is called supercritical. If $R > 0$, the bifurcating limit cycle is unstable and the Hopf bifurcation is called subcritical.*

## 7. Influence of Fear Effect

In this section, we shall discuss the effect of fear parameter on each of the population where the interior equilibrium point exists and is locally asymptotically stable.

Let us consider the following system without fear effect:

$$\begin{aligned} \frac{dn}{dt} &= n(1-n) - \frac{\alpha n p_s}{a+n} \\ \frac{dp_s}{dt} &= r p_s \left\{1 - \frac{h(p_s+\eta p_i)}{n+m}\right\} - \frac{\beta p_s p_i}{b+p_s} - d_1 p_s \\ \frac{dp_i}{dt} &= \frac{\beta p_s p_i}{b+p_s} - d_2 p_i \end{aligned} \tag{7.1}$$

Let $\underline{E^*} = (\underline{n^*}, \underline{p_s^*}, \underline{p_i^*})$ be the equilibrium point of system (7.1), where $\underline{n^*} = \frac{-\underline{A_4}+\sqrt{\underline{A_4}^2-4\underline{A_5}}}{2}$, of which $\underline{A_4} = a-1$, $\underline{A_5} = \frac{\alpha\beta d_2}{\beta-d_2} - a$ and $\underline{p_s^*} = \frac{bd_2}{\beta-d_2}$, $\underline{p_i^*} = \left(r - \frac{rh\underline{p_s^*}}{\underline{n^*}+m} - d_1\right) / \left(\frac{rh\eta}{\underline{n^*}+m} + \frac{\beta}{b+\underline{p_s^*}}\right)$.

### 7.1. Influence of fear effect on prey population

As $A_5 = \frac{\alpha b d_2}{\beta-d_2} + \frac{kb^2d_2^2}{(\beta-d_2)^2} - a > \frac{\alpha b d_2}{\beta-d_2} - a = \underline{A_5}$. Hence, $\frac{-\underline{A_4}+\sqrt{\underline{A_4}^2-4\underline{A_5}}}{2} > \frac{-A_4+\sqrt{A_4^2-4A_5}}{2}$, i.e., $\underline{n^*} > n^*$.

So, for any fixed $k$, the fear effect can decrease the prey population. Since $n^*$ is a continuous function of $k$, we have

$$\frac{dn^*}{dk} = -\frac{\alpha b^2 d_2^2}{(\beta - d_2)^2 \sqrt{A_4^2 - 4A_5}} < 0. \tag{7.2}$$

Thus $n^*$ is a decreasing function of $k$, i.e., if we increase the value of fear parameter, it can decrease the prey population when $\underline{E^*}$ is stable.

### 7.2. Influence of fear effect on susceptible predator population

Since $p_s^* = \underline{p_s^*} = \frac{bd_2}{\beta - d_2}$, both are independent of fear parameter $k$, then it has no effect on susceptible predator population.

### 7.3. Influence of fear effect on infected predator population

Again, we have $\underline{p_i^*} = \left(r - \frac{rh\underline{p_s^*}}{\underline{n^*}+m} - d_1\right) / \left(\frac{rh\eta}{\underline{n^*}+m} + \frac{\beta}{b+\underline{p_s^*}}\right)$ is always less than $p_i^*$, i.e. $\underline{p_i^*} < p_i^*$ for a given set of parameters. Also,

$$\frac{dp_i^*}{dk} = \frac{\left(\frac{rh\eta}{n^*+m} + \frac{\beta}{b+p_s^*}\right)\frac{rhp_s^*}{(n^*+m)^2} + \left(r - \frac{rhp_s^*}{n^*+m} - d_1\right)\frac{rh\eta}{(n^*+m)^2}}{\left(\frac{rh\eta}{n^*+m} + \frac{\beta}{b+p_s^*}\right)^2} \frac{dn^*}{dk} < 0. \tag{7.3}$$

Thus $p_i^*$ is also a decreasing function of $k$, i.e., fear parameter can decrease the infected predator population, when $\underline{p_i^*}$ exists and locally asymptotically stable.

## 8. Numerical Simulations

In this section, extensive numerical simulations have been performed for various hypothetical set of parameters to determine the dynamics of the system (2.2). The time series diagram, phase plane diagram, and bifurcation diagram of system (2.2) are demonstrated to validate the analytical findings. This study not only provides local stability and Hopf bifurcation, but also exhibit the feasibility of several complex dynamical behaviors, including limit-cycle and chaos.

Let us consider the set of parameters as

$$k = 1.2,\ \alpha = 0.3,\ a = 0.05,\ \eta = 0.03,\ \beta = 0.4,\ b = 0.8\ d_1 = 0.15,\ d_2 = 0.2. \quad (I)$$

to verify the stability diagram of $E_1$, $E_2$, and $E_3$. Taking $r = 0.1$, $h = 0.6$, and $m = 0.2$, we get the stability diagram of $E_1(1, 0, 0)$ (Fig. 1(a)). Now, keeping $m$ fixed, considering $r = 0.6$ and $h = 0.3$, then it will satisfy the stability condition of $E_2(0, \tilde{p_s}, 0)$ and the diagram is given in Fig. 1(b). Again, keeping the same value of $m$ and $h$ as in the case of $E_2$ and taking $r = 0.2$, the existence and stability conditions of $E_3$ are satisfied and Fig. 1(c), 1(d) and 1(e) show the stable behavior of $E_3(\hat{n}, \hat{p_s}, 0) = (0.4626, 0.5522, 0)$.

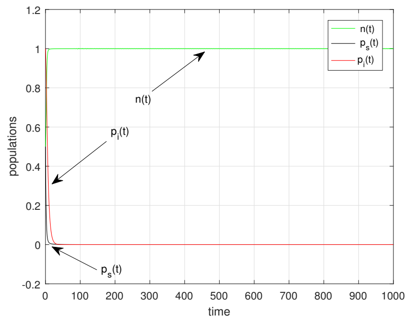



**Fig. 1(a)**

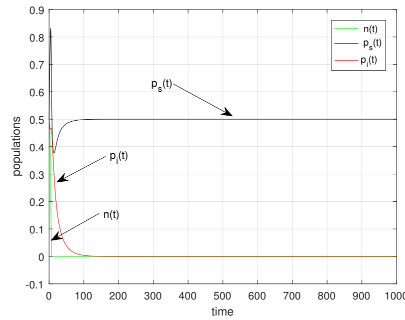



**Fig. 1(b)**

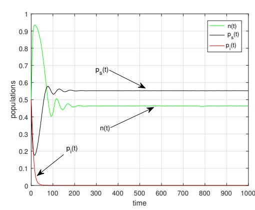


**Fig. 1(c)**

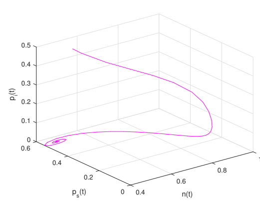


**Fig. 1(d)**

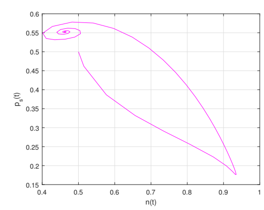


**Fig. 1(e)**

**Fig. 1.** *Stable behavior of the system at (a) $E_1$, (b) $E_2$, (c) $E_3$. Fig. (d) and (e) shows the phase portrait of $E_3$ in 3D and 2D.*

Now we consider the set of parameter values as

$$k = 1.2,\ \alpha = 0.3,\ a = 0.05,\ r = 0.4,\ h = 0.6,\ \eta = 0.03,\ m = 0.3,\ \beta = 0.4,$$
$$b = 0.8,\ d_1 = 0.1,\ d_2 = 0.15. \qquad (II)$$

Then the existence conditions of $E^*$ and the conditions of Theorem 5.5(a) is satisfied and the interior equilibrium point $E^*(0.8228, 0.36, 0.5272)$ is locally asymptotically stable. The stability behavior and phase portrait of that equilibrium point are shown in Figs. 2(a) and 2(b) respectively.

Now let us show the bifurcation diagrams one by one. At $r = d_1$, $E_3$ collides with $E_1$ whereas for $r < d_1$, $E_1$ is stable and $E_3$ does not exist but for $r > d_1$, $E_1$ is unstable and $E_3$ exists. So, taking $r$ as a bifurcation parameter, we obtain a transcritical bifurcation around $E_1$ at $r = d_1 = r_{[TC_1]} = 0.15$. Fig. 3(a) depicts the transcritical bifurcation diagram around $E_1$.

Also, from the stability condition of $E_3$, if we take $\beta = \frac{d_2(b+\hat{p_s})}{\hat{p_s}}$ keeping the other inequality as it is, then by changing the value of $\beta$, a transcritical bifurcation occurs around $E_3$ at $\beta = 0.4897 = \beta_{[TC_2]}$. Fig. 3(b) depicts the transcritical bifurcation diagram around $E_3$.

Here the fear constant $k$ plays an important role in the dynamics of the underlying system. The system undergoes a Hopf bifurcation taking $k$ as a

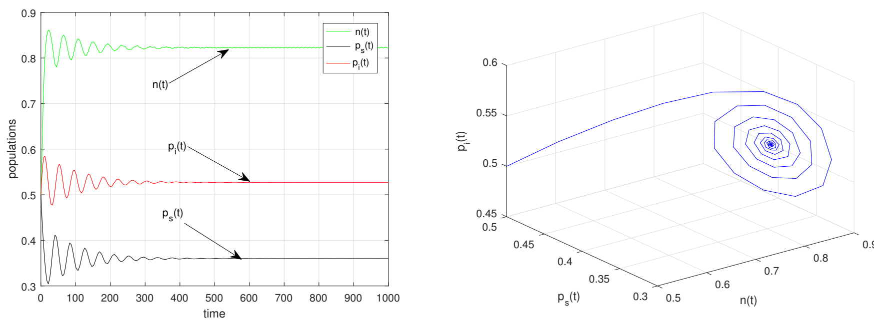



**Fig. 2(a)** **Fig. 2(b)**

**Fig. 2.** *Stable behavior of the system at* $E^*(0.8228, 0.36, 0.5272)$ *with initial value* $(n(0), p_s(0), p_i(0)) = (0.5, 0.5, 0.5)$. *(a) shows the stable behavior with respect to time* $t$ *and (b) shows the stable phase portrait.*

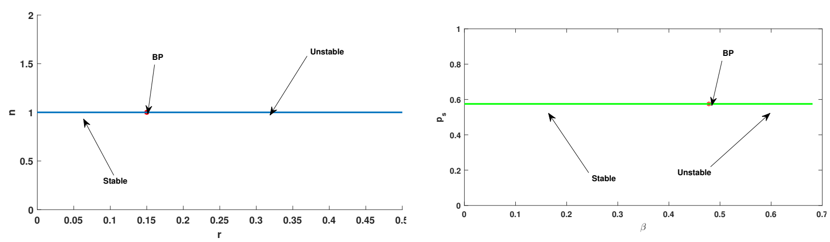



**Fig. 3(a)** **Fig. 3(b)**

**Fig. 3.** $(a)$ *shows transcritical bifurcation occurs around* $E_1$ *and* $(b)$ *shows transcritical bifurcation occurs around* $E_3$.

bifurcation parameter. Using the values of the parameters mentioned in the list $(II)$, it is observed that $E^*$ is locally asymptotically stable when $k > k^*$ and unstable when $k < k^*$, i.e. the system undergoes a Hopf bifurcation around $E^*$ at $k = k^* = 1.12$. Since $\frac{d}{dk}[\mathrm{Re}(\lambda(k))]_{k=k^*} = -0.01040059 < 0$, i.e., the real part of the eigenvalue is monotonically decreasing. So as the parameter $k$ crosses its critical value $k^*$ from lower to higher level, the real part of the eigenvalue becomes negative. Now considering the analysis in Section 6.2.1, we observed the value of $R = -0.0079562 < 0$ which implies that the obtained Hopf bifurcation is transcritical bifurcation. Fig. 4(a) and (b) depicts the unstable behavior of $n^*, p_s^*, p_i^*$ in time and unstable phase portrait of the system (2.2). Now, let us find out the system dynamics subject to the change in the value of $k$ keeping the other parameters as before to satisfy Theorem 6.2. The bifurcation diagram is presented in Fig. 5(a), (b) and (c) for variations in value of $k$ over [0.5,1.15] which shows that the system undergoes stable dynamics for $k > 1.12$ and unstable behavior for $k < 1.12$.

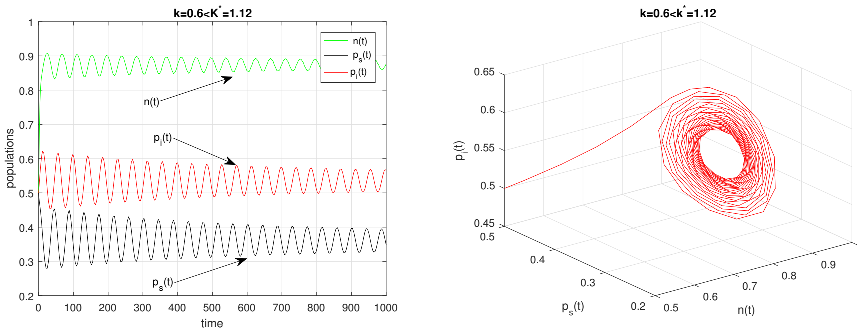



**Fig. 4(a)** **Fig. 4(b)**

**Fig. 4.** *Occurrence of oscillatory behavior and limit cycle around* $E^*$ *when* $k = 0.6 < k^* = 1.12$ .

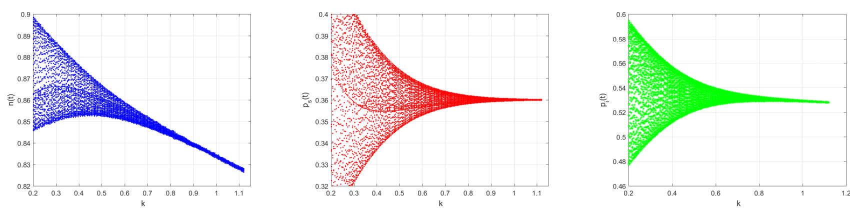



**Fig. 5(a)** **Fig. 5(b)** **Fig. 5(c)**

**Fig. 5.** *bifurcation diagram for the parameter* $k \in [0.2, 1.15]$ *and other parameters as given in list II.*

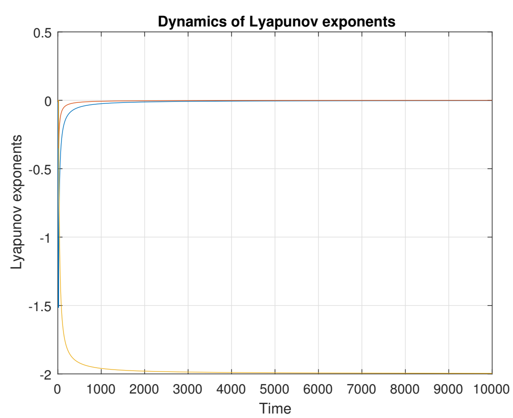



**Fig. 6.** *Lyapunov exponent for increasing time with* $k = 0.6$*, other parameters are same as in list II.*

To show whether the system exhibits chaotic behavior for lower values of $k$, we have plotted the Lyapunov exponents of the system for $k = 0.6$ (Fig. 6). Lyapunov exponent is an average exponential rate of convergence or divergence of two nearby trajectories ([12], [43]). In a three-dimensional system, the spectra of Lyapunov exponents $\lambda_i, i = 1, 2, 3$, must be $(i)$ $(+, 0, -)$ for chaotic solution; $(ii)$ $(0, 0, -)$ for quasi-periodic solution or two tours; $(iii)$ $(0, -, -)$ for periodic solution or limit limit cycle and $(iv)$ $(-, -, -)$ for a fixed point solution. Here we get a negative and two zero Lyapunov exponent for $k = 0.6$ (Fig. 5), indicating quasi-periodic nature of the system.

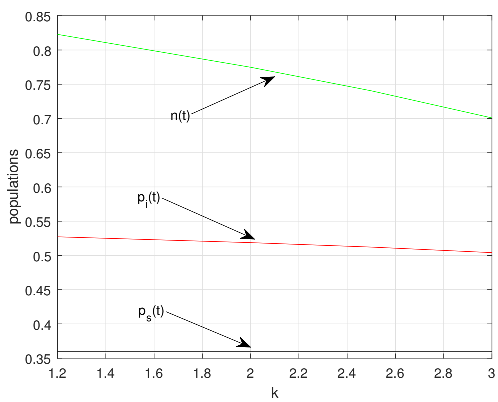
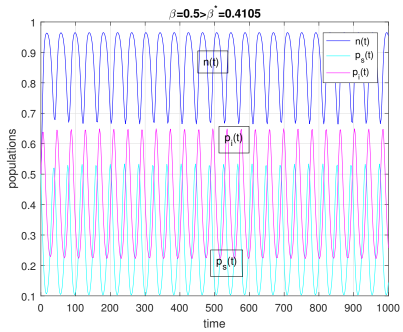



**Fig. 7.** *Influence of fear effect* $(k)$ *on the system* (2.2) .

Also the influence of fear effect on the prey, susceptible predator and infected predator populations shown in the Fig. 7. As discussed in article 8., prey population will decrease very fast along with the increasing value of $k$, susceptible predator population remains constant and infected predator population will also decrease but rate is slow.

Again the transmission coefficient from susceptible predator to infected predator has a great influence on the dynamics of the present system (2.2). It is observed that $E^*$ is locally asymptotically stable when $\beta < \beta^*$ and unstable when $\beta > \beta^*$, i.e. the system undergoes a Hopf bifurcation around $E^*$ at $\beta = \beta^* = 0.4105$. Fig. 8(a) and (b) depicts the unstable behavior of $n^*, p_s^*, p_i^*$ in time and unstable phase portrait of the system (2.2). Now, let us find out the system dynamics subject to the change in the value of $\beta$ keeping the other parameters as before to satisfy Theorem 6.2. The bifurcation diagram is presented in Fig. 9(a), (b) and (c) for variations in value of $\beta$ over [0.4,0.52] which shows that the system undergoes stable dynamics for $\beta < 0.4105$ and unstable behavior for $\beta > 0.4105$.



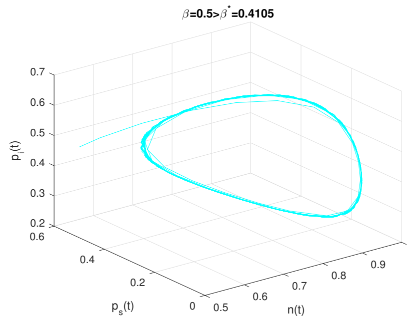
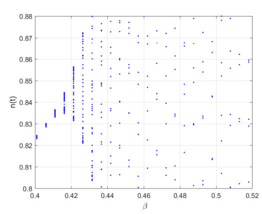
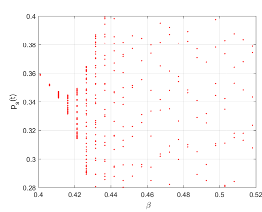




**Fig. 8(a)** **Fig. 8(b)**

**Fig. 8.** *Occurrence of oscillatory behavior and limit cycle around* $E^*$ *when* $\beta = 0.5 > \beta^* = 0.4105$.





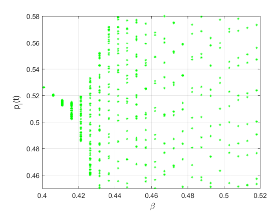


**Fig. 9(a)** **Fig. 9(b)** **Fig. 9(c)**

**Fig. 9.** *bifurcation diagram for the parameter* $\beta \in [0.4, 0.52]$ *and other parameters as given in list* $II$.

## 9. Discussion

In this paper, we have formulated a modified Lesli-Gower eco-epidemiological model with disease spreading in the predator only and also considered the cost of fear on prey population growth. The main objective of this consideration is to study the influence of anti-predator behavior due to fear of predators analytically and numerically. We have first observed the dynamical behavior of the model system for variation of fear factor in the prey population and finally observed the role of transmission coefficient from susceptible predator to infected predator with the help of numerical simulations. The model proposed here is ecologically well-behaved as fulfilling the positivity and boundedness of the prey and both the predators. The predation rate plays an important role in the extinction of the prey population. Also if the growth rate of predator lies below the parasite independent death rate of predators then the susceptible predators will wash out. And if the death rate of infected predator exceeds the maximal transmission rate, then it is not possible for the infected predator to survive. Analytically, we study the local and global stability analysis of the equation of the model, and show the model exhibits Hopf bifurcation and limit cycle. Here, we have observed that high levels of fear effect can stabilize the eco-epidemic model by excluding the periodic solution, which also confirmed the earlier result by Wang, 2016 [39]. Also, if the system (2.2) has a unique positive equilibrium, the fear effect can reduce the prey population, as the level of fear $k$ increases, the prey density gradually decreases. We also investigated the impact of disease transmission coefficient of susceptible to infected predator on the dynamics of the system when the system shows limit cycle oscillations about the interior equilibrium. We observed that if we increase the level of transmission coefficient then the system switches it's dynamics from stable to limit cycle oscillation.

As days go, researchers are showing their interest on predator-prey model with effect of fear on prey population. But, most of the cases have dealt with ecological problems. Here, we have incorporated fear effect on a modified Lesli-Gower eco-epidemiological model with disease in predator. It can be exhibits a rich dynamics. There are also several proceedings that should be cultivated. For example, how the system will endorse if the infected predator will also able to capture the prey population. Also we can improve our model by incorporating the gestation delay as a part of future work to make it more realistic.

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## References

1. R.M. Anderson and R.M. May, *Regulation and stability of host-parasite population interactions: I. Regulatory processes*, Journal of Animal Ecology **47** (1978), 219-247.
2. M.A. Aziz-Alaoui and M.D. Okiye, *Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling type II schemes*, Appl. Math. Lett. **16** (2003), 1069-1075.
3. G. Birkhoff and G.C. Rota, *Ordinary Differential Equations*, Ginn, Boston, 1982.
4. J. Carr, *Applications of Center Manifold Theory*, Springer, New York, 1982.
5. J. Chattopadhyay and O. Arino, *A predator-prey model with disease in the prey*, Nonlinear Analysis **36** (1999), 747-766.
6. S. Creel, D. Christianson, S. Liley, and J.A. Winnie, *Predation risk affects reproductive physiology and demography of elk*, Science **315** (2007), pp 960.
7. Das, K.P., *Disease-induced chaotic oscillations and its possible control in a predator-prey system with disease in predator*, Differ. Equ. Dyn. Syst. (2015), doi:10.1007/s12591-015-0249-7.
8. H.I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.
9. H.I. Freedman and S. Ruan, *Uniform Persistence in Functional Differential Equations*, J. Diff. Eq. **115** (1995), 173-192.
10. T.C. Gard and T.G. Hallam, *Persistence in food web-1, Lotka- Volterra food chains*, Bull. Math. Bio. **41** (1979), 302-315.
11. D. Greenhalgh and M. Haque, *A predator-prey model with disease in the prey species only*, Math. Meth. Appl. Sci. **30** (2006), 911-929, DOI: 10.1002/mma.815.

12. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, NY, 1983.
13. R. Gupta and P. Chandra, *Bifurcation analysis of modified Lesli-Gower predator-prey model with Michaelis-Menten type harvesting*, J. Math. Anal. Appl. **398** (2013), 278-295.
14. K.P. Hadeler and H.I. Freedman, *Predator-prey populations with parasitic infection*, J. Math. Biol. **27** (1989), 609-631.
15. L. Han, Z. Ma and H.W. Hethcote, *Four predator prey models with infectious disease*, Math. Comp. Model **34** (2001), 849-858.
16. M. Haque, S. Sarwardi, S. Preston and E. Venturino, *Effect of delay in a Lotka-Volterra type predator-prey model with a transmission disease in the predator species*, Math. Biosci. **234** (2011), 47-57.
17. H. Hethcote, W. Wang, L. Han and Z. Ma, *A predator-prey model with infected prey*, Theor. Popul. Biol. **66** (2004), 259-268.
18. F. Hua, K.E. Sieving, R.J. Fletcher, et al., *Increased perception of predation risk to adults and offspring alters avian reproductive strategy and performance*, Behav. Ecol. **25** (2014), 509-519.
19. T.K. Kar, A. Gorai and S. Jana, *Dynamics of pest and its predator model with disease in the pest and optimal use of pesticide*, J. Theor. Biol. **310** (2012), 187-198.
20. P.H. Lesli, *A Stochastic model for studying the properties of certain biological systems by numerical methods*, Biometrika **45** (1958), 16-31.
21. P.H. Lesli, *Some further notes on the use of matrices in population mathematics*, Biometrika **35** (1948), 213-245.
22. A. Maiti and G.P. Samanta, *Deterministic and stochastic analysis of a prey-dependent predator-prey system*, Int. J. Math. Educ. Sci. Technol. **36** (2005), 65-83.
23. A. Mondal, A.K. Pal and G.P. Samanta, *On the dynamics of evolutionary Lesli-Gower predator-prey eco-epidemiological model with disease in predator*, Ecol. Genet. Genom. **10** (2019), 100034. doi: 10.1016/j.egg.2018.11.002.
24. A. Nindjin, M. Aziz-Alaoui and M. Cadivel, *Analysis of a predator-prey model with modified Lesli-Gower and Holling-type II schemes with time delay*, Nonlinear Anal. R. World Appl. **7** (2006), 1104-1118.
25. J.L. Orrock and R.J. Fletcher, *An island-wide predator manipulation reveals immediate and long-lasting matching of risk by prey*, Proceed. of the Royal Soc. of London B: Biol. Sc. **281** (2014), 1784, 2014039. doi: 10.1098/rspb.2014.0391.
26. P. Panday, N, Pal, S. Samanta, et. al., *Stability and bifurcation analysis of a three-species food chain model with fear*, Int. J. Bifurc. Chaos Appl. Sci. Eng. **28** (2018), 1850009. doi: 10.1142/S0218127418500098.
27. A.K. Pal and G.P. Samanta, *Stability analysis of an eco-epidemiological model incorporating a prey refuge*, Nonlinear Anal. Model. Contrl. **15** (2010), 473-491.
28. A.K. Pal and G.P. Samanta, *A Ratio-dependent Eco-epidemiological Model Incorporating a Prey Refuge*, Univ. J. Applied Maths. **1** (2013), 86-100, doi: 10.13189/ujam.2013.010208.
29. S. Pal, S. Majhi, S. Mandal, et. al., *Role of fear in a predator-prey model with Beddington-DeAngelis functional response*, Z. Naturforsh. A **74** (2019), 301-327.
30. L. Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, NY, 2001.
31. S. Ruan and D. Xiao, *Global analysis in a predator-prey system with nonmonotonic functional response*, SIAM J Appl. Math. **61** (2001), 1445-1472.
32. S. Sarwardi, M. Haque and E. Venturino, *A Leslie-Gower Holling-type II ecoepidemic model*, Journal of Applied Mathematics and Computing **35** (2011), 263-280.
33. S.K. Sasmal and Y. Takeuchi, *Dynamics of a predator-prey system with fear and group defense*, J. Math. Anal. Appl. **481** (2020), doi: 10.1016/j.jmaa.2019.123471.
34. A. Sha, S. Samanta, M. Martcheva, et al., *Backward bifurcation, oscillations and chaos in an eco-epidemiological model with fear effect*, J. Biol. Dyn. **13** (2019), 301-327.

35. B.K. Singh and J. Chattopadhyay, *The role of virus infection in a simple phytoplankton zooplankton system*, J. Theor. Biol. **231** (2004), 545-559.
36. J.P. Tripathi and S. Abbas, *Almost Periodicity of a Modified Lesli-Gower Predator-Prey System with Crowley-Martin functional Response*, Springer Proceedings in Mathematics and Statistics **143** (2015).
37. E. Venturino, *Epidemics in predator-prey models: disease in the prey*, In: O. Arino, D. Axelrod, M. Kimmel, M. Langlais, editors, Mathematical population dynamics: analysis of heterogeneity, Theory of epidemics **1** (1995), 381-393.
38. E. Venturino, *Epidemics in predator-prey models: disease in the predators*, IMA J. Math. Appl. Med. Biol. **19** (2002), 285-205.
39. X. Wang, L.Y. Zanette and X. Zou, *Modeling the fear effect in predator-prey interactions*, J. Math. Biol. **73** (2016), 1179-1204.
40. X. Wang and X. Zou, *Modeling the fear effect in predator-prey interactions with adaptive avoidance of predators*, Bull. Math. Biol. **79** (2017), 1325-1359.
41. J. Wang, Y. Cai, S. Fu, et. al., *The effect of the fear factor on the dynamics of a predator-prey model incorporating the prey refuge*, Chaos **29** (2019), doi: 10.1063/1.5111121.
42. S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Second Edition, Springer, New York, 2003.
43. A. Wolf, J.B. Swift, H.L. Swinney and J.A. Vastano, *Determining Lyapunov exponents from a time series*, Physica D **16** (1985), 285-317.
44. Y. Xiao and L. Chen, *Modelling and analysis of a predator-prey model with disease in the prey*, Mathematical Biosciences **171** (2001), 59-82.
45. L.Y. Zanette, A.F. White, M.C. Allen, et al. *Perceived predation risk reduces the number of offspring songbirds produce per year*, Science **334** (2011), 1398-1401.
46. Y. Zhu and K. Wang, *Existence and global attractivity of positive periodic solutions for a predator-prey model with modified Lesli-Gower Holling-type II schemes*, J. Math. Anal. Appl. **384** (2011), 445-454.

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