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CIRCLE ACTIONS ON ORIENTED MANIFOLDS WITH FEW FIXED POINTS

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Abstract. Let the circle act on a compact oriented manifold with a discrete fixed point set. At each fixed point, there are positive integers called **weights**, which describe the local action of S^1 near the fixed point. In this paper, we provide the author's original proof that only uses the Atiyah-Singer index formula for the classification of the weights at the fixed points if the dimension of the manifold is 4 and there are at most 4 fixed points, which made the author possible to give a classification for any finite number of fixed points.

1. Introduction

The purpose of this paper is to make some contribution for studying a circle action on an oriented manifold with isolated fixed points by using a simple method, which only uses the Atiyah-Singer index formula.

Let the circle act on an oriented manifold M. Assume that M has an isolated fixed point. Since the dimensions of M and its fixed point set have the same parity, the dimension of M is even; let $\dim M = 2n$. Suppose that p is an isolated fixed point. The tangent space T_pM to M at p decomposes into real 2-dimensional irreducible S^1-equivariant real vector spaces $L_1, \cdots, L_n$, each of which is isomorphic to a complex 1-dimensional S^1-equivariant space with action given as multiplication by $g^{w_{pi}}$, where w_{pi} is a positive integer, $1 \leq i \leq n$. The n positive integers $w_{p1}, \cdots, w_{pn}$ are called the **weights** at p. Let $\epsilon(p) = +1$ if the orientation on M agrees with the orientation on the representation space at p and $\epsilon(p) = -1$ otherwise, and call $\epsilon(p)$ the **sign** of p. Let $\Sigma_p = \{\epsilon(p), w_{p1}, \cdots, w_{pn}\}$ be the **fixed point data** of p. The **fixed point data** of M is a collection of the fixed point data Σ_p for all fixed points $p \in M^{S^1}$.

Let M be a compact oriented manifold. The L-genus of M is the genus belonging to the power series $\frac{\sqrt{z}}{\tanh \sqrt{z}}$. The **signature** $\mathrm{sign}(M)$ of M is the

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analytic index of the signature operator on M. The Atiyah-Singer index theorem states that the analytic index of the operator is equal to the topological index of the operator, and the L-genus of M is equal to the signature of M. If the circle acts on M, we define the equivariant index of the operator for any element of the circle. The equivariant index is rigid; it does not depend on the choice of an element of S^1 and is equal to the signature of M. As a consequence, we obtain the following formula:

Theorem 1.1. [Atiyah-Singer index theorem] [AS] *Let the circle act on a $2n$-dimensional compact oriented manifold M with a discrete fixed point set. Then the signature of M is*

$$sign(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \frac{1+t^{w_{pi}}}{1-t^{w_{pi}}} \tag{1}$$

and is a constant, where t is an indeterminate.

Taking $t=0$ in Equation 1, it follows that

$$\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p). \tag{2}$$

Moreover, we can convert Equation 1 as follows.

$$\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \frac{1+t^{w_{pi}}}{1-t^{w_{pi}}} = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} [(1+t^{w_{pi}})(\sum_{j=0}^{\infty} t^{jw_{pi}})]$$

and hence

$$\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} (1 + 2\sum_{j=1}^{\infty} t^{jw_{pi}}) \tag{3}$$

Let the circle on a compact oriented manifold M with a discrete fixed point set. In this paper, we introduce a simple method to classify the weights at the fixed points of M.

Method 1.2.
(1) In Equation 3, consider the terms that have smallest positive exponent. Since $\text{sign}(M)$ is a constant, the coefficients of the terms with smallest positive exponent must sum to 0. In Equation 3, cancel out 'all' the terms from this observation.
(2) Consider the remnant; in the remnant, consider the terms with smallest positive exponent. Since $\text{sign}(M)$ is a constant, those terms must sum to 0; in the remnant, cancel out 'all' the terms from this observation.
(3) Repeat.

The classification results for other types of S^1-manifolds with few fixed points can be found in [J1], [J2], [J3], [J4], [K1], [K2], [K3], [M], [PT], etc.

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2. Classification

Consider a circle action on a compact oriented manifold M with a discrete fixed point. Suppose that $\dim M > 0$. Then there cannot be exactly one fixed point. In [J3], the author proved that if $\dim M = 4$, there are lower and upper bounds on the signature of M in terms of the number of fixed points; see Corollary 1.3 of [J3]. The result extends to an arbitrary dimension.

Lemma 2.1. *Let the circle act on a compact oriented manifold M with k fixed points, where $k > 0$ is an integer. Suppose that* $\dim M > 0$*. Then $k \geq 2$ and the signature of M satisfies $|sign(M)| \leq k - 2$.*

Proof. Let $\dim M = 2n > 0$. First, suppose that there is exactly one fixed point p. By Theorem 1.1,

$$\operatorname{sign}(M) = \epsilon(p) \prod_{i=1}^{n} (1 + 2 \sum_{j=1}^{\infty} t^{j w_{pi}})$$

must be a constant for all indeterminates t. However, since $\dim M > 0$ and $k = 1$, the right hand side cannot be a constant. Therefore, if $\dim M > 0$, $k \geq 2$.

From now on, assume that $k \geq 2$. Taking $t = 0$ in Theorem 1.1, we have

$$\operatorname{sign}(M) = \textstyle\sum_{p \in M^{S^1}} \epsilon(p).$$

Since $\epsilon(p)$ is either $+1$ or -1, this implies $-k \leq \operatorname{sign}(M) \leq k$.

Suppose that $\operatorname{sign}(M) = k$, i.e., $\epsilon(p) = +1$ for all $p \in M^{S^1}$. By Equation 3, we have

$$\operatorname{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} (1 + 2 \sum_{j=1}^{\infty} t^{j w_{pi}}) = \sum_{p \in M^{S^1}} \prod_{i=1}^{n} (1 + 2 \sum_{j=1}^{\infty} t^{j w_{pi}}).$$

The signature of M is a constant, but the right hand side cannot be a constant for all indeterminates t, which is a contradiction. Therefore, $\operatorname{sign}(M) \neq k$. Similarly, we can show that $\operatorname{sign}(M) \neq -k$. Therefore, there must exist at least one fixed point p such that $\epsilon(p) = +1$. Similarly, there must exist at least one fixed point q such that $\epsilon(q) = -1$. Since $\operatorname{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p)$, this implies that $2 - k \leq \operatorname{sign}(M) \leq k - 2$. ☐

As an immediate consequence of Lemma 2.1, we obtain a lower bound on the number of fixed points.

Corollary 2.2. *Let the circle act on a compact oriented manifold M. If $sign(M) \neq 0$, then there are at least $|sign(M)| + 2$ fixed points.*

Moreover, the bound is sharp; for any integer k, there exists a 4-dimensional compact oriented manifold M equipped with a circle action, such that $\text{sign}(M) = k$ and it has $|k|+2$ fixed points; see [J3]. If there are exactly 2 fixed points, in [K3] Kosniowski classified the weights at the fixed points. In this case, the weights at the fixed points agree with those for a rotation on S^{2n}. This is reproved in [L]. In this paper, we give another simple proof.

Theorem 2.3. *Let the circle act on a compact oriented manifold M with exactly 2 fixed points p and q. Then $\epsilon(p) = -\epsilon(q)$ and the weights at p and q agree.*

Proof. Since there are exactly 2 fixed points, by Lemma 2.1, $\text{sign}(M) = 0$. By Equation 2, it follows that $\epsilon(p) = -\epsilon(q)$. Without loss of generality, let $\epsilon(p) = +1$. By Equation 3,

$$0 = \text{sign}(M) = \prod_{i=1}^{n}(1 + 2\sum_{j=1}^{\infty} t^{jw_{pi}}) - \prod_{i=1}^{n}(1 + 2\sum_{j=1}^{\infty} t^{jw_{qi}})$$

We follow Method 1.2. Without loss of generality, let $w_{p1} \leq \cdots \leq w_{pn}$ and $w_{q1} \leq \cdots \leq w_{qn}$. In the expression $\prod_{i=1}^{n}(1 + 2\sum_{j=1}^{\infty} t^{jw_{pi}})$ that p contributes, the term that has smallest positive exponent is $2t^{w_{p1}}$. In the expression $-\prod_{i=1}^{n}(1+2\sum_{j=1}^{\infty} t^{jw_{qi}})$ that q contributes, the term that has the smallest positive exponent is $-2t^{w_{q1}}$. Since the signature of M is a constant, this implies that $w_{p1} = w_{q1}$. Therefore, cancel out $(1+2\sum_{j=1}^{\infty} t^{jw_{p1}})$ and $(1+2\sum_{j=1}^{\infty} t^{jw_{q1}})$ in the equation above. Then we are left with

$$0 = \prod_{i=2}^{n}(1 + 2\sum_{j=1}^{\infty} t^{jw_{pi}}) - \prod_{i=2}^{n}(1 + 2\sum_{j=1}^{\infty} t^{jw_{qi}})$$

Considering terms with smallest positive exponents, it follows that $w_{p2} = w_{q2}$. By induction, the theorem follows. $\square$

If $\dim M = 4$ and there are 3 fixed points, Li classified the weights at the fixed points [L]. By using Method 1.2, we give a simple proof.

Theorem 2.4. *Let the circle act on a 4-dimensional compact oriented manifold with 3 fixed points. Then either*

(1) *sign(M) $= 1$ and the fixed point data of M is $\{-, a, b\}$, $\{+, a, a+b\}$, $\{+, b, a+b\}$ for some positive integers a and b, or*
(2) *sign(M) $= -1$ and the fixed point data of M is $\{+, a, b\}$, $\{-, a, a+b\}$, $\{-, b, a+b\}$ for some positive integers a and b.*

Proof. By Lemma 2.1 and Equation 2, $\text{sign}(M) = \pm 1$. Suppose that $\text{sign}(M) = +1$. By Equation 2, we can label fixed points by p, q, r so that $\epsilon(p) = -1$, $\epsilon(q) = \epsilon(r) = +1$. By Equation 3,

$$1=\operatorname{sign}(M)=-\prod_{i=1}^{2}(1+2\sum_{j=1}^{\infty}t^{jw_{p,i}})+\prod_{i=1}^{2}(1+2\sum_{j=1}^{\infty}t^{jw_{q,i}})+\prod_{i=1}^{2}(1+2\sum_{j=1}^{\infty}t^{jw_{r,i}})-$$
$$(1+2\sum_{j>0}t^{jw_{p,1}}+2\sum_{j>0}t^{jw_{p,2}}+4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}})+(1+2\sum_{j>0}t^{jw_{q,1}}+2\sum_{j>0}t^{jw_{q,2}}+$$
$$4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}})+(1+2\sum_{j>0}t^{jw_{r,1}}+2\sum_{j>0}t^{jw_{r,2}}+4\sum_{j,k>0}t^{jw_{r,1}+kw_{r,2}}).$$

We follow Method 1.2. Without loss of generality, let $w_{x,1}\leq w_{x,2}$ for all $x\in M^{S^1}$. The exponent of the first nonconstant term is $w_{p,1}$, $w_{q,1}$, or $w_{r,1}$. Since the signature of M is a constant, this implies that $w_{p,1}=w_{q,1}$ or $w_{p,1}=w_{r,1}$. By changing the role of q and r if necessary, assume that $w_{q,1}\leq w_{r,1}$. Then $w_{p,1}=w_{q,1}$. The equation simplifies to

$$0=-(2\sum_{j>0}t^{jw_{p,2}}+4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}})+(2\sum_{j>0}t^{jw_{q,2}}+4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}})+$$
$$(2\sum_{j>0}t^{jw_{r,1}}+2\sum_{j>0}t^{jw_{r,2}}+4\sum_{j,k>0}t^{jw_{r,1}+kw_{r,2}}).$$

In the remnant, the exponent of the first nonconstant term is $w_{p,2}$, $w_{q,2}$, or $w_{r,1}$.

Suppose that $w_{p,2}=w_{q,2}$. Since $w_{p,j}=w_{q,j}$ for $j=1,2$, the equation above simplifies to

$$0=2\sum_{j>0}t^{jw_{r,1}}+2\sum_{j>0}t^{jw_{r,2}}+4\sum_{j,k>0}^{\infty}t^{jw_{r,1}+kw_{r,2}},$$

which cannot hold. Therefore, it follows that $w_{p,2}=w_{r,1}$ and $w_{p,2}<w_{q,2}$. Then the equation simplifies to

$$0=-4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}}+(2\sum_{j>0}t^{jw_{q,2}}+4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}})+(2\sum_{j>0}t^{jw_{r,2}}+$$
$$4\sum_{j,k>0}t^{jw_{r,1}+kw_{r,2}}).$$

In the remnant, the exponent of the first nonconstant term is $w_{p,1}+w_{p,2}$, $w_{q,2}$, or $w_{r,2}$. However, since $t^{w_{p,1}+w_{p,2}}$ has coefficient -4 and both of $t^{w_{q,2}}$ and $t^{w_{r,2}}$ have coefficients 2, it follows that both of $w_{q,2}$ and $w_{r,2}$ must equal $w_{p,1}+w_{p,2}$. This proves the theorem when $\operatorname{sign}(M)=+1$. The case that $\operatorname{sign}(M)=-1$ is similar. $\square$

Using Method 1.2, we can classify the fixed point data when $\dim M=4$ and there are 4 fixed points.

Theorem 2.5. *Let the circle act on a 4-dimensional compact oriented manifold M with 4 fixed points.*

(1) *If sign(M) = 2, the fixed point data of M is* $\{-, a, b\}$, $\{+, a, a+b\}$, $\{+, b, a+2b\}$, $\{+, a+b, a+2b\}$ *for some positive integers a and b.*
(2) *If sign(M) = −2, the fixed point data of M is* $\{+, a, b\}$, $\{-, a, a+b\}$, $\{-, b, a+2b\}$, $\{-, a+b, a+2b\}$ *for some positive integers a and b.*

Proof. First, suppose that $\operatorname{sign}(M) = 2$. Since $\operatorname{sign}(M) = 2$, by Equation 2, there is 1 fixed point p with sign -1 and there are 3 fixed points q, r, s with sign $+1$. Without loss of generality, assume that $w_{x,1} \leq w_{x,2}$ for all $x \in M^{S^1}$ and $w_{q,1} \leq w_{r,1} \leq w_{s,1}$. By Equation 3,

$$2 = \operatorname{sign}(M) = -(1 + 2\sum_{j>0} t^{jw_{p,1}} + 2\sum_{j>0} t^{jw_{p,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}}) + (1 + 2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (1 + 2\sum_{j>0} t^{jw_{r,1}} + 2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}) + (1 + 2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

By comparing the exponents of the first nonconstant terms, it follows that $w_{p,1} = w_{q,1}$. Cancel out the constant terms and all the terms $2\sum_{j>0} t^{jw_{p,1}}$ and $2\sum_{j>0} t^{jw_{q,1}}$. In the remnant, the exponent of the first nonconstant term is $w_{p,2}$, $w_{q,2}$, or $w_{r,1}$. Suppose that $w_{p,2} = w_{q,2}$. The equation simplifies to

$$0 = (2\sum_{j>0} t^{jw_{r,1}} + 2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}) + (2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}),$$

which cannot hold. Therefore, it follows that $w_{p,2} = w_{r,1}$ and $w_{p,2} < w_{q,2}$. Canceling out the terms $-2\sum_{j>0} t^{jw_{p,2}}$ and $2\sum_{j>0} t^{jw_{r,1}}$, we are left with

$$0 = -4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}} + (2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}) + (2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

In the remnant, the exponent of the first nonconstant term is $w_{p,1} + w_{p,2}$, $w_{q,2}$, $w_{r,2}$, $w_{s,1}$, or $w_{s,2}$. Since $t^{w_{p,1}+w_{p,2}}$ has coefficient -4, exactly two of $w_{q,2}$, $w_{r,2}$, $w_{s,1}$, and $w_{s,2}$ equal $w_{p,1} + w_{q,2}$. Suppose that $w_{p,1} + w_{p,2} = w_{q,2} = w_{r,2}$. Then the equation above simplifies to

$$0 = 2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}},$$

which cannot hold. Therefore, one of the following holds.

(1) $w_{p,1} + w_{p,2} = w_{q,2} = w_{s,1}$ and $w_{p,1} + w_{p,2} < w_{r,2}$.
(2) $w_{p,1} + w_{p,2} = w_{r,2} = w_{s,1}$ and $w_{p,1} + w_{p,2} < w_{q,2}$.
(3) $w_{p,1} + w_{p,2} = w_{s,1} = w_{s,2}$.

Suppose that Case (1) holds. Since $w_{p,1} = w_{q,1}$ and $w_{p,1} + w_{p,2} = w_{q,2}$, $-4\sum_{j>k>0} t^{jw_{p,1}+kw_{p,2}}$ cancel out with $4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}$. Therefore, the equation above simplifies to

$$0 = -4\sum_{0<j<k} t^{jw_{p,1}+kw_{p,2}} + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}) + (2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

In the remained, the exponent of the first nonconstant term is $w_{p,1}+2w_{p,2}$, $w_{r,2}$, or $w_{s,2}$. Because $t^{w_{p,1}+2w_{p,2}}$ has coefficient -4, it follows that $w_{p,1} + 2w_{p,2} = w_{r,2} = w_{s,2}$. The fixed point data of M is then

$$\{-, w_{p,1}, w_{p,2}\}, \{+, w_{p,1}, w_{p,1} + w_{p,2}\}, \{+, w_{p,2}, w_{p,1} + 2w_{p,2}\}, \{+, w_{p,1} + w_{p,2}, w_{p,1} + 2w_{p,2}\}.$$

If we let $w_{p,1} = a$ and $w_{p,2} = b$, then this belongs to the conclusion of the theorem with $a \leq b$.

Suppose that Case (2) holds. Since $w_{p,2} = w_{r,1}$ and $w_{p,1} + w_{p,2} = w_{r,2}$, $-4\sum_{0<j<k} t^{jw_{p,1}+kw_{p,2}}$ cancel out with $4\sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}$. Therefore, the equation above simplifies to

$$0 = -4\sum_{j>k>0} t^{jw_{p,1}+kw_{p,2}} + (2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

In the remained, the exponent of the first nonconstant term is $2w_{p,1}+w_{p,2}$, $w_{q,2}$, or $w_{s,2}$. Because $t^{2w_{p,1}+w_{p,2}}$ has coefficient -4, it follows that $2w_{p,1} + w_{p,2} = w_{q,2} = w_{s,2}$. The fixed point data of M is then

$$\{-, w_{p,1}, w_{p,2}\}, \{+, w_{p,1}, 2w_{p,1} + w_{p,2}\}, \{+, w_{p,2}, w_{p,1} + w_{p,2}\}, \{+, w_{p,1} + w_{p,2}, 2w_{p,1} + w_{p,2}\},$$

If we let $w_{p,1} = b$ and $w_{p,2} = a$, then this belongs to the other half of the conclusion of the theorem with $b \leq a$.

Suppose that Case (3) holds. The equation above simplifies to

$$0 = -4\sum_{j,k>0, j\neq k} t^{jw_{p,1}+kw_{p,2}} + (2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{p,2}+kw_{r,2}}) + 4\sum_{j,k>0} t^{(j+k)(w_{p,1}+w_{p,2})}.$$

The exponent of the first nonconstant term is $2w_{p,1} + w_{p,2}$, $w_{q,2}$, or $w_{r,2}$, which has coefficient -4, 2, and 2. Therefore, $2w_{p,1} + w_{p,2} = w_{q,2} = w_{r,2}$ and so

$$0 = -4 \sum_{j,k>0, j\neq k, j\neq 2k} t^{jw_{p,1}+kw_{p,2}} + 4 \sum_{j,k>0} t^{jw_{p,1}+k(2w_{p,1}+w_{p,2})} + 4 \sum_{j,k>0} t^{jw_{p,2}+k(2w_{p,1}+w_{p,2})} + 4 \sum_{j,k>0} t^{(j+k)(w_{p,1}+w_{p,2})}.$$

In the equation, the coefficient of $t^{2w_{p,1}+2w_{p,2}}$ is not zero, which leads to a contradiction.

This proves the theorem when $\text{sign}(M) = 2$. The case that $\text{sign}(M) = -2$ is analogous. $\square$

Theorem 2.6. *Let the circle act on a 4-dimensional compact oriented manifold M with 4 fixed points. If $\text{sign}(M) = 0$, the fixed point data of M is*

$$\{-, a, b\}, \{-, c, d\}, \{+, a, b\}, \{+, c, d\}$$

for some positive integers $a, b, c,$ and d.

Proof. By Equation 2, there are two fixed points p and q with sign -1 and two fixed points r and s with sign $+1$. Without loss of generality, assume that $w_{x,1} \leq w_{x,2}$ for all $x \in M^{S^1}$, $w_{p,1} \leq w_{q,1}$, and $w_{r,1} \leq w_{s,1}$. By Equation 3,

$$0 = \text{sign}(M) = -(1 + 2\sum_{j>0} t^{jw_{p,1}} + 2\sum_{j>0} t^{jw_{p,2}} + 4 \sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}}) - (1 + 2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4 \sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (1 + 2\sum_{j>0} t^{jw_{r,1}} + 2\sum_{j>0} t^{jw_{r,2}} + 4 \sum_{j,k>0} t^{jw_{r,1}+kw_{r,2}}) + (1 + 2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4 \sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

By comparing exponents of the first nonconstant terms, we have $w_{p,1} = w_{r,1}$.

Suppose that $w_{p,2} = w_{r,2}$. Then the equation above simplifies to

$$0 = -(1 + 2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4 \sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (1 + 2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4 \sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}})$$

and hence it follows that $\{w_{q,1}, w_{q,2}\} = \{w_{s,1}, w_{s,2}\}$; this case covers the conclusion of the theorem.

Therefore, from now on, assume that $w_{p,2} \neq w_{r,2}$. By changing the roles p, q and r, s, we may assume that $w_{p,2} < w_{r,2}$. Cancelling out constant terms and the terms $-2\sum_{j>0} t^{jw_{p,1}}$ and $2\sum_{j>0} t^{jw_{r,1}}$, the equation above simplifies to

$$0 = -(2\sum_{j>0} t^{jw_{p,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}}) - (2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{r,2}}) + (2\sum_{j>0} t^{jw_{s,1}} + 2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{s,1}+kw_{s,2}}).$$

Comparing exponents of the first nonconstant terms, we have $\min\{w_{p,2}, w_{q,1}\} = \min\{w_{r,2}, w_{s,1}\}$. Since $w_{p,2} < w_{r,2}$, it follows that $\min\{w_{p,2}, w_{q,1}\} = w_{s,1}$.

Suppose that (1) $w_{p,2} = w_{s,1}$. Then Equation 3 simplies to

$$0 = -4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}} - (2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{r,2}}) + (2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{p,2}+kw_{s,2}}).$$

Assume that (1-a) $w_{p,1} + w_{p,2} < w_{q,1}$. This implies that $w_{r,2} = w_{s,2} = w_{p,1} + w_{p,2}$. The equation then simplifies to

$$0 = -(2\sum_{j>0} t^{jw_{q,1}} + 2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}),$$

which cannot hold.

Assume that (1-b) $w_{p,1} + w_{p,2} = w_{q,1}$. Then the coefficient of $t^{w_{q,1}}$ cannot be zero, which leads to a contradiction.

Assume that (1-c) $w_{p,1} + w_{p,2} > w_{q,1}$. Then $w_{q,1} = w_{r,2}$ or $w_{q,1} = w_{s,2}$.

First, suppose that (1-c-i) $w_{q,1} = w_{r,2}$. The equation above simplifies to

$$0 = -4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}} - (2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{q,1}} + (2\sum_{j>0} t^{jw_{s,2}} + 4\sum_{j,k>0} t^{jw_{p,2}+kw_{s,2}}).$$

It follows that $w_{q,2} = w_{s,2}$. The equation above simplifies further to

$$0 = -\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}} - \sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}} + \sum_{j,k>0} t^{jw_{p,1}+kw_{q,1}} + \sum_{j,k>0} t^{jw_{p,2}+kw_{q,2}},$$

which cannot hold since $w_{p,2} < w_{r,2} = w_{q,1} \leq w_{q,2}$.

Second, suppose that (1-c-ii) $w_{q,1} = w_{s,2}$. The equation above simplifies to

$$0 = -4\sum_{j,k>0} t^{jw_{p,1}+kw_{p,2}} - (2\sum_{j>0} t^{jw_{q,2}} + 4\sum_{j,k>0} t^{jw_{q,1}+kw_{q,2}}) + (2\sum_{j>0} t^{jw_{r,2}} + 4\sum_{j,k>0} t^{jw_{p,1}+kw_{r,2}}) + 4\sum_{j,k>0} t^{jw_{p,2}+kw_{s,2}}.$$

This implies that $w_{q,2} = w_{r,2}$. Simplifying further,

$$0=-\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}}-\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}}+\sum_{j,k>0}t^{jw_{p,1}+kw_{q,2}}+\sum_{j,k>0}t^{jw_{p,2}+kw_{q,1}}.$$

Since $w_{p,2} < w_{r,2} = w_{q,2}$, we must have $w_{p,1} = w_{q,1}$. Then the fixed point data of M is

$$\{-,w_{p,1},w_{p,2}\},\{-,w_{p,1},w_{q,2}\},\{+,w_{p,1},w_{q,2}\},\{+,w_{p,2},w_{p,1}\}$$

and hence this belongs to the conclusion of the theorem.

Suppose that (2) $w_{q,1} = w_{s,1}$. The equation above then simplifies to

$$0=-(2\sum_{j>0}t^{jw_{p,2}}+4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}})-(2\sum_{j>0}t^{jw_{q,2}}+4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}})+ (2\sum_{j>0}t^{jw_{r,2}}+4\sum_{j,k>0}t^{jw_{p,1}+kw_{r,2}})+(2\sum_{j>0}t^{jw_{s,2}}+4\sum_{j,k>0}t^{jw_{q,1}+kw_{s,2}}).$$

Comparing exponents of the first nonconstant terms, since $w_{p,2}$ cannot equal $w_{r,2}$ ($w_{p,2} < w_{r,2}$), we must have $w_{p,2} = w_{s,2}$ (Alternatively, if $w_{q,2} = w_{s,2}$, we reach a contradiction). Then we have

$$0=-4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}}-(2\sum_{j>0}t^{jw_{q,2}}+4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}})+(2\sum_{j>0}t^{jw_{r,2}}+ 4\sum_{j,k>0}t^{jw_{p,1}+kw_{r,2}})+4\sum_{j,k>0}t^{jw_{q,1}+kw_{p,2}}.$$

It follows that $w_{q,2} = w_{r,2}$, and hence

$$0=-4\sum_{j,k>0}t^{jw_{p,1}+kw_{p,2}}-4\sum_{j,k>0}t^{jw_{q,1}+kw_{q,2}}+4\sum_{j,k>0}t^{jw_{p,1}+kw_{q,2}}+ 4\sum_{j,k>0}t^{jw_{q,1}+kw_{p,2}}.$$

Since $w_{p,2} < w_{s,2} = w_{q,2}$, we must have $w_{p,1} = w_{q,1}$. The fixed point data of M is then

$$\{-,w_{p,1},w_{p,2}\},\{-,w_{p,1},w_{q,2}\},\{+,w_{p,1},w_{q,2}\},\{+,w_{p,1},w_{p,2}\}$$

and hence this belongs to the conclusion of the theorem. □

Combining Theorems 2.5 and 2.6, we classify the fixed point data of a 4-dimensional compact oriented S^1-manifold with 4 fixed points.

Theorem 2.7. *Let the circle act on a 4-dimensional compact oriented manifold M with 4 fixed points. Then exactly one of the following holds.*

(1) *sign(M) $= 2$ and the fixed point data of M is $\{-,a,b\}$, $\{+,a,a+b\}$, $\{+,b,a+2b\}$, $\{+,a+b,a+2b\}$ for some positive integers a and b.*
(2) *sign(M) $= 0$ and the fixed point data of M is $\{-,a,b\}$, $\{-,c,d\}$, $\{+,a,b\}$, $\{+,c,d\}$ for some positive integers $a,b,c,$ and d.*

(3) *sign*$(M) = -2$, *the fixed point data of* M *is* $\{+, a, b\}$, $\{-, a, a+b\}$, $\{-, b, a+2b\}$, $\{-, a+b, a+2b\}$ *for some positive integers* a *and* b.

Proof. By Lemma 2.1 and Equation 2, the signature of M satisfies $\text{sign}(M) = -2$, 0, or 2. If $\text{sign}(M) = \pm 2$, the theorem follows from Theorem 2.5. If $\text{sign}(M) = 0$, the theorem follows from Theorem 2.6. □

An example of a manifold of Case (2) of Theorem 2.7 is an equivariant connected sum of two 6-spheres equipped with rotations. A manifold of Case (3) of Theorem 2.7 can be constructed as follows. Let the circle act on the complex projective space $\mathbb{CP}^2$ by

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : g^a z_1 : g^{a+b} z_2]$$

for all $g \in S^1 \subset \mathbb{C}$, for some positive integers a and b. The fixed points are $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, and $p_3 = [0 : 0 : 1]$ which have $\{a, a+b\}$, $\{-a, b\}$, $\{-b, -a-b\}$ as complex weights, and hence the fixed point data $\{+, a, a+b\}$, $\{-, a, b\}$, $\{+, b, a+b\}$. Reverse the orientation of $\mathbb{CP}^2$ (which results in changing the sign of every fixed point). Next, we blow up p_1 by identifying a neighrborhood of p_1 with $\mathbb{C}^2$ with action $g \cdot (w_1, w_2) = (g^{-a} w_1, g^{a+b} w_2)$. The blowing up replaces the fixed point p_1 with $\mathbb{CP}^1$ on which the circle extends to act with two fixed points q_1 and q_2 that have fixed point data $\{-, a, 2a+b\}$, $\{-, a+b, 2a+b\}$. The resulting manifold M has fixed points q_1, q_2, p_2, p_3 that have fixed point data $\{-, a, 2a+b\}$, $\{-, a+b, 2a+b\}$, $\{+, a, b\}$, $\{-, b, a+b\}$, as required. If we reverse the orientation of M, a manifold of Case (1) is obtained. For more details, see [J3].

Remark 1. We discuss the author's original motivation for the classification of the fixed point data of a circle action on a 4-dimensional compact oriented manifold with any finite number of fixed points. For this, consider the case $\text{sign}(M) = 2$ in Theorem 2.7. In this case, the weight $a + b$ occurs as the sum of other two weights a and b. Similarly, the weight $a + 2b$ occurs as the sum of other two weights b and $a+b$. From this, the author made an observation on the relationships betweeh the weights at the fixed points, that any weight is either independent of other weights, or occurs as the sum of other two weights. The author wonders if this phenomenon always holds in higher dimensions, or even for other types of S^1-manifolds with isolated fixed points. From this observation, the author was able to classify the fixed point data of a 4-dimensional compact oriented S^1-manifold with any finite number of fixed points, combining with other tools; see [J3].

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