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ON THE CONVERGENCE OF AN OPTIMIZATION ALGORITHM BASED ON NONLINEAR OPERATORS

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Abstract. In this paper, an equilibrium problems involving a finite family of maximal monotone operators and inverse-strongly monotone operators are introduced and investigated. A strong convergence theorem of common solutions is obtained in Hilbert spaces.

1. Introduction

In this paper, H is assumed to be a real Hilbert space with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ and C is assumed to a convex and closed nonempty set in H. For all $x \in H$, there exists a unique vector in C, denoted by $P_C x$, such that $\|x - P_C x\| = \min_{y\in C} \|x - y\|$. P_C is called the metric or nearest point projection from H onto C. We have the following two essential properties of the prjection $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x \in H, y \in H$ and $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2$, $\forall x \in H$, $y \in C$.

Let $B : C \times C \to \mathbb{R}$ be a bifunction. Consider the following generalized variational inequality, which was first introduced by Fan [6]

$$\text{Find } y^* \in C \text{ such that } B(y^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The above problem is called the equilibrium in the sense of Blum and Oettli [1]. From now on, one always calls (1) an equilibrium problem. The set of all solutions to problem (1) is presented by $Sol(B)$, i.e., $Sol(B) := \{y^* \in C : B(y^*, y) \geq 0, \forall y \in C\}$. Problem (1) is quite general. Indeed, it includes a number of mathematical problems as special cases, such as, variational inequality problems, saddle-point problems, minimization problems, complementarity problem, etc; see, e.g., [3, 8, 9, 10, 11]. In addition to the theoretical importance, it also provides a general and unified framework of a number of real-world practice problems, such as, signal processing, medical imaging, traffic and transportation etc. From the viewpoint of numerical analysis, variational solution methods

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have been suggested and investigated for solving problem (1). For established and recent results, one refers to [12, 17, 18, 21, 22] and the references therein.

Next, we introduce the other problem, the zero-point problem. Find a zero point of the sum of monotone operators. In this paper, we will consider the case of two operators, that is,

$$0 \in (A + B)(x), \tag{2}$$

where A and B are two maximal monotone operators (see below). It is known that a stationary solution to the initial value problem of the evolution equation

$$\begin{cases} 0 \in Fu + \frac{\partial u}{\partial t}, \\ u_0 = u(0), \end{cases}$$

could be recast as (2) if the operator F can be rewritten as $F = A + B$. This problem, which can be viewed as a significant mathematical modelling in medical imaging, has extensively studied via resolvent techniques; see, e.g., [2, 4, 5, 19] and the references therein.

Many problems can be formulated as a problem of the form (2). Splitting methods, including, the Douglas-Rachford splitting method, the Peaceman-Rachford splitting method and the forward-backward splitting method, are popular and have been employed to approximating zero points of the sum of two or more monotone operators; see, e.g., [5, 19, 21, 24] and the references therein.

In this paper, we consider the following convex feasibility problem, which consists of finding a solution to a bifunction equilibrium problem and the zero problem involving two finite families of maximal monotone operators

$$Sol(B) \cap \big(\cap_{m=1}^{N} (A_m + B_m)^{-1}(0) \big), \tag{3}$$

where B is an equilibrium problem, N is positive integer, and A_m, B_m are two maximal monotone operators. We consider a projection algorithm [20] for the above problem and establish a strong convergence theorem, that is, convergence in norm, in the framework of real Hilbert spaces without compact conditions on operators and any subset of H, and without the aid of contractive conditions. The main results presented in this paper extend some recently results in the literatures in this field.

We organize this paper as follows: Section 2 is devoted to necessary tools, properties, definitions, and lemmas. The strong convergence theorem is studied and obtained in Section 3. Some subresults on classical variational inequalities are presented in the last section, Section 4.

2. Preliminaries

Let $A : C \to H$ be a single-valued mapping. One recalls that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be ν-inverse-strongly monotone if there exists a constant $\nu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \nu \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

The classical monotone variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

It is known fixed-point methods are efficient to solve the above monotone variational inequality. The solution set of the variational inequality problem is denoted by $VI(C, A)$. The variational inequality is an efficient tool for dealing with a number of real-world problems and it has been extensively studied via fixed-point methods; see, e.g., [7, 12, 14, 23, 24] and the references therein.

Let $B : H \to 2^H$ be set-valued mapping with domain $Dom(B) = \{x \in H : Bx \neq \emptyset\}$ and range $Ran(B) = \{Bx : x \in Dom(B)\}$. One recalls that B is monotone if, for all $x_1, x_2 \in Dom(B)$, $y_1 \in Bx_1$ and $y_2 \in Bx_2$, $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$. Let $Graph(B) = \{(x, y) : y \in Bx\}$ be the graph of B. Mapping B is said to be maximal if $Graph(B)$ is not contained in the graph of any other monotone mapping properly.

Let S be a mapping on H. S is said to be firmly nonexpansive if $\|Sx-Sy\|^2 \leq \langle x - y, Sx - Sy\rangle$, $\forall x, y \in C$. It is known that the metric projection is firmly nonexpansive. S is said to be nonexpansive iff $\|Sx-Sy\| \leq \|x-y\|$, $\forall x, y \in C$. It is known that if C is bounded, then the set of fixed points of every nonexpansive mapping is nonempty. In addition, the set of fixed points is convex and closed. Let I denote the identity mapping on H. One can define a single-valued mapping $Res^B_\mu : H \to H$ by $Res^B_\mu = (I+\mu B)^{-1}$ for any $\mu > 0$. $Res^B_\mu : H \to H$ is call the resolvent of B. Res^B_μ is firmly nonexpansive and $B^{-1}0 = Fix(Res^B_\mu)$, where $Fix(Res^B_\mu)$ denotes the fixed-point set of Res^B_μ.

To study equilibrium problem (1), one assumes that B satisfies the following restrictions:

(R1) $B(x, x) = 0$, for all $x \in C$;

(R2) $B(x, y) + B(y, x) \leq 0$, for all $x, y \in C$;

(R3) $\limsup_{t\downarrow 0} B(tz + (1 - t)x, y) \leq B(x, y)$, for each $x, y, z \in C$;

(R4) $y \mapsto B(x, y)$ is convex and weakly lower semi-continuous for each $x \in C$.

Lemma 2.1. [1] *Let $B : C \times C \to \mathbb{R}$ be a bifunction satisfying* (R1)-(R4). *Then, for any $\eta > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, z - x\rangle + \eta B(z, y) \geq 0, \quad \forall y \in C.$$

Further, define

$$Res^B_\eta x := \Big\{z \in C : \langle y - z, z - x\rangle + \eta B(z, y) \geq 0,\ \forall y \in C\Big\}$$

for any $\eta > 0$ and $x \in H$. Then, the following assertions hold:

(a) *Res^B_η is single-valued;*

(b) *Res^B_η is firmly nonexpansive;*

(c) *$Fix(Res^B_\eta) = Sol(B)$ is convex and closed, where $Fix(Res^B_\eta)$ denotes the set of fixed points of mapping Res^B_η.*

Lemma 2.2. [13] *Let $A : C \to H$ be a single-valued monotone mapping, and $B : H \to H$ a maximal set-valued monotone operator. Then $Fix(Res^B_\mu(I - \mu A)) = (A + B)^{-1}(0)$, which is convex and closed.*

3. Main results

Theorem 3.1. *Let C be a closed and convex nonempty subset of a real Hilbert space and let N be some positive integer. Let B be a bifunction with restrictions (R1)-(R4). Let $B_m : C \to H$ be a maximal monotone mapping and let $A_m : C \to H$ be a ν_m-inverse-strongly monotone mapping for all $1 \le m \le N$. Assume that $\Phi := \left(\cap_{m=1}^N (A_m + B_m)^{-1}(0)\right) \cap Sol(B) \ne \emptyset$ and $\{x_n\}$ is a vector sequence defined by*

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ \langle \lambda - y_n, y_n - x_n\rangle + \eta_n B(y_n, \lambda) \ge 0, \quad \forall \lambda \in C_n, \\ z_n = \alpha_n \sum_{m=1}^N \varphi_{n,m} Res^{B_m}_{\mu_{n,m}}(I - \mu_{n,m}A_m)y_n + (1 - \alpha_n)x_n, \\ C_{n+1} = \{w \in C_n : \|x_n - w\| \ge \|z_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 0, \end{cases}$$

where the control sequences $\{\alpha_n\}$, $\{\eta_n\}$, $\{\mu_{n,m}\}$ and $\{\varphi_{n,m}\}$ satisfy the following restrictions: $\alpha_n \in [a, 1)$ with $a \in \mathbb{R}$ being some constant, $\{\eta_n\}$ is a real position sequence such that $\liminf_{n\to\infty} \eta_n > 0$, $0 < d \le \mu_{n,m} \le e < 2\nu_m$. Then the sequence $\{x_n\}$ generated above converges to $P_\Phi x_1$ in norm.

Proof. From Lemma 2.1, one has that $Sol(B)$ is convex and closed. From Lemma 2.2, one has that $(A_m + B_m)^{-1}(0))$ is convex and closed for each $1 \le m \le N$. Hence, one concludes that Φ is a convex and closed set so that the projection is well-defined.

Next, one shows that each C_n is convex and closed. From the construction of C_n, one easily sees that each C_n is closed. For the convexity, one observes that $C_1 = H$ is convex. One supposes that for some positive integer i, C_i is convex. One next only needs to C_{i+1} is convex. Indeed, $\|x_n - w\| \le \|z_n - w\|$ is equivalent to $2\langle z_n - x_n, w\rangle \le \|z_n\|^2 - \|x_n\|^2$. Letting w_1 and w_2 be two vectors in C_{i+1}, one next shows that $\bar{w} = (1-t)w_1 + tw_2$, where t is a real in $(0, 1)$, is in C_{i+1}. In view of the construction of C_{i+1}, one has $w_1 \in C_i$, $w_2 \in C_i$, $\|x_n - w_1\| \le \|z_n - w_1\|$, and $\|x_n - w_2\| \le \|z_n - w_2\|$, that is, $2\langle z_n - x_n, w_1\rangle \le \|z_n\|^2 - \|x_n\|^2$ and $2\langle z_n - x_n, w_2\rangle \le \|z_n\|^2 - \|x_n\|^2$. It follows that $2\langle z_n - x_n, \bar{w}\rangle \le \|z_n\|^2 - \|x_n\|^2$, which implies that $\bar{w} \in C_{i+1}$. This proves the convexity of C_{i+1}. Therefore, C_n is convex and closed so that the nearest projection on it is well-defined.

One next shows that the solution set lies in C_n. It is clear $\Phi \subset C_1 = H$. Let $\Phi \subset C_i$. Next, for the same i, one shows that $\Phi \subset C_{i+1}$. For each $p \in \Phi \subset C_i$, one sees from Lemma 2.1 that

$$\|y_i - p\| = \|Res_{\eta_i}^{B} x_i - Res_{\eta_i}^{B} p\| \leq \|x_i - p\|.$$

Since each A_m is a ν_m-inverse-strongly monotone, one sees that $I - \mu_{n,m}A_m$ is nonexpansive. Indeed,

$$\begin{aligned}
&\|(I - \mu_{n,m}A_m)x - (I - \mu_{n,m}A_m)y\|^2 \\
&= \|x - y\|^2 + \mu_{n,m}^2\|A_m x - A_m y\|^2 - 2\mu_{n,m}\langle x - y, A_m x - A_m y\rangle \\
&\leq \|x - y\|^2 - \mu_{n,m}(2\nu_m - \mu_{n,m})\|A_m x - A_m y\|^2 \\
&\leq \|x - y\|^2, \quad \forall x, y \in C,
\end{aligned}$$

that is, $I - \mu_{n,m}A_m$ is nonexpansive. Setting $e_n := \sum_{m=1}^{N} \varphi_{n,m} t_{n,m}$, where

$$t_{n,m} = Res_{\mu_{n,m}}^{B_m}(I - \mu_{n,m}A_m)y_n,$$

one has

$$\begin{aligned}
\|e_i - p\| &= \|\sum_{m=1}^{N} \varphi_{i,m} Res_{\mu_{i,m}}^{B_m}(I - \mu_{i,m}A_m)y_i - \sum_{m=1}^{N} \varphi_{i,m} Res_{\mu_{i,m}}^{B_m}(I - \mu_{i,m}A_m)p\| \\
&\leq \sum_{m=1}^{N} \varphi_{i,m}\|Res_{\mu_{i,m}}^{B_m}(I - \mu_{i,m}A_m)y_i - Res_{\mu_{i,m}}^{B_m}(I - \mu_{i,m}A_m)p\| \\
&\leq \|x_i - p\|.
\end{aligned}$$

This implies

$$\|x_i - p\| \geq \alpha_i\|e_i - p\| + (1 - \alpha_i)\|x_i - p\| \geq \|z_i - p\|.$$

This indicates that $\Phi \subset C_n$. From the construction of $\{x_n\}$, one sees $x_n = P_{C_n}x_1$. In view of $\Phi \subset C_n$, one concludes that $\|x_1 - x_n\| \leq \|x_1 - p\|$, $\forall p \in \Phi$. Fixing a special vector in Φ, one obtains that

$$\|x_1 - x_n\| \leq \|x_1 - P_\Phi x_1\|.$$

This clearly shows that $\{x_n\}$ is a bounded vector sequence.

Since the framework of space is Hilbert, one concludes that there exists a subsequence of sequence $\{x_n\}$, $\{x_{n_j}\}$, which converges to x^* weakly. Observe that

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&= 2\langle x_{n+1} + x_n - x_n - x_1, x_1 - x_n\rangle + \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 \\
&= 2\langle x_{n+1} - x_n, x_1 - x_n\rangle + \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2
\end{aligned}$$

due to $x_{n+1} \in C_n$ and $x_n = P_{C_n}x_1$. In view of this, one has $\lim_{n\to\infty}\|x_n - x_{n+1}\|^2 = 0$ since $\lim_{n\to\infty}\|x_n - x_1\|^2$ exists. Thanks to $x_{n+1} \in C_{n+1}$, one obtains

$$\|x_n - x_{n+1}\| \geq \|z_n - x_{n+1}\|,$$

which yields that $\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0$. This further shows that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Fixing $p \in \Phi$, one gets from Lemma 2.1 that

$$\begin{aligned} 2\|y_n - p\|^2 &= 2\|Res^B_{\eta_n}x_n - Res^B_{\eta_n}p\|^2 \\ &\le 2\langle x_n - p, y_n - p\rangle \\ &= \|x_n - p\|^2 - \|y_n - x_n\|^2 + \|y_n - p\|^2, \end{aligned}$$

which indicates $\|y_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - x_n\|^2$. Observe that

$$\begin{aligned} \|z_n - p\|^2 &\le \alpha_n\|e_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|t_{n,m} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n\|y_n - x_n\|^2. \end{aligned}$$

It follows that

$$\alpha_n\|y_n - x_n\|^2 \le M\|x_n - z_n\|,$$

where $M \ge \max\{\|x_n - p\| + \|z_n - p\|\}$ is some real constant. This indicates that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Observe that

$$\begin{aligned} &\|(I - \mu_{n,m}A_m)y_n - (I - \mu_{n,m}A_m)p\|^2 \\ &= \mu_{n,m}^2\|A_m y_n - A_m p\|^2 - 2\mu_{n,m}\langle y_n - p, A_m y_n - A_m p\rangle + \|y_n - p\|^2 \\ &\le \|y_n - p\|^2 - (2\nu_m - \mu_{n,m})\mu_{n,m}\|A_m y_n - A_m p\|^2 \\ &\le \|x_n - p\|^2 - (2\nu_m - \mu_{n,m})\mu_{n,m}\|A_m y_n - A_m p\|^2, \quad \forall 1 \le m \le N. \end{aligned}$$

It follows that

$$\begin{aligned} &\|z_n - p\|^2 \\ &\le \alpha_n\|e_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|t_{n,m} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|(I - \mu_{n,m}A_m)y_n - (I - \mu_{n,m}A_m)p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 - \alpha_n \sum_{m=1}^{N} \varphi_{n,m}(2\nu_m - \mu_{n,m})\mu_{n,m}\|A_m y_n - A_m p\|^2. \end{aligned}$$

Thus

$$\alpha_n \sum_{m=1}^{N} \varphi_{n,m}(2\nu_m - \mu_{n,m})\mu_{n,m}\|A_m y_n - A_m p\|^2 \le \|x_n - p\|^2 - \|z_n - p\|^2.$$

From the restriction on $\{\mu_{n,m}\}$, one arrives at

$$\lim_{n\to\infty} \|A_m y_n - A_m p\| = 0, \quad \forall 1 \le m \le N.$$

Observe that

$$\begin{aligned}
&2\|t_{n,m} - p\|^2\\
&\le 2\langle (I - \mu_{n,m}A_m)y_n - (I - \mu_{n,m}A_m)p, t_{n,m} - p\rangle\\
&= \|(I - \mu_{n,m}A_m)y_n - (I - \mu_{n,m}A_m)p\|^2 + \|t_{n,m} - p\|^2\\
&\quad - \|(t_{n,m} - p) - (I - \mu_{n,m}A_m)y_n + (I - \mu_{n,m}A_m)p\|^2\\
&\le \|y_n - p\|^2 + \|t_{n,m} - p\|^2 - \|t_{n,m} + \mu_{n,m}(A_m y_n - A_m p) - y_n\|^2\\
&\le \|y_n - p\|^2 + \|t_{n,m} - p\|^2 - \|t_{n,m} - y_n\|^2 - \mu_{n,m}^2\|A_m y_n - A_m p\|^2\\
&\quad + 2\mu_{n,m}\|t_{n,m} - y_n\|\|A_m y_n - A_m p\|\\
&\le \|y_n - p\|^2 + \|t_{n,m} - p\|^2 - \|t_{n,m} - y_n\|^2 + 2\mu_{n,m}\|t_{n,m} - y_n\|\|A_m y_n - A_m p\|.
\end{aligned}$$

Hence,

$$\|t_{n,m} - p\|^2 \le \|y_n - p\|^2 - \|t_{n,m} - y_n\|^2 + 2\mu_{n,m}\|t_{n,m} - y_n\|\|A_m y_n - A_m p\|.$$

It follows that

$$\begin{aligned}
\|z_n - p\|^2 &\le \alpha_n\|e_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2\\
&\le \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|t_{n,m} - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2\\
&\le \|x_n - p\|^2 - \alpha_n \sum_{m=1}^{N} \varphi_{n,m}\|t_{n,m} - y_n\|^2\\
&\quad + 2\alpha_n \sum_{m=1}^{N} \varphi_{n,m}\mu_{n,m}\|t_{n,m} - y_n\|\|A_m y_n - A_m p\|
\end{aligned}$$

This yields that $\lim_{n\to\infty} \|t_{n,m} - y_n\| = 0$, $\forall 1 \le m \le N$ and $\lim_{n\to\infty} \|e_n - y_n\| = 0$. Observe that

$$\langle \lambda - y_n, y_n - x_n\rangle + \eta_n B(y_n, \lambda) \ge 0, \quad \forall \lambda \in C_n.$$

By use of restriction (R2), one sees that

$$\langle \lambda - y_n, y_n - x_n\rangle \ge \eta_n B(\lambda, y_n), \quad \forall \lambda \in C_n.$$

By replacing n by n_j, one concludes from restriction (R4) that $B(\lambda, x^*) \le 0$, $\forall \lambda \in C_{n_j}$. Observe that $c\lambda + (1-c)x^* \in C_{n_j}$, where c is a real constant in $(0, 1)$. So,

$$B(c\lambda + (1 - c)x^*, x^*) \le 0.$$

This further concludes that

$$\begin{aligned}0 &= B(c\lambda + (1-c)x^*, c\lambda + (1-c)x^*)\\ &\leq cB(c\lambda + (1-c)x^*, \lambda) + (1-c)B(c\lambda + (1-c)x^*, x^*)\\ &\leq cB(c\lambda + (1-c)x^*, \lambda),\end{aligned}$$

that is, $B(c\lambda + (1-c)x^*, \lambda) \geq 0$, $\forall \lambda \in C_{n_j}$. Letting $c \downarrow 0$ and using restriction (R3), one asserts that $B(x^*, \lambda) \geq 0$, $\forall \lambda \in C_{n_j}$. This yields that $x^* \in Sol(B)$.

Now, one is in a position to show that $x^* \in (A_m + B_m)^{-1}(0)$. Since $t_{n,m} = (I + \mu_{n,m} B_m)^{-1}(I - \mu_{n,m} A_m) y_n$, one finds that

$$\frac{y_n - t_{n,m}}{\mu_{n,m}} - A_m y_n \in B_m t_{n,m}.$$

Note that B_m is monotone for each m. For any $(\psi_m, \omega_m) \in Graph(B_m)$, one asserts that

$$\langle t_{n,m} - \psi_m, \frac{y_n - t_{n,m}}{\mu_{n,m}} - A_m y_n - \omega_m \rangle \geq 0.$$

Since both $\{y_{n_j}\}$ and $\{t_{n_j,m}\}$ converge to x^* weakly, one has

$$\langle x^* - \psi_m, -A_m x^* - \omega_m \rangle \geq 0.$$

This indicates $-A_m x^* \in B_m x^*$, that is,

$$x^* \in (A_m + B_m)^{-1}(0).$$

Finally, one proves that $\{x_n\}$ converges strongly to $P_\Phi x_1$. Observe that

$$\|x_1 - P_\Phi x_1\| \leq \|x_1 - x^*\| \leq \liminf_{j\to\infty} \|x_1 - x_{n_j}\| \leq \limsup_{j\to\infty} \|x_1 - x_{n_j}\| \leq \|x_1 - P_\Phi x_1\|,$$

which yields that

$$\lim_{j\to\infty} \|x_1 - x_{n_j}\| = \|x_1 - x^*\| = \|x_1 - P_\Phi x_1\|.$$

In view of the fact that H is a Hilbert space, one concludes that $\{x_{n_j}\}$ converges to $P_\Phi x_1$ in norm. Therefore, $\{x_n\}$ converges to $P_\Phi x_1$ in norm. The proof is completed. □

Let $f : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Let ∂f be the subdifferential of function f, which is defined by

$$\partial f(x) = \{y \in H : f(z) - f(x) \geq \langle z - x, y \rangle, \quad z \in H\}, \quad \forall x \in H.$$

Rockafellar [15, 16] proved that the subdifferential ∂f is a maximal monotone mapping and $0 \in \partial f(x)$ iff $f(x) = \min_{y\in H} f(y)$. Define the indicator function I_C of set C by

$$I_C(x) = \begin{cases} 0, & x \in C,\\ +\infty, & x \notin C.\end{cases}$$

Since I_C is proper, lower semicontinuous and convex on H, one finds that the subdifferential ∂I_C of I_C is set-valued maximal monotone. For the maximal monotone mapping, one can define its resolvent mapping $Res_{\mu}^{I_C}$, i.e., $Res_{\mu}^{I_C} := (I + \mu\partial I_C)^{-1}$. Letting $x = Res_{\mu}^{I_C} y$, we find that

$$\begin{aligned} y \in x + \mu\partial I_C x &\Longleftrightarrow y \in x + \mu N_C x \\ &\Longleftrightarrow \langle y - x, z - x\rangle \le 0, \forall z \in C \\ &\Longleftrightarrow x = P_C y, \end{aligned}$$

where

$$N_C x := \{e \in H : \langle e, z - x\rangle, \forall z \in C\}.$$

From Theorem 3.1, one has the following result on the variational inequality and the equilibrium problem.

Corollary 3.2. *Let C be a closed and convex nonempty subset of a real Hilbert space and let N be some positive integer. Let B be a bifunction with restrictions (R1)-(R4). Let $B_m : C \to H$ be a maximal monotone mapping and let $A_m : C \to H$ be a ν_m-inverse-strongly monotone mapping for all $1 \le m \le N$. Assume that $\Phi := \cap_{m=1}^{N} VI(C, A_m) \cap Sol(B) \ne \emptyset$ and $\{x_n\}$ is a vector sequence defined by*

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ \langle \lambda - y_n, y_n - x_n\rangle + \eta_n B(y_n, \lambda) \ge 0, \quad \forall \lambda \in C_n, \\ z_n = \alpha_n \sum_{m=1}^{N} \varphi_{n,m} P_C(I - \mu_{n,m}A_m)y_n + (1 - \alpha_n)x_n, \\ C_{n+1} = \{w \in C_n : \|x_n - w\| \ge \|z_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where the control sequences $\{\alpha_n\}$, $\{\eta_n\}$, $\{\mu_{n,m}\}$ and $\{\varphi_{n,m}\}$ satisfy the following restrictions: $\alpha_n \in [a, 1)$ with $a \in \mathbb{R}$ being some constant, $\{\eta_n\}$ is real position sequence such that $\liminf_{n\to\infty} \eta_n > 0$, $0 < d \le \mu_{n,m} \le e < 2\nu_m$. Then the sequence $\{x_n\}$ generated above converges to $P_\Phi x_1$ in norm.

If $B(x, y) \equiv 0$, the following result is not hard to derive.

Corollary 3.3. *Let C be a closed and convex nonempty subset of a real Hilbert space and let N be some positive integer. Let $B_m : C \to H$ be a maximal monotone mapping and let $A_m : C \to H$ be a ν_m-inverse-strongly monotone mapping for all $1 \le m \le N$ with $\Phi := \cap_{m=1}^{N} (A_m + B_m)^{-1}(0) \ne \emptyset$. Let $\{x_n\}$ be a vector sequence defined by*

$$\begin{cases} x_1 \in C, C_1 = H, \\ z_n = \alpha_n \sum_{m=1}^{N} \varphi_{n,m} Res_{\mu_{n,m}}^{B_m} (I - \mu_{n,m}A_m)x_n + (1 - \alpha_n)x_n, \\ C_{n+1} = \{w \in C_n : \|x_n - w\| \ge \|z_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where the control sequences $\{\alpha_n\}$, $\{\mu_{n,m}\}$ and $\{\varphi_{n,m}\}$ satisfy the following restrictions: $\alpha_n \in [a, 1)$ with $a \in \mathbb{R}$ being some constant, $0 < d \le \mu_{n,m} \le e < 2\nu_m$. Then the sequence $\{x_n\}$ generated above converges to $P_\Phi x_1$ in norm.

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