East Asian Math. J.
Vol. 36 (2020), No. 5, pp. 555–560
http://dx.doi.org/10.7858/eamj.2020.037

A NEW COMBINATION THEOREM FOR RELATIVELY HYPERBOLIC GROUPS

Jan Kim and Donghi Lee*

Abstract. We prove a new combination theorem for relatively hyperbolic groups by analyzing diagrams over HNN-extensions of relatively hyperbolic groups.

1. Introduction

We recall Osin's definition [3] of relatively hyperbolic groups among many equivalent definitions of relatively hyperbolic groups. Let G be a group, $\mathbb{H} = \{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, X a subset of G. Suppose that X is a relative generating set for $(G, \mathbb{H})$, namely, G is generated by the set $\big(\bigcup_{\lambda\in\Lambda} H_\lambda\big) \cup X$ (for convenience, we assume that $X = X^{-1}$). Then G can be regarded as the quotient group of the free product

$$F = (*_{\lambda\in\Lambda}\tilde{H}_\lambda) * F(X),$$

where the groups $\tilde{H}_\lambda$ are isomorphic copies of H_λ, and $F(X)$ is the free group generated by X. Let $\mathcal{H}$ be the disjoint union

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} (\tilde{H}_\lambda \setminus \{1\}). \tag{1}$$

For every $\lambda \in \Lambda$, we denote by S_λ the set of all words over the alphabet $\tilde{H}_\lambda \setminus \{1\}$ that represent the identity in F. Then we may describe G as a *relative presentation*

$$\langle X, \mathcal{H} \,|\, S_\lambda, \lambda \in \Lambda, \mathcal{R}\rangle \tag{2}$$

with respect to the collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$, where $\mathcal{R} \subseteq F$. For brevity, we often use the following shorthand for presentation (2):

$$\langle X, \mathbb{H} \,|\, \mathcal{R}\rangle. \tag{3}$$

Received June 19, 2020; Accepted August 17, 2020.
2010 *Mathematics Subject Classification.* 20F06.
Key words and phrases. relatively hyperbolic group, HNN-extension, combination theorem.
The second author was supported by a 2-Year Research Grant of Pusan National University.
*Corresponding author.

©2020 The Youngnam Mathematical Society
(pISSN 1226-6973, eISSN 2287-2833)

If both the sets $\mathcal{R}$ and X are finite, relative presentation (2) or (3) is said to be *finite* and the group G is said to be *finitely presented relative to the collection of subgroups* $\mathbb{H}$.

For every word w in the alphabet $X \cup \mathcal{H}$ representing the identity in the group G, there exists an expression

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i f_i \tag{4}$$

with the equality in the group F, where $R_i \in \mathcal{R}$ and $f_i \in F$ for $i = 1, \ldots, k$. The smallest possible number k in a presentation of the form (4) is called the *relative area* of w and is denoted by $Area^{rel}(w)$.

Definition 1. A group G is said to be *hyperbolic relative to a collection of subgroups* $\mathbb{H}$ if G admits a relatively finite presentation (2) with respect to $\mathbb{H}$ satisfying a *linear relative isoperimetric inequality.* That is, there is a constant $C > 0$ such that for any cyclically reduced word w in the alphabet $X \cup \mathcal{H}$ representing the identity in G, we have

$$Area^{rel}(w) \leq C|w|_{X\cup\mathcal{H}},$$

where the symbol $|w|_{X\cup\mathcal{H}}$ means the word length of w over $X\cup\mathcal{H}$. The constant C is called an *isoperimetric constant* of relative presentation (2).

Let G be a group that is hyperbolic relative to a collection of subgroups $\mathbb{H} = \{H_\lambda\}_{\lambda\in\Lambda}$. Suppose that there exists a monomorphism $\iota : H_\mu \to H_\nu$ for some $\mu, \nu \in \Lambda$. Osin [4] proved that if $\mu \neq \nu$ and H_μ is finitely generated, then the HNN-extension

$$G^* = \langle G, t \,|\, t^{-1}ht = \iota(h),\ h \in H_\mu\rangle$$

is hyperbolic relative to $\mathbb{H} \setminus \{H_\mu\}$. Our new combination theorem covers the case when $\mu = \nu$, and is stated as follows. (cf. For finitely generated groups, a similar result was obtained by Dahmani [1].)

Theorem 1.1. *Suppose that a group G is hyperbolic relative to a collection of subgroups $\mathbb{H} = \{H_\lambda\}_{\lambda\in\Lambda}$. Assume in addition that there exists a monomorphism $\iota : K_\mu \to H_\mu$ for some $\mu \in \Lambda$, where K_μ is a subgroup of H_μ and it need not be finitely generated. Then the HNN-extension*

$$G^* = \langle G, t \,|\, t^{-1}kt = \iota(k),\ k \in K_\mu\rangle \tag{5}$$

is hyperbolic relative to the collection $\mathbb{H}\setminus\{H_\mu\}\cup\{H_\mu^*\}$, where $H_\mu^* = \langle H_\mu, t\rangle \leq G^*$.

In particular, if $G = \langle X, \mathbb{H} \,|\, \mathcal{R}\rangle$ is a relative presentation of G with respect to the collection of subgroups $\mathbb{H}$, then $G^* = \langle X, \mathbb{H} \setminus \{H_\mu\} \cup \{H_\mu^*\} \,|\, \mathcal{R}\rangle$, and these two relative presentations have the same isoperimetric constant.

As an immediate corollary, we obtain

Corollary 1.2. *Suppose that a group* $G = \langle X, H_\lambda, \lambda \in \Lambda \,|\, \mathcal{R}\rangle$ *is hyperbolic relative to* $\{H_\lambda\}_{\lambda\in\Lambda}$. *Then the group* $G^\bullet$ *defined by a relative presentation* $G^\bullet = \langle X, H^\bullet_\lambda, \lambda \in \Lambda \,|\, \mathcal{R}\rangle$ *is hyperbolic relative to* $\{H^\bullet_\lambda\}_{\lambda\in\Lambda}$, *where* $H^\bullet_\lambda \cong H_\lambda \times A_\lambda$ *for some (finitely or infinitely generated) free abelian group* A_λ *for each* $\lambda \in \Lambda$.

2. Proof of Theorem 1.1

A word in an alphabet is called *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (w), we denote the cyclic word associated with a cyclically reduced word w. Also, by $(u) \equiv (w)$, we mean the *visual equality* of two cyclic words (u) and (w). For other terminology and notation used throughout this section, we refer the reader to [4, Sections 2 and 3].

Let us fix a finite relative presentation

$$G = \langle X, H_\lambda, \lambda \in \Lambda \,|\, \mathcal{R}\rangle \tag{6}$$

of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. Clearly HNN-extension (5) has a finite relative presentation

$$G^* = \langle X, H^*_\mu, H_\lambda, \lambda \in \Lambda \setminus \{\mu\} \,|\, \mathcal{R}\rangle \tag{7}$$

in view of shorthand (3).

For $\mathcal{H}$ defined as in (1), let

$$\mathcal{H}^* = \mathcal{H} \setminus (\tilde{H}_\mu \setminus \{1\}) \sqcup (\tilde{H}^*_\mu \setminus \{1\}),$$

where $\tilde{H}^*_\mu$ is an isomorphic copy of H^*_μ. Also let w be a cyclically reduced word in the alphabet $X \cup \mathcal{H}^*$ such that w represents the identity in G^*. We use the symbol $\|w\|$ to mean the word length of w over $X \cup \mathcal{H}^*$. Not only for w but also for any element in G^*, the symbol $\|\cdot\|$ will be used to mean its word length over $X \cup \mathcal{H}^*$. Let C be an isoperimetric constant of relative presentation (6). Then we will show that

$$Area^{rel}(w) \le C\|w\|. \tag{8}$$

By van Kampen's Lemma, there is a reduced van Kampen diagram Δ over presentation (7) such that a boundary label of Δ is visually equal to w (cf. [2]). In particular, we can take Δ so that Δ has the least number of $\mathcal{R}$-cells among all van Kampen diagrams over (7) with a boundary label w. If $Area^{rel}(\Delta)$ denotes the number of $\mathcal{R}$-cells in Δ, this implies that $Area^{rel}(w) = Area^{rel}(\Delta)$. So in order to show (8), it suffices to show

$$Area^{rel}(\Delta) \le C\|w\|. \tag{9}$$

A cell in a diagram over presentation (7) is called a *t-cell* if it corresponds to a relation of the form $t^{-1}kt = \iota(k)$, where $k \in K_\mu$. They are shown on Figure 1(a). A configuration of t-cells, as shown on Figure 1(b), we call a *t-annulus*.

Claim. *We may assume that* Δ *does not contain a* t*-annulus.*

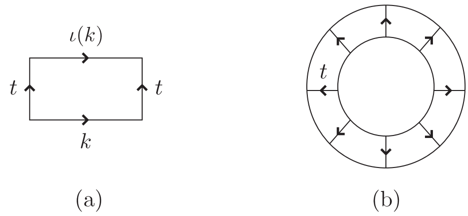

FIGURE 1.

Proof of Claim. Suppose that Δ contains a t-annulus. Take an innermost t-annulus T in Δ, meaning that there is not another t-annulus inside T. Then the label of the internal contour p of T represents the identity in H_μ. Since ι is a monomorphism, the label of the external contour q of T also represents the identity in H_μ. This implies that the circular subdiagram, say D, bounded by the contour p consists of only H_μ-cells and that we may replace $D \sqcup T$ with D' consisting of only H_μ-cells with the contour q. By repeating this process to remove all t-annuli from Δ, we obtain a new van Kampen diagram Δ' such that $(\mathbf{Lab}(\partial\Delta')) \equiv (\mathbf{Lab}(\partial\Delta))$ and $Area^{rel}(\Delta') = Area^{rel}(\Delta)$ (see Figure 2), where **Lab** is a labeling function. Hence we may rename Δ' as Δ. □

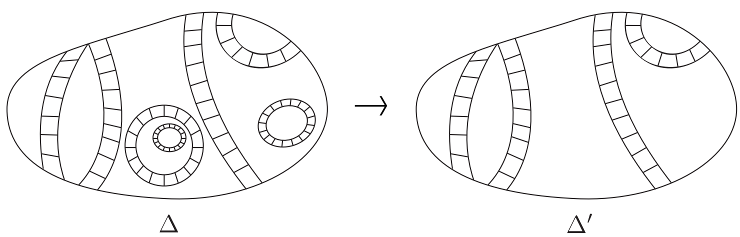

FIGURE 2. Removing all t-annuli from Δ

By Claim, t-cells can only form t-strips, and these t-strips must end on the boundary of Δ. To show inequality (9), we proceed by induction on the number of t-strips in Δ. If there is no t-strip in Δ, then Δ is a van Kampen diagram over (6), and hence (9) holds.

Now assume that Δ contains at least one t-strip. Take any t-strip, say T, in Δ. Let Δ_1 and Δ_2 be the subdiagrams lying in the left and right of T, respectively, so that $\Delta = \Delta_1 \sqcup T \sqcup \Delta_2$ (see Figure 3).

Clearly t-cells belong to H_μ^*-cells, so they are not counted in $Area^{rel}(\Delta)$. Hence

$$Area^{rel}(\Delta) = Area^{rel}(\Delta_1) + Area^{rel}(\Delta_2). \tag{10}$$

Moreover, for each $i = 1, 2$, note that Δ_i is a van Kampen diagram over (7) which has the smallest relative area among all van Kampen diagrams over (7)

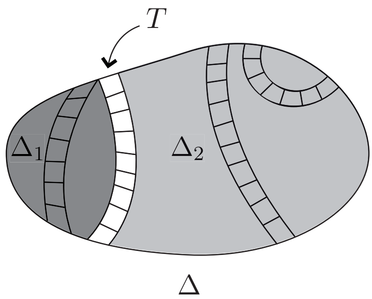

FIGURE 3. $\Delta = \Delta_1 \sqcup T \sqcup \Delta_2$

with the same boundary label as Δ_i. Then by the induction hypothesis, we have

$$Area^{rel}(\Delta_i) \leq C\|\mathbf{Lab}(\partial\Delta_i)\| \tag{11}$$

for all $i = 1, 2$.

Let $(\mathbf{Lab}(\partial\Delta_i)) \equiv (w_i k_i)$, where w_i is a reduced word over $X \cup \mathcal{H}^*$ and k_i is a reduced word over $\tilde{H}_\mu^*$ for all $i = 1, 2$, so that $(w) \equiv (\mathbf{Lab}(\partial\Delta)) \equiv (w_1 t^{\pm 1} w_2 t^{\mp 1})$ (see Figure 4).

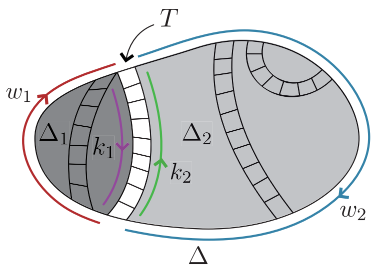

FIGURE 4. $(w) \equiv (\mathbf{Lab}(\partial\Delta)) \equiv (w_1 t^{\pm 1} w_2 t^{\mp 1})$

Put $w_i \equiv w_{ib}\bar{w}_i w_{ie}$, where $\|w_{ib}\| = \|w_{ie}\| = 1$ for all $i = 1, 2$. Note that for each $i = 1, 2$,

$$\begin{cases} \|\mathbf{Lab}(\partial\Delta_i)\| = \|w_i\| - 1 & \text{if both } w_{ib} \text{ and } w_{ie} \text{ belong to } \tilde{H}_\mu^*; \\ \|\mathbf{Lab}(\partial\Delta_i)\| = \|w_i\| & \text{if either } w_{ib} \text{ or } w_{ie} \text{ but not both belongs to } \tilde{H}_\mu^*; \\ \|\mathbf{Lab}(\partial\Delta_i)\| = \|w_i\| + 1 & \text{if neither } w_{ib} \text{ nor } w_{ie} \text{ belongs to } \tilde{H}_\mu^*. \end{cases}$$

Let $Y = \{w_{1b}, w_{1e}, w_{2b}, w_{2e}\} \cap \tilde{H}^*_\mu$. It then follows that

$$\begin{cases} \|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w_1\| + \|w_2\| - 2 & \text{if } |Y| = 4; \\ \|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w_1\| + \|w_2\| - 1 & \text{if } |Y| = 3; \\ \|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w_1\| + \|w_2\| & \text{if } |Y| = 2; \\ \|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w_1\| + \|w_2\| + 1 & \text{if } |Y| = 1; \\ \|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w_1\| + \|w_2\| + 2 & \text{if } |Y| = 0. \end{cases}$$

In view of $(w) \equiv (\mathbf{Lab}(\partial\Delta)) \equiv (w_1 t^{\pm 1} w_2 t^{\mp 1})$, note also that

$$\begin{cases} \|w\| = \|w_1\| + \|w_2\| - 2 & \text{if } |Y| = 4; \\ \|w\| = \|w_1\| + \|w_2\| - 1 & \text{if } |Y| = 3; \\ \|w\| = \|w_1\| + \|w_2\| & \text{if } |Y| = 2; \\ \|w\| = \|w_1\| + \|w_2\| + 1 & \text{if } |Y| = 1; \\ \|w\| = \|w_1\| + \|w_2\| + 2 & \text{if } |Y| = 0. \end{cases}$$

Therefore, in any of five cases, we have

$$\|\mathbf{Lab}(\partial\Delta_1)\| + \|\mathbf{Lab}(\partial\Delta_2)\| = \|w\|.$$

This together with (10) and (11) finally yields (9), which completes the proof of Theorem 1.1.

Acknowledgement

The authors are heartily grateful to an anonymous referee for very careful reading and valuable remarks.

References

[1] F. Dahmani, *Combination of convergence groups*, Geom. and Topol. **7** (2003), 933–963.
[2] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977.
[3] D. Osin, *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems*, Memoirs Amer. Math. Soc. **179** (2006), no. 843, vi+100 pp.
[4] D. Osin, *Relative Dehn funtions of amalgamated products and HNN-extensions*, Contemp. Math. **394** (2006), 209–220.

Jan Kim
Department of Mathematics
Pusan National University
San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Korea
E-mail address: jankim@pusan.ac.kr

Donghi Lee
Department of Mathematics
Pusan National University
San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Korea
E-mail address: donghi@pusan.ac.kr