East Asian Math. J.
Vol. 36 (2020), No. 5, pp. 537–545
http://dx.doi.org/10.7858/eamj.2020.035

POLYNOMIAL INVARIANTS FOR VIRTUAL KNOTS VIA VIRTUALIZATION MOVES

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Abstract. We investigate some polynomial invariants for virtual knots via virtualization moves. We define two types of polynomials $W_G(t)$ and $S^2_G(t)$ from the definition of the index polynomial for virtual knots. And we show that they are invariants for virtual knots on the quotient ring $\mathbf{Z}[t^{\pm 1}]/I$ where I is an ideal generated by $t^2 - 1$.

1. Introduction

In [5], Kauffman introduced virtual knot theory as a generalization of classical knot theory in the sense that if two classical link diagrams are equivalent as virtual links, then they are equivalent as classical links. A virtual link diagram is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information, called virtual crossings. A virtual link is an equivalence class of such link diagrams by the generalized Reidemeister moves, which consist of (classical) Reidemeister moves of type R_1, R_2 and R_3 and virtual Reidemeister moves of type VR_1, VR_2, VR_3 and the *semivirtual move* VR_4 as shown in Figure 1.

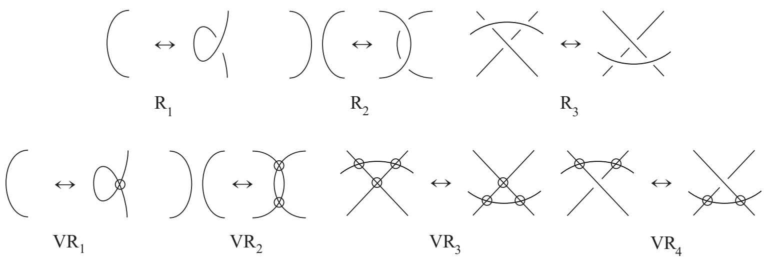

Figure 1. Generalized Reidemeister moves

Received May 29, 2020; Accepted July 22, 2020.
2010 *Mathematics Subject Classification.* Primary 57M25. Secondary 57M27.
Key words and phrases. virtual knot; Gauss diagram; index polynomial; way virtualization; sign virtualization.
This work was supported by a 2-Year Research Grant of Pusan National University.
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(pISSN 1226-6973, eISSN 2287-2833)

And we consider the following oriented Reidemeister moves instead of the classical Reidemeister moves to prove the main result. In [6], Polyak proved all oriented Reidemeister moves are generated by the following four oriented Reidemeister moves I_a, I_b, II_a, III_a shown in Figure 2.

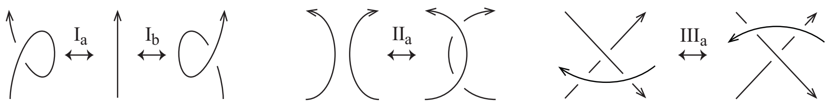

FIGURE 2. Oriented Reidemeister moves for virtual knot diagrams

Let us investigate the corresponding Reidemeister moves I_a, I_b, II_a, III_a and $III_{a'}$ in Gauss diagrams shown in Figure 3. Using these moves we can find a proper ideal to define polynomial invariants for virtual knots.

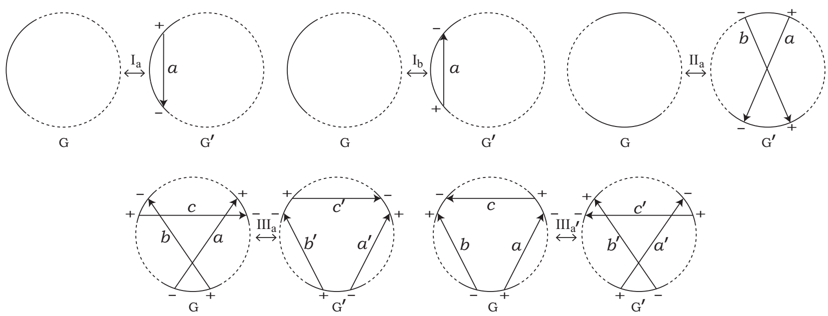

FIGURE 3. Reidemeister moves I_a, I_b, II_a, III_a and $III_{a'}$ for Gauss diagrams

In [5], Kauffman showed the bracket polynomial is preserved under the virtualization moves which are not invariants for virtual knots. For example, the bracket polynomial could not detected the difference between the virtual trefoil knot K and the other virtual knot K' via a virtualization move at a real crossing of K. But it is known that K and K' are not equivalent by the index polynomial which is an invariant for virtual knots. In [4], the index polynomial for checkerboard colorable flat virtual knots are elements in $\mathbf{Z}[x^2]$ and we showed the Miyazawa knot and its flat virtual knot diagram are not checkerboard colorable by applying a virtualization move at a flat crossing of the flat virtual knot diagram. For the next step, we investigate the virtualization moves and polynomial invariants for virtual knots.

This paper is organized as follows. Section 2 reviews some basic definitions and properties for virtual knots. We introduce the definition of the index polynomial and two types of virtualization moves. In Section 3, we consider polynomials from two virtualization moves and prove these polynomials are invariants on the quotient ring $\mathbf{Z}[t^{\pm 1}]/I$ where I is an ideal generated by $t^2 - 1$.

2. Preliminaries

We begin this section with some definitions and results which are needed throughout this paper.

Let D be a virtual knot diagram with m real crossings. A *Gauss diagram* G is a counter-clockwise oriented circle S^1 with m chords joining each pair of points corresponding to each real crossing of D. Each classical knot diagram has a corresponding Gauss diagram, but the converse is not true. We have the following correspondence between virtual knot diagrams and Gauss diagrams.

Theorem 2.1. [2] *A Gauss diagram uniquely defines a virtual knot isotopy class.*

Let G be a Gauss diagram, c be a chord of G and D be a corresponding virtual knot diagram. Since the preimages of the overcrossing and the undercrossing of D are connected by a chord directed from the preimage of the overcrossing to the preimage of the undercrossing in a circle with an counter-clockwise orientation, we assign a sign to each chord according to the sign of the corresponding real crossing of D. For each chord c of G, we assign the signs of endpoints of the chord c as shown in Figure 4.

If $R(c)$ is the collection of endpoints except those of c on the *right side of the chord* c, then the *intersection index of* c denoted by $i(c)$ is the sum of signs of endpoints in $R(c)$.

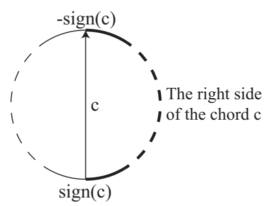

FIGURE 4. The sign of endpoints of c

Definition 1. Let D be a virtual knot diagram and G be a Gauss diagram of D. We define the index polynomial for G as $f_G(t) = \sum_{c \in C(G)} sign(c)(t^{i(c)} - 1)$ where $C(G)$ is the set of chords of G.

It is known that the index polynomial $f_G(t)$ is an invariant for Gauss diagrams. In fact, the polynomial $f_G(t)$ is same as the index polynomial $f_D(t)$ for D.

From now on, we introduce two types of virtualization moves at a real crossing of a virtual knot diagram in [1]. Let D be a virtual knot diagram and c be a real crossing of D. Then there are two virtualization moves at c as shown in Figure 5. For convenience, the corresponding chord of c in a Gauss diagram G of D is denoted by the same letter c. The Gauss diagram G_c^* is obtained from G by the *way virtualization move* at a chord c of G in Figure 5. Similarly, the Gauss diagram obtained by the *sign virtualization move* at a chord c as the following figure is denoted by G_c°.

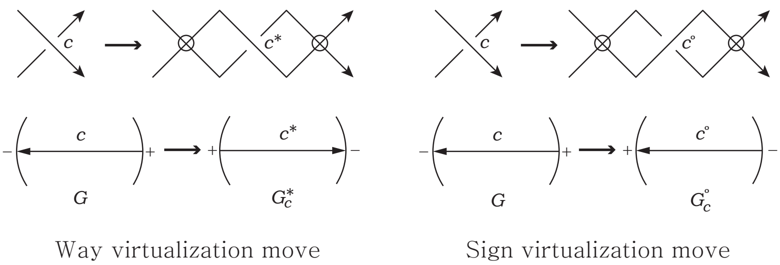

Figure 5. Virtualization moves

3. Two types of polynomials via virtualization moves

For each chord c of a Gauss diagram G, we define polynomials with Gauss diagrams G_c^* and G_c° and consider invariants for virtual knot diagrams.

Definition 2. Let G be a Gauss diagram. We define a polynomial $W_G(t)$ as

$$W_G(t) = \sum_{c\in C(G)} sign(c) f_{G_c^*}(t)$$

where $C(G)$ is the set of chords of G and $f_{G_c^*}(t)$ is the index polynomial for G_c^*.

And we define a polynomial $S_G(t)$ as

$$S_G(t) = \sum_{c\in C(G)} \sum_{d\in I(c)} sign(d)(t^{i_c(d)} - 1)$$

where $C(G)$ is the set of chords of G, $I(c)$ is the set of chords which are intersected with c in G and c itself, and $i_c(d)$ is the intersection index of d in G_c°.

In [3], let G be a Gauss diagram and $C(G)$ be the set of all chords of G. Define a subset of $C(G)$ for each non-negative integer n as

$$C_n(G) = \{c \in C(G) | ind(c) = kn \text{ for some integer } k\}.$$

Definition 3. [3] For each c in $C_n(G)$ and non-negative integer n, we define the integer $d_n(c)$ as the sum of signs of endpoints in $R(c)$ whose chords belong to $C_n(G)$, and we define the n-th polynomial for G as $Z^n_G(t) = \sum_{c \in C_n(G)} sign(c)(t^{d_n(c)} - 1)$.

Similarly, we define the polynomials $W^n_G(t)$ and $S^n_G(t)$ for a non-negative integer n if we put the set $C_n(G)$ and $I_n(c)$ instead of $C(G)$ and $I(c)$ for each chord c in $C_n(G)$. Then we get the following lemmas and the main theorem.

Lemma 3.1. *Let G be a Gauss diagram and H be a Gauss diagram obtained from G by a single move II_a as Figure 3. Then the difference between $W_G(t)$ and $W_H(t)$ has the form $(t^2-1)g(t)$ for some laurent polynomial $g(t)$. Similarly, the difference between $S^2_G(t)$ and $S^2_H(t)$ has the form $(t^2-1)h(t)$ for some laurent polynomial $h(t)$.*

Proof. Suppose that the number of chords of H is greater than the one of G, and the new chords of H by the II_a move are called a and b as Figure 3. And the chords intersected with chords a and b in H are denoted by c'.

Then the terms related to a and b chords in $W_H(t)$ are

$$sign(a) f_{H^*_a}(t) = +\{p(t) + (t^{-i(a)} - 1) - (t^{i(b)+2} - 1) + \sum_{c' \in I(a)} sign(c')(t^{i(c') \pm 2} - 1)\}$$

$$sign(b) f_{H^*_b}(t) = -\{p(t) + (t^{i(a)+2} - 1) - (t^{-i(b)} - 1) + \sum_{c' \in I(b)} sign(c')(t^{i(c') \mp 2} - 1)\}$$

where $p(t)$ is the sum of terms related to chords of H which are not intersected with the chord a and b. Then $sign(a) f_{H^*_a}(t) + sign(b) f_{H^*_b}(t) = (t^2 - 1)\{-2t^{-\alpha}(t^\alpha + t^{\alpha-1} + \cdots + 1)(t^\alpha - t^{\alpha-1} + \cdots + 1) + \sum_{c' \in I} sign(c') t^{i(c')-2}(t^2+1)\}$ where $\alpha = i(a) = i(b)$ and $I = I(a) = I(b)$. Since the other terms in $W_H(t)$ are same as the corresponding terms in $W_G(t)$, $W_H(t) - W_G(t) = (t^2-1)g(t)$ for some laurent polynomial $g(t)$.

Similarly, the terms related to a and b chords in $S_H(t)$ are

$$\sum_{d \in I(a)} sign(d)(t^{i_a(d)} - 1) = -(t^{i(a)} - 1) - (t^{i(b)+2} - 1) + \sum_{c' \in I(a)} sign(c')(t^{i(c') \pm 2} - 1)$$

$$\sum_{d \in I(b)} sign(d)(t^{i_b(d)} - 1) = +(t^{i(a)+2} - 1) + (t^{i(b)} - 1) + \sum_{c' \in I(b)} sign(c')(t^{i(c') \mp 2} - 1).$$

The sum of the above two terms is $\sum_{c' \in I} sign(c')(t^{i(c') \pm 2} + t^{i(c') \mp 2} - 2)$ where $I = I(a) = I(b)$. It could be same as $\sum_{c' \in I} sign(c')(t^2-1)(t^{i(c')} + t^{i(c')-2} + 2t^{i(c')-4} + \cdots + 2)$ if $i(c')$ are even for all chord c' in I. Since the other terms in $S^2_H(t)$ are same as the corresponding terms in $S^2_G(t)$, $S^2_H(t) - S^2_G(t)$ is equal to $(t^2-1)h(t)$ for some laurent polynomial $h(t)$.

□

Lemma 3.2. *Let G be a Gauss diagram and H be a Gauss diagram obtained from G by a single move III_a as Figure 3. Then the difference between $W_G(t)$ and $W_H(t)$ has the form $(t^2-1)g(t)$ for some laurent polynomial $g(t)$. Similarly, the difference between $S_G^2(t)$ and $S_H^2(t)$ has the form $(t^2-1)h(t)$ for some laurent polynomial $h(t)$.*

Proof. The chords of G related to the III_a move are called a, b, and c and the corresponding chords of H are called a', b' and c' as Figure 3.

Then the terms related to a, b, and c chords in $W_G(t)$ are

$$sign(a)f_{G_a^*}(t) = -\{p(t) - (t^{-i(a)} - 1) + (t^{i(b)+2} - 1) + (t^{i(c)-2} - 1)+$$
$$\sum_{c_1\in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)} - 1) + \sum_{c_2\in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2} - 1)+$$
$$\sum_{c_3\in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)\pm 2} - 1)\},$$

$$sign(b)f_{G_b^*}(t) = +\{p(t) - (t^{i(a)+2} - 1) + (t^{-i(b)} - 1) + (t^{i(c)+2} - 1)+$$
$$\sum_{c_1\in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)\pm 2} - 1) + \sum_{c_2\in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2} - 1)+$$
$$\sum_{c_3\in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)} - 1)\},\ and$$

$$sign(c)f_{G_c^*}(t) = +\{p(t) - (t^{i(a)-2} - 1) + (t^{i(b)-2} - 1) + (t^{-i(c)} - 1)+$$
$$\sum_{c_1\in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)\pm 2} - 1) + \sum_{c_2\in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)} - 1)$$
$$+ \sum_{c_3\in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)\pm 2} - 1)\}$$

where $p(t)$ is the sum of terms related to chords of G which are not intersected with the chord a, b and c.

Then the terms related to a', b', and c' chords in $W_H(t)$ are

$$sign(a')f_{H_{a'}^*}(t) = -\{p(t) - (t^{-i(a)} - 1) + (t^{i(b)} - 1) + (t^{i(c)} - 1)+$$
$$\sum_{c_1'\in I(b')\cap I(c')} sign(c_1)(t^{i(c_1)} - 1) + \sum_{c_2'\in I(a')\cap I(c')} sign(c_2)(t^{i(c_2)\pm 2} - 1)$$
$$+ \sum_{c_3'\in I(a')\cap I(b')} sign(c_3)(t^{i(c_3)\pm 2} - 1)\},$$

$$sign(b')f_{H_{b'}^*}(t) = +\{p(t) - (t^{i(a)} - 1) + (t^{-i(b)} - 1) + (t^{i(c)} - 1)+$$
$$\sum_{c_1'\in I(b')\cap I(c')} sign(c_1)(t^{i(c_1)\pm 2} - 1) + \sum_{c_2'\in I(a')\cap I(c')} sign(c_2)(t^{i(c_2)\pm 2} - 1)+$$

$$\sum_{c_3' \in I(a')\cap I(b')} sign(c_3)(t^{i(c_3)}-1)\},\ and$$

$$sign(c')f_{H^*_{c'}}(t) = +\{p(t) - (t^{i(a)}-1) + (t^{i(b)}-1) + (t^{-i(c)}-1) +$$
$$\sum_{c_1' \in I(b')\cap I(c')} sign(c_1)(t^{i(c_1)\pm 2}-1) + \sum_{c_2' \in I(a')\cap I(c')} sign(c_2)(t^{i(c_2)}-1) +$$
$$\sum_{c_3' \in I(a')\cap I(b')} sign(c_3)(t^{i(c_3)\pm 2}-1)\}$$

where $p(t)$ is the sum of terms related to chords of H which are not intersected with the chord a', b' and c'.

Since the other terms in $W_G(t)$ are same as the corresponding terms in $W_H(t)$, $W_H(t) - W_G(t) = t^{i(a)}(t^2-2+t^{-2}) + t^{i(b)-2}(t^4-1) + t^{i(c)-2}(t^4-1) = (t^2-1)g(t)$ for some laurent polynomial $g(t)$.

Similarly, the terms related to a, b and c in $S_G(t)$ are that

$$\sum_{d\in I(a)} sign(d)(t^{i_a(d)}-1) = +(t^{i(a)}-1) + (t^{i(b)+2}-1) + (t^{i(c)-2}-1)$$
$$+ \sum_{c_2 \in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2}-1) + \sum_{c_3 \in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)\pm 2}-1),$$

$$\sum_{d\in I(b)} sign(d)(t^{i_b(d)}-1) = -(t^{i(a)+2}-1) - (t^{i(b)}-1) + (t^{i(c)+2}-1)$$
$$+ \sum_{c_1 \in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)\pm 2}-1) + \sum_{c_2 \in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2}-1),\ and$$

$$\sum_{d\in I(c)} sign(d)(t^{i_c(d)}-1) = -(t^{i(a)-2}-1) + (t^{i(b)-2}-1) - (t^{i(c)}-1)$$
$$+ \sum_{c_1 \in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)\pm 2}-1) + \sum_{c_3 \in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)\pm 2}-1)\}.$$

The terms related to a', b' and c' in $S_H(t)$ are that

$$\sum_{d\in I(a')} sign(d)(t^{i_{a'}(d)}-1) = +(t^{i(a)}-1)$$
$$+ \sum_{c_2 \in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2}-1) + \sum_{c_3 \in I(a)\cap I(b)} sign(c_3)(t^{i(c_3)\pm 2}-1),$$

$$\sum_{d\in I(b')} sign(d)(t^{i_{b'}(d)}-1) = -(t^{i(b)}-1)$$
$$+ \sum_{c_1 \in I(b)\cap I(c)} sign(c_1)(t^{i(c_1)\pm 2}-1) + \sum_{c_2 \in I(a)\cap I(c)} sign(c_2)(t^{i(c_2)\pm 2}-1),\ and$$

$$\sum_{d \in I(c')} sign(d)(t^{i_{c'}(d)} - 1) = -(t^{i(c)} - 1)$$

$$+ \sum_{c_1 \in I(b) \cap I(c)} sign(c_1)(t^{i(c_1) \pm 2} - 1) + \sum_{c_3 \in I(a) \cap I(b)} sign(c_3)(t^{i(c_3) \pm 2} - 1)\}.$$

Since the other terms in $S_G(t)$ are same as the corresponding terms in $S_H(t)$, $S_H(t) - S_G(t) = +(t^{i(a)+2} + t^{i(a)-2} - 2) - (t^{i(b)+2} + t^{i(b)-2} - 2) - (t^{i(c)-2} + t^{i(c)+2} - 2)$. Thus, $S_H^2(t) - S_G^2(t)$ is equal to $(t^2 - 1)h(t)$ for some laurent polynomial $h(t)$. $\square$

In fact, we get the $III_{a'}$ move from III_a move by the changing the signs of chords. Then we get the same conclusion for the $III_{a'}$ move. And we prove the following theorem by these lemmas.

Theorem 3.3. *Let D be a virtual knot diagram and G be a Gauss diagram. Then the polynomial $W_G(t)$ and $S_G^2(t)$ are invariants for virtual knot diagrams on the quotient ring $\mathbf{Z}[t^{\pm 1}]/I$ where I is the ideal generated by $t^2 - 1$.*

For non-negative integer n, $W_G^n(t)$ is also invariant for virtual knots on the quotient ring $\mathbf{Z}[t^{\pm 1}]/I$ where I is the ideal generated by $t^2 - 1$.

Remark 1. Let D be a virtual knot diagram and G be a Gauss diagram of D. If D is obtained from a classical knot diagram by virtualization moves, $W_G(t)$ and $S_G(t)$ could be zero. If D is checkerboard colorable, $W_G(t)$ and $S_G(t)$ could be constant on the quotient ring $\mathbf{Z}[t^{\pm 1}]/I$ where I is the ideal generated by $t^2 - 1$ since all real crossings of D have even intersection indices.

Example 3.4. *Let D be a Miyazawa knot diagram and G be a Gauss diagram of D in Figure 6.*

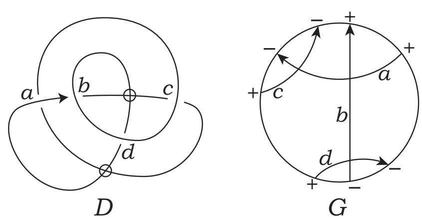

FIGURE 6. Miyazawa knot

$W_G(t)$ is equal to $-2t^{-2} + 4t^{-1} - 2$. Then $W_G(t)$ is equivalent to $4t - 4$ in $P[t]/I$ where I is the ideal generated by $t^2 - 1$. Therefore, D is non-trivial and not checkerboard colorable.

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