A METHOD FOR SOLVING OF LINEAR SYSTEM WITH NORMAL COEFFICIENT MATRICES

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ABSTRACT. This study aims to generalize MINRES-N2 method [1]. It means that we tend to obtain an algorithm to transfer each normal matrix - that its eigenvalues belong to an algebraic curve of low degree k- to its condensed form through using a unitary similarity transformation. Then, we aim to obtain a method to solve a system of linear equations that its coefficient matrix is equal to such a matrix by utilizing it. Finally this method is compared to the well-known GMRES method through using numerical examples. The results obtained through examples show that the given method is more efficient than GMRES.

1. INTRODUCTION

A condensed form for normal matrices was introduced by Elsner and Ikramov in [1]. In fact, they indicated that each normal matrix is unitarily equivalent to a block tridiagonal matrix as shown in the following form:

$$\begin{pmatrix} H_{11} & H_{12} & & \\ H_{21} & H_{22} & H_{23} & \\ & H_{32} & H_{33} & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \qquad (1.1)$$

If eigenvalues of the matrix belong to an algebraic curve of low degree k, size of blocks does not exceed k (see [1, 2, 3, 4, 5, 6]). After that, Dana , Zykov , and Ikramov in [7] designed the obtained method to reach the condensed form of normal matrices in form of an algorithm for normal matrices that their eigenvalues belong to an algebraic curve of second degree, and developed a method to solve systems of linear equations with coefficient matrix as above which they called MINRES-N2 (see [7]).

In this paper, by expanding MINRES-N2 method, we obtained an algorithm which is efficient for normal matrices their eigenvalues of which belong to an algebraic curve of low degree k. Finally, we determine for which k our method is more efficient than GMRES method by studying several numerical examples.

We assume $A \in \mathbf{C}^{n \times n}$ is a square matrix, for a desired non-zero vector, in which $q_1 \in \mathbf{C}^n$, and consider the followings sequence in $\mathbf{C}^n$:

$$q_1, \ Aq_1, A^*q_1, A^2q_1, AA^*q_1, A^{*2}q_1, \ldots \tag{1.2}$$

where A^* is adjoint of matrix A. Sequence (1.2) is named generalized power sequence generated by A and q_1. Each vector as

$$u_{\alpha\beta}(A, A^*) = A^\alpha(A^*)^\beta q_1 \qquad , \qquad \alpha + \beta = i.$$

is called a vector out of vectors creating ith layer of the sequence (α and β are natural numbers). By such a naming, $i+1$ of the vector is available in ith layer. This indicates that the sequence (1.2) is made of some layers with lengths of 1, 2, 3, and so on, respectively. It is proved in [1] that diagonal blocks H_{ii} in the condensed form (1.1) have respectively equal sizes with width of layers 0, 1, 2, 3,... of the sequence (1.2). As can be seen in [1], if

$$f(A, A^*) = 0,$$

in which $f(x, y)$ is a polynomial of degree k (k is a small positive integer), then, spectrum of A belongs to an algebraic curve of degree k, in such a mode, size of H_{ii} of (1.1) is equal to k for $i \geq k$.

As noted before, authors [1] have found a method to reach condensed form of normal matrix A. we state this method as an algorithm so that we can use it for our ultimate destination. Assume that q_1 is a unit vector in $\mathbf{C}^n$, and also assume that L_i is subspace generated by vectors of first i layers of the sequence (1.2), This subspace is ith generalized Krylov subspace generated by A and q_1, and express it as follows:

$$L_i = span\{u_{\alpha\beta}(A, A^*)q_1 : \alpha + \beta \leq i\}.$$

Let $dim(L_i) = l_i$, define $w_i = l_i - l_{i-1}$, and agree $w_0 = 1$. If the sequence (1.2) generates $\mathbf{C}^n$ space, orthogonalization (1.2) makes a orthonormal basis $q_1, \ldots, q_n$ for $\mathbf{C}^n$. Put

$$Q = (q_1 \ q_2 \ \ldots \ q_n).$$

The above orthogonalization process results in matrix H, that

$$AQ = QH,$$

is H of a matrix in form of (1.1) and diagonal block H_{ii} has size of w_i (see [1]).

The block tridiagonal of H means that for orthogonalization of vector Aq-while q is a vector in ith layer- to the previous layers it is just enough to orthogonality it to vectors of $i-1, i$ and $i+1$ layers and not to all of the previous vectors. In other words, the depth of the recursion defining the next vector q does not exceed $\omega_{i-1} + \omega_i + \omega_{i+1}$.

The algorithm MINRES-N, which we propose, combines the orthogonalization of sequence (1.2) construct the entire basis $q_1, \ldots, q_n$ and the entire matrix H, in the following way:

1- Choice a unit vector $q_1 \in \mathbf{C}^n$.

2- Set $p_1 = 0$, $l_1 = 1$ and $s = 1$.

3- For $t = l_s, \ldots, l_s + P_s$ the vector Aq_t is orthogonalized to

$$q_{l_s-P_s}, \ldots, q_{t+P_s};$$

that is,

$$w_t = Aq_t - h_{l_s-P_s,t}q_{l_s-P_s} - \cdots - h_{t+P_s,t}q_{t+P_s},$$

where

$$h_{i,t} = (Aq_t, q_i), \qquad i = l_1 - p_s, \ldots, t + p_s.$$

Now, we set

$$h_{t+p_s+1,t} = \| w_t \|_2, \qquad q_{t+p_s+1} = w_t/h_{t+p_s+1,t}$$

(We assume that $h_{t+p_s+1,t} \neq 0$; similar assumptions are made at the subsequent steps.)

Now the vector $A^*q_{l_s+p_s}$ is orthogonalized to $q_{l_s-p_s}, \ldots, q_{l_s+2p_s+1}$; thus,

$$\hat{w}_{l_s+p_s} = A^*q_{l_s+p_s} - \overline{h_{l_s+p_s,l_s-p_s}}q_{l_s-p_s} - \ldots - \overline{h_{l_s+p_s,l_s+2p_s+1}}q_{l_s+2p_s+1}.$$

Coefficients are calculating by the formulas

$$h_{l_s+p_s,i} = \overline{(A^*q_{l_s+p_s}, q_i)}, \qquad l_s - p_s \leqslant i \leqslant l_s + 2p_s + 1.$$

After calculating $\hat{w}_{l_s+p_s}$, we set

$$h_{l_s+p_s,l_s+2p_s+2} = \| \hat{w}_{l_s+p_s} \|_2, \qquad q_{l_s+2p_s+2} = \hat{w}_{l_s+p_s}/h_{l_s+p_s,l_s+2p_s+2}.$$

Entires of H, which are obtained in this step, are repeated.

4- For $s = 2, \ldots,$ until $l_s + 2p_s + 2 \leq n$,

$$p_s = p_{s-1} + 1, \qquad l_s = l_{s-1} + p_{s-1} + 1 \qquad .$$

5- For $s = 2, \ldots,$ until $l_s + 2p_s + 2 \leq n$, repeat the third step.

For each s; the 5-th step produce the vectors

$$q_{l_s+p_s+1}, \ldots, q_{l_s+2p_s+2},$$

which, in combination with $q_1, \ldots, q_{l_s+p_s}$, form an orthonormal basis in the subspace

$$L_s = \text{span}\{q_1, \ldots, q_{l_s+2p_s+2}\},$$

spanned by the first $s + 1$ segments of sequence (1.2). This completes the processing of the s-th layer in this sequence.

Algorithm 1. Algorithm for reducing an admissible normal matrix A to its condensed form H.

choice q_1, with $\|q_1\|_2 = 1$,
$l = 1$
$p = 0$
$while \ l + 2p + 2 \leq n$
$\quad for \ t = l \ to \ l + p$
$\quad\quad z = Aq_t$
$\quad\quad for \ i = l - p \ to \ t + p$
$\quad\quad\quad h_{i,t} = q_i^* z$
$\quad\quad\quad z = z - h_{i,t}q_i$
$\quad\quad end \ for$
$\quad\quad h_{t+p+1,t} = \|z\|_2$
$\quad\quad q_{t+p+1} = z/h_{t+p+1,t}$
$\quad end \ for$
$\quad z = A^* q_{l+p}$
$\quad for \ i = l - p \ to \ l + 2p + 1$
$\quad\quad h_{l+p,i} = q_i^* z$
$\quad\quad z = z - h_{l+p,i}q_i$
$\quad end \ for$
$\quad h_{l+p,l+2p+2} = \|z\|_2$
$\quad h_{l+2p+2} = z/g_{l+p,l+2p+2}$
$\quad p = p + 1$
$\quad l = l + p$
$end \ while$

2. METHOD MINRES-Nκ FOR NORMAL MATRICES

Assume that the spectrum of a normal matrix A belongs to an algebraic curve

$$f(x, y) = 0$$

of order k. This means that A itself satisfies the equality

$$f\left(\frac{A + A^*}{2}, \frac{A - A^*}{2i}\right) = 0.$$

Multiplying both sides of this relation by a vector q_1, we can reduce it to the form

$$\alpha_1 A^k q_1 + \alpha_2 A^{k-1} A^* q_1 + \ldots + \alpha_{k+1}(A^*)^k q_1 + y I_n = 0. \tag{2.1}$$

where the vector y belongs to the subspace L_{k-1} and at least one of the coefficients $\alpha_1, \ldots, \alpha_{k+1}$ is nonzero. Eq.(2.1) signifies that layer

$$A^k q_1, A^{k-1} A^* q_1, A^{k-2}(A^*)^2 q_1, \ldots, A(A^*)^{k-1} q_1, (A^*)^k q_1,$$

can add at most k vectors to a basis of L_{k-1}; i.e., $\omega_k \leq k$. It follows that $\omega_t \leq k$, for all $t > k$ (see [[1], therem 1]). Let N_i and M_i be the numbers of nonzero entries in the ith column and the ith row of H, respectively. Now, bounds

$$N_i \leq \omega_{s-1} + \omega_s + \omega_{s+1}, \qquad M_i \leq \omega_{s-1} + \omega_s + \omega_{s+1},$$

yield $N_i \leq 3k$, $M_i \leq 3k$.

Hence, the condensed form H can be considered a band matrix. Depending on the order of the vectors within each layer, its bandwidth is $3k$ or $3k - 1$,.

We can seek a solution to system $Ax = b$ by constructing an orthonormal system $q_1, q_2, q_3, \ldots$ the discussion in the preceding section implies that the condensed form H of A is a block tridiagonal matrix, while all the subsequent diagonal blocks have an order of k. The orthonormal sequence $q_1, q_2, q_3, \ldots$ is constructed in the following way:

1- An appropriate choice of the initial vector is

$$q_1 = \frac{1}{\|b\|_2} b.$$

2- Set $p_1 = 0$, $l_1 = 1$ and $s = 1$.

3- For $t = l_s, \ldots, l_s + P_s$ the vector Aq_t is orthogonalized to

$$q_{l_s - P_s}, \ldots, q_{t+P_s};$$

that is,

$$w_t = Aq_t - h_{l_s - P_s, t} q_{l_s - P_s} - \cdots - h_{t+P_s, t} q_{t+P_s},$$

where

$$h_{i,t} = (Aq_t, q_i), \qquad i = l_s - p_s, \ldots, t + p_s.$$

Now, we set

$$h_{t+p_s+1, t} = \| w_t \|_2, \qquad q_{t+p_s+1} = w_t / h_{t+p_s+1, t}$$

(We assume that $h_{t+p_s+1, t} \neq 0$; similar assumptions are made at the subsequent steps.)

Now the vector $A^* q_{l_s + p_s}$ is orthogonalized to $q_{l_s - p_s}, \ldots, q_{l_s + 2p_s + 1}$; thus,

$$\hat{w}_{l_s + p_s} = A^* q_{l_s + p_s} - \overline{h_{l_s + p_s, l_s - p_s}} q_{l_s - p_s} - \ldots - \overline{h_{l_s + p_s, l_s + 2p_s + 1}} q_{l_s + 2p_s + 1}.$$

Coefficients are calculating by the formulas

$$h_{l_s + p_s, i} = \overline{(A^* q_{l_s + p_s}, q_i)}, \qquad l_s - p_s \leqslant i \leqslant l_s + 2p_s + 1.$$

After calculating $\hat{w}_{l_s + p_s}$, we set

$$h_{l_s + p_s, l_s + 2p_s + 2} = \| \hat{w}_{l_s + p_s} \|_2, \qquad q_{l_s + 2p_s + 2} = \hat{w}_{l_s + p_s} / h_{l_s + p_s, l_s + 2p_s + 2}.$$

Entires of H, which are obtained in this step, are repeated.

4- If $k = 2$ go to 7-th step, otherwise go to 5-th step.

5- Set,

$$p_s = p_{s-1} + 1, \qquad l_s = l_{s-1} + p_s \qquad s = 2, \ldots, k - 1.$$

6- For $s = 2, \ldots, k - 1$, repeat the third step.

For $s = 2, \ldots, k - 1$, the 6-th step produce the vectors

$$q_{l_s + p_s + 1}, \ldots, q_{l_s + 2p_s + 2},$$

which, in combination with $q_1, \ldots, q_{l_s + p_s}$ form an orthonormal basis in the subspace

$$L_s = \text{span}\{q_1, \ldots, q_{l_s + 2p_s + 2}\},$$

spanned by the first $s + 1$ segments of sequence (1.2). This completes the processing of the s-th layer in this sequence.

7- Set $d_1 = l_{k-1} + p_{k-1} + 1,$, $u_1 = d_1 - k + 1$, and $v = 1$.

8- For $t = d_v, \ldots, d_v + k - 1$, the vector Aq_t is orthogonalized to $q_{u_v}, \ldots, q_{t+k-1}$; that is, for $t = d_v, \ldots, d_v + k - 1$, we set

$$w_t = Aq_t - h_{u_v,t}q_{u_v} -, \ldots, -h_{t+k-1,t}q_{t+k-1}.$$

Here,

$$h_{r,t} = (Aq_t, q_r), \qquad r = u_v, \ldots, t + k - 1.$$

After finding w_t, we set

$$h_{t+k,t} = \| w_t \|_2, \qquad q_{t+k} = w_t / h_{t+k,t}.$$

The 8-th step produce the vectors $q_{d_v+k}, \ldots, q_{d_v+2k-1}$, which, in combination with $q_1, \ldots, q_{d_v+k-1}$ form an orthonormal basis in the subspace

$$L_{v+k-1} = \text{span}\{q_1, \ldots, q_{d_v+2k-1}\},$$

spanned by first $v + k$ segments of sequence (1.2). This completes the processing of the $(v + k - 1)$-th layer in this sequence.

9- Set,

$$d_v = d_{v-1} + k, \qquad u_v = d_v - k, \qquad v = 2, 3, \ldots.$$

10- Until $d_v + 2k - 1 \leq n$, repeat the 8-th step, for $v = 2, 3, \ldots$.

Algorithm 2. Algorithm for reducing an admissible normal matrix A to its condensed form H, that the spectrum of A belong to a plane algebraic curve of degree k.

$q_1 = \frac{b}{\|b\|_2}$
$l = 1$
$p = 0$
$while\ p + 2 \leq k$
$\quad for\ t = l\ to\ l + p$
$\qquad z = Aq_t$
$\qquad for\ i = l - p\ to\ t + p$
$\qquad\quad h_{i,t} = q_i^* z$
$\qquad\quad z = z - h_{i,t} q_i$
$\qquad end\ for$
$\qquad h_{t+p+1,t} = \|z\|_2$
$\qquad q_{t+p+1} = z / h_{t+p+1,t}$
$\quad end\ for$
$\quad z = A^* q_{l+p}$
$\quad for\ i = l - p\ to\ l + 2p + 1$
$\qquad h_{l+p,i} = q_i^* z$
$\qquad z = z - h_{l+p,i} q_i$
$\quad end\ for$
$\quad h_{l+p,l+2p+2} = \|z\|_2$
$\quad q_{l+2p+2} = z / h_{l+p,l+2p+2}$
$\quad p = p + 1$
$\quad l = l + p$
$end\ while$
$\quad for\ t = l\ to\ l + k - 1$
$\qquad z = Aq_t$
$\qquad if\ t \leq l + k - 1$
$\qquad\quad s = l - k + 1$
$\qquad else$
$\qquad\quad s = l - k$
$\qquad end\ if$
$\qquad for\ i = s\ to\ t + k - 1$
$\qquad\quad h_{i,t} = q_i^* z$
$\qquad\quad z = z - h_{i,t} q_i$
$\qquad end\ for$
$\qquad h_{t+k,t} = \|z\|_2$
$\qquad q_{t+k} = z / h_{t+k,t}$
$\quad end\ for$
$\quad r = 0$

$$l = l + k$$
$$while \; r \leq n$$
$$\quad for \; t = l \; to \; l + k - 1$$
$$\qquad z = Aq_t$$
$$\qquad for \; i = l - k \; to \; t + k - 1$$
$$\qquad\quad h_{i,t} = q_i^* z$$
$$\qquad\quad z = z - h_{i,t}q_i$$
$$\qquad end \; for$$
$$\qquad h_{t+k,t} = \|z\|_2$$
$$\qquad q_{t+k} = z/h_{t+k,t}$$
$$\quad end \; for$$
$$\quad l = l + k$$
$$\quad r = t + k$$
$$end \; while$$

Assume that the m steps in the algorithm for reducing a matrix to its condensed form have already been completed (that is, the first m columns of the condensed form H have been completed). (The normalization $q_1 = b/\|b\|_2$ is considered as the first step.) We assume that m is an integer: $m = (\ell - k + 1)k + \frac{k(k+1)}{2}$, where $\ell > k - 1$. Then, the vectors $q_1, \ldots, q_m$ constitute a basis in the generalized Krylov subspace $L_\ell(A, b)$. We partition the matrix H as in

$$H = \begin{pmatrix} H_m & H_{m,n-m} \\ H_{n-m,m} & H_{n-m,n-m} \end{pmatrix}.$$

A part of this partitioning, namely,

$$\begin{pmatrix} H_m \\ H_{n-m,m} \end{pmatrix},$$

corresponds to the first m columns of the matrix depicted in the figure.

The use of expansion

$$x_m = \eta_1 q_1 + \ldots + \eta_m q_m. \tag{2.2}$$

reduces the search for x_m, which is the vector in L_ℓ, minimizing the 2-norm of the residual, searching the $m-$dimensional vector $y_m = (\eta_1, \ldots, \eta_m)^T$. The calculation of y_m is equivalent to solving least squares problem

$$min_{y \in C^m} \| \begin{pmatrix} H_m \\ H_{n-m,m} \end{pmatrix} y - \|b\|_2 e_1 \|_2. \tag{2.3}$$

Taking into account the form of matrix

$$\begin{pmatrix} H_m \\ H_{n-m,m} \end{pmatrix}, \tag{2.4}$$

we can solve this problem as follows:

1. Using the rotation R_{12}, eliminates the entry (2,1) in matrix (2.4).

2. For $j = 2, \ldots, m$ eliminate the subdiagonal entries in column j of matrix (2.4) using the appropriate Householder matrices P_j.

3. Apply the same transformations i.e., the rotation R_{12} and the Householder matrices $P_2, \ldots, P_m$, to the right-hand side of problem (2.3).

This process results in the matrix

$$\begin{pmatrix} T_m \\ 0 \end{pmatrix}$$

and in the vector z_m, which now has $m + k$ nonzero components :

$$z_m = (\varsigma_1^{(m)}, \ldots, \varsigma_{m+1}^{(m)}, \varsigma_{m+2}^{(m)}, \ldots, \varsigma_{m+k}^{(m)}, 0, \ldots, 0)^T.$$

The vector y_m is calculated as before (i.e., by solving a triangular system of equations); then, x_m is calculatedwith the use of (2.2). This calculation makes no sense until the stopping criterion is satisfied.

The norm of the current residual r_m is determind by the last k components z_m :

$$\|r_m\|_2 = (|\varsigma_{m+1}^{(m)}|^2 + |\varsigma_{m+2}^{(m)}|^2 + \ldots + |\varsigma_{m+k}^{(m)}|^2)^{1/2}.$$

If $\|r_m\|_2$ does not satisfy the given stopping criterion, then a new calculation of the norm of the residual is performed after the next k steps, when the orthonormal basis $q_1, \ldots, q_{m+k}$ in the subspace $L_{\ell+1}$ has been obtained. k additional columns are attached to the matrix of problem (2.3). It is evident from the figure that these columns are changed only by the application of the Householder matrices $P_{m-2k+1}, \ldots, P_{m-4}, P_{m-3}, P_{m-2}, P_{m-1}$ and P_m. After these columns have been updated, they determine the new Householder matrices $P_{m+1}, P_{m+2}, \ldots, P_{m+k}$. The new vector z_{m+2} has $m + 2k$ nonzero components; moreover, its first m components are the same as in z_m. The remaining $2k$ components are calculated by the formula

$$\begin{pmatrix} \varsigma_{m+1}^{(m+1)} \\ \varsigma_{m+2}^{(m+1)} \\ \varsigma_{m+3}^{(m+1)} \\ \varsigma_{m+4}^{(m+1)} \\ \vdots \\ \varsigma_{m+2k}^{(m+1)} \end{pmatrix} = P_{m+k} \cdots P_{m+2} P_{m+1} \begin{pmatrix} \varsigma_{m+1}^{(m)} \\ \vdots \\ \varsigma_{m+k}^{(m)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

this description shows that the cost of the transition from $\|r_m\|_2$ to $\|r_{m+1}\|_2$ is independent of m.

3. NUMERICAL RESULTS

In this section, we discuss some numerical experiments in which MINRES-Nk was compared with GMRES.

Our normal matrix A must satisfy an additional condition, namely, its spectrum must lie on a curve of low degree k.

In all of our experiments, the order of systems was 2000. The right-hand sides were generated as pseudorandom vectors with components distributed uniformly on (0, 1). The calculations were performed on a 2 Duo E630 OEM 1.86 GHz PC with core memory of 1024 Mb. GMRES was represented by the Matlab procedure gmres; for MINRES-Nk, we designed our own Matlab procedure. To terminate the iteration, we used the condition

$$\|b - Ax_m\|_2 = \|r(x_m)\|_2 < \varepsilon,$$

where ε is a given positive scalar. In all of our experiments, $\varepsilon = 10^{-8}$ and m is equal to number of iteration steps.

Example 3.1. *The eigenvalues of A are uniformly distributed on the $y^2 = x^2 + 9$ corresponding to x in the interval (5,6). The number of iteration steps in MINRES-N2 is 4 and in GMRES is 9. Times: 0.11 s for MINRES-N2 against 0.37 s for GMRES. The results are shown in the following table and figure .*
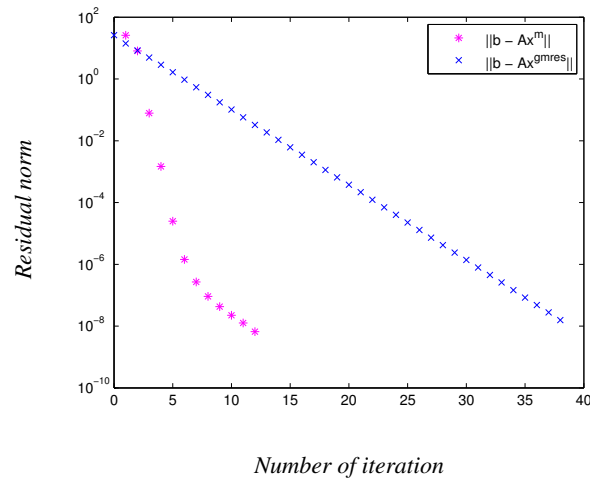
Method	ϵ	m	$t(s)$
MINRES-N2	10^{-8}	4	0.11
GMRES	10^{-8}	9	0.37

Number of iteration
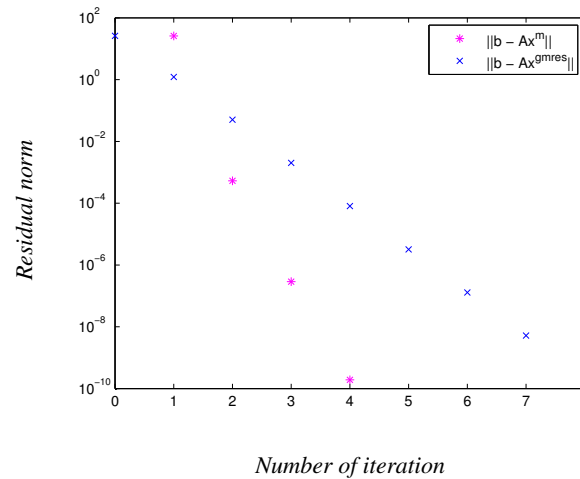
Example 3.2. *The eigenvalues of A are uniformly distributed on the $y = x^3 + 3x^2 + 2$ corresponding to x in the interval (10,25). The number of iteration steps in MINRES-N3 is 12 and*
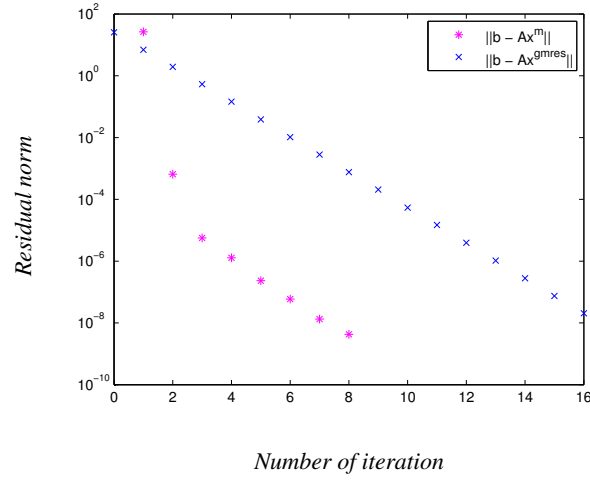
in GMRES is 39. Times: 0.11 s for MINRES-N3 against 0.47 s for GMRES. The results are shown in the following table and figure .

Method	ϵ	m	t(s)
MINRES-N3	10^{-8}	12	0.11
GMRES	10^{-8}	39	0.47

Number of iteration

Example 3.3. *The eigenvalues of A are uniformly distributed on the $x^2y^2 = 1$ corresponding to x in the interval (5,15). The number of iteration steps in MINRES-N4 is 8 and in GMRES is 17. Times: 0.17 s for MINRES-N4 against 0.25 s for GMRES. The results are shown in the following table and figure .*

| Method	ϵ	m	t(s)
MINRES-N4	10^{-8}	8	0.17
GMRES	10^{-8}	17	0.25

Example 3.4. *The spectrum of A is uniformly distributed on the*

$$y = \begin{cases} x^5 + x^2 & x > 0 \\ -(x^5 + x^2) & x < 0 \end{cases}$$

corresponding to x in $(10, 20) \cup (-20, -10)$. *The number of iteration steps in MINRES-N5 is 14 and in GMRES is 59. Times: 0.28 s for MINRES-N5 against 0.83 s for GMRES. The results are shown in the following table and figure .*

| Method	ϵ	m	t(s)
MINRES-N5	10^{-8}	14	0.28
GMRES	10^{-8}	59	0.83
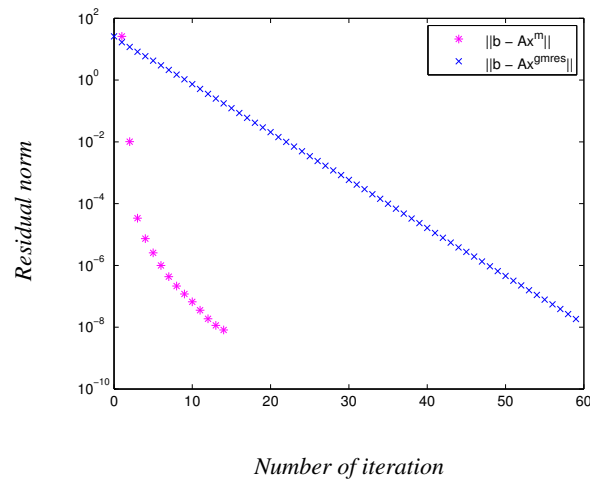
Example 3.5. *The spectrum of A is uniformly distributed on the*

$$y = \begin{cases} x^6 + x & x > 0 \\ -(x^6 + x) & x < 0 \end{cases}$$

corresponding to x in $(10, 20) \cup (-20, -10)$. The number of iteration steps in MINRES-N6 is 24 and in GMRES is 84. Times: 1.31 s for MINRES-N6 against 1.56 s for GMRES. The results are shown in the following table and figure .

Method	ϵ	m	t(s)
MINRES-N6	10^{-8}	24	1.31
GMRES	10^{-8}	84	1.56

Number of iteration
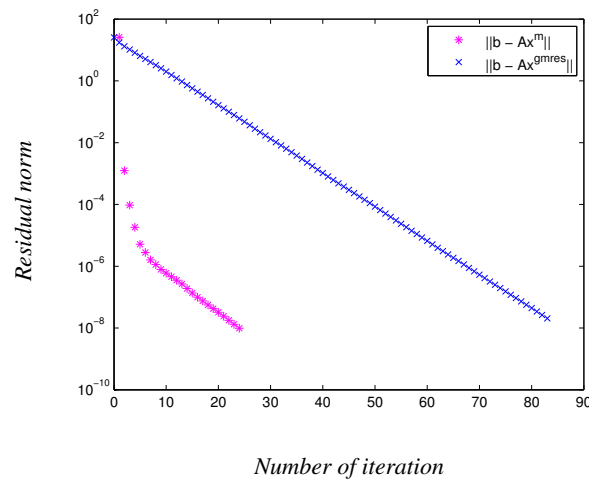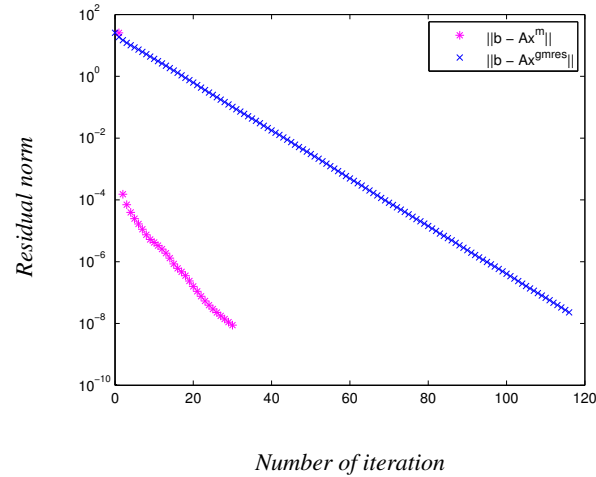
Example 3.6. *The eigenvalues of A are uniformly distributed on the $y = x^7 + 3x^2 + 2$ corresponding to x in the interval (10,25). The number of iteration steps in MINRES-N7 is 117 and in GMRES is 251. Times: 9.58 s for MINRES-N7 against 12.01 s for GMRES. The results are shown in the following table and figure .*

| Method	ϵ	m	t(s)
MINRES-N7	10^{-8}	117	9.58
GMRES	10^{-8}	251	12.01

Number of iteration
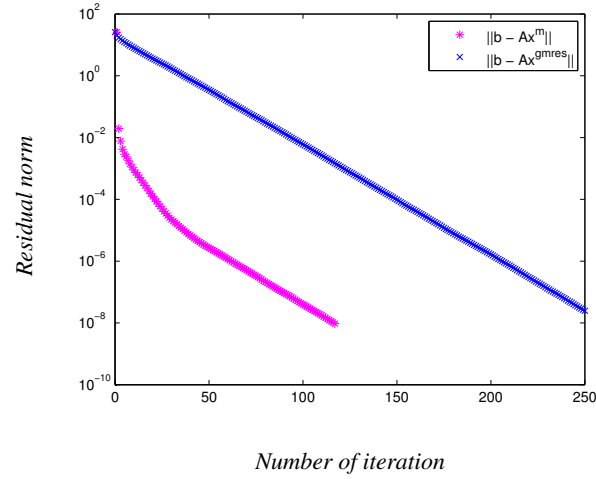
Example 3.7. *The eigenvalues of A are uniformly distributed on the $y = x^8 + x^5 + 20$ corresponding to x in the interval (-11,-6). The number of iteration steps in MINRES-N8 is 28 and in GMRES is 118. Times: 2.80 s for MINRES-N8 against 2.85 s for GMRES. The results are shown in the following table and figure .*

Method	ϵ	m	$t(s)$
MINRES-N8	10^{-8}	28	2.80
GMRES	10^{-8}	118	2.85

Number of iteration

Example 3.8. *The eigenvalues of A are uniformly distributed on the* $y = x^9 + 3x^5 + 20$ *corresponding to* x *in the interval (-8,-3). The number of iteration steps in MINRES-N9 is 192 and in GMRES is 691. Times: 24.46 s for MINRES-N9 against 90.05 s for GMRES. The results are shown in the following table and figure .*

Method	ϵ	m	$t(s)$
MINRES-N9	10^{-8}	192	24.46
GMRES	10^{-8}	691	90.05

Number of iteration
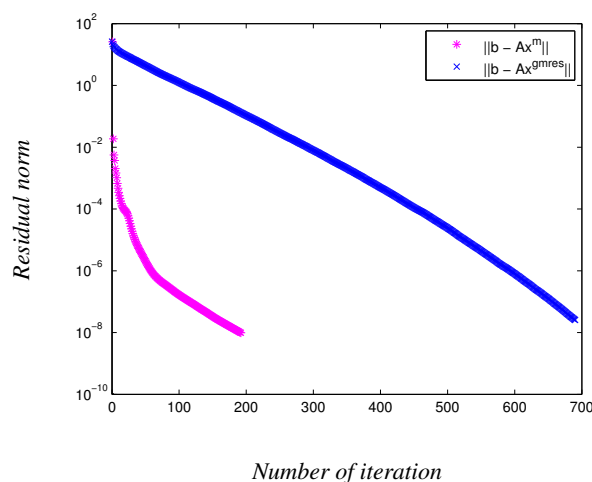
4. CONCLUSION

This study represents a method for solving of linear system with normal cofficient matrices. As for presented method in [1] for reach the condensed form of normal matrices, that their eigenvalues belong to an algebraic curve of second degree and MINRES-N2 method. Here, we are going to expand the so-call method. Also an algorithm is obtained which belong to an algebraic curve of low degree k.

Then the obtained results show that for each ks, our generalized method is more efficient than GMRES method.

The advantage of our method compared to GMRES method in small ks, is due to lower number of arithmetic operations in calculating the condensed form of matrix as well as the applied orthogonalization process in this paper. Based on numerous experiments, it is concluded that such an advantage exists for ks lower than 10.

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