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MULTI-BLOCK BOUNDARY VALUE METHODS FOR ORDINARY DIFFERENTIAL AND DIFFERENTIAL ALGEBRAIC EQUATIONS

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ABSTRACT. In this paper, multi-block generalized backward differentiation methods for numerical solutions of ordinary differential and differential algebraic equations are introduced. This class of linear multi-block methods is implemented as multi-block boundary value methods ($\mathrm{MB_2VMs}$). The root distribution of the stability polynomial of the new class of methods are determined using the Wiener-Hopf factorization of a matrix polynomial for the purpose of their correct implementation. Numerical tests, showing the potential of such methods for output of multi-block of solutions of the ordinary differential equations in the new approach are also reported herein. The methods which output multi-block of solutions of the ordinary differential equations on application, are unlike the conventional linear multistep methods which output a solution at a point or the conventional boundary value methods and multi-block methods which output only a block of solutions per step. The $\mathrm{MB_2VMs}$ introduced herein is a novel approach at developing very large scale integration methods (VLSIM) in the numerical solution of differential equations.

1. INTRODUCTION

Consider the numerical solution of the stiff problem (see [7]),

$$\begin{aligned} y'(t) = f(t, y(t)), \quad t \in (t_0, T), \quad y(t_0) = y_0, \quad y(t) \in R^v, \\ f(t, y(t)) \in R^v, \quad t \in R, \quad v = 1, 2, ..., \end{aligned} \tag{1.1}$$

in ordinary differential equations (ODEs) and the differential algebraic equations (DAEs)

$$\begin{aligned} y'(t) = f(t, x(t), y(t)), \quad x(t_0) = x_0, \quad y(t_0) = y_0, \\ 0 = g(t, x(t), y(t)). \end{aligned} \tag{1.2}$$

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The ODEs in (1.1) and DAEs in (1.2) arise in the modelling of constrained mechanical systems, biological system, circuits theory and chemical reaction kinetics [1, 2] etc. The initial value problems (IVPs) in (1.1) and the DAEs in (1.2) can be solved numerically by the conventional linear multistep methods of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j}; \qquad n=0,1,\cdots; \quad k\geq 1, \tag{1.3}$$

due to relatively cheap implementation, but is limited by Dahlquist order stability barrier in [3]. In this regard, there have been several modification of the conventional LMM (1.3) restricted by Dahlquist theorems to obtain different classes of methods (including hybrid methods and block methods) with high order along with having $A-$stability properties. Example of such methods can be found in [7, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15]. However, in Amodio et al [16], Brugnano and Trigiante ([18, 19, 20] and references therein), provides a compelling approach, where the continuous IVPs (1.1) and (1.2) is approximated by a means of discrete boundary value problems (BVPs) using boundary value methods (BVMs) based on linear multistep formulas in (1.3) of the form,

$$\begin{gathered}\sum_{j=-k_1}^{k_2}\alpha_j y_{n+j} = h\sum_{j=-k_1}^{k_2}\beta_j f_{n+j}; \quad n=0,1,\cdots; \quad k_1+k_2=k,\\ \underbrace{y_0, y_1, \cdots, y_{k_1-1}}_{\text{(a1)}} \qquad \underbrace{y_{k_1}, \cdots, y_{N-k_2}}_{\text{solution values to be generated by the BVM}} \qquad \underbrace{y_{N-k_2+1}, \cdots, y_{n+N}}_{\text{(a2)}}\end{gathered} \tag{1.4}$$

This approach defines BVMs with (k_1, k_2)-boundary conditions, if the root distribution of the stability polynomial of the main method in (1.4) is of the type $(k_1, 0, k_2)$. Here k_1 is the number of roots lying inside the unit circle and k_2 is the number of roots lying outside the unit circle of the stability polynomial of the main methods in (1.4). Examples of such class of methods includes; the generalized backward differentiation formulas (GBDFs) [19, 20],

$$\begin{gathered}\sum_{j=0}^{k}\alpha_j y_{n+j} = hf_{n+u}; \qquad n=0,1,\cdots; \quad k\geq 1,\\ y_0, y_1, y_2, \cdots, y_{k_1-1}, \quad y_{N-k_2+1}, \cdots, y_N \quad (fixed).\end{gathered} \tag{1.5}$$

This is a generalization of the conventional backward differentiation formulas in [1] with $u(=k_1)$ defined as (see [19]),

$$u = \begin{cases} \frac{k+2}{2}; & k \quad even \\ \frac{k+1}{2}; & k \quad odd \end{cases}$$

The BVMs in (1.4) provides the numerical solution $\{y_{n+k_1}, \cdots, y_{n+N-k_2}\}$ of the ODEs in (1.1) and (1.2) given the boundary solutions in (1.5). The generalized Adams methods (GAMs), extended trapeziodal rule of first (ETRs) and of second kinds (ETR$_2$s), and top order methods (TOMs) and a comprehensive theory for these classes of methods, along with the generalization of zero-stability and A-stability of linear multistep methods (LMM) from the theory of initial value methods (IVMs) in (1.3) to BVMs in (1.4) can be found in [19]. The generalized second derivative linear multistep methods based on the methods of Enright [22] have been considered in [21]. The second derivative generalized extended backward differentiation formulas (SDGEBDFs) is,

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=k}^{2k-1}\beta_j f_{n+j} + h^2 f'_{n+k}, \quad \forall k \geq 1, \qquad n = 0,1,2,\cdots,$$

$$\underbrace{y_{n+1},\cdots,y_{n+k-1}}_{\text{(a)}} \qquad \underbrace{y_{n+k},\cdots,y_{n+N-k+1}}_{\text{solution values to be generated by the SDBVM}} \qquad \underbrace{y_{n+N-k+2},\cdots,y_{n+N}}_{\text{(b)}}$$

for stiff problems of (1.1) have been proposed in [23] as the main formula. The solution output (a) and (b) are to be provided or replaced by second derivative linear multistep formulas (SDLMF) as equations at the points $t_{n+1},\cdots,t_{n+k-1}$ and $t_{n+N-k+2}$, $t_{n+N-k+3},\cdots,t_{n+N}$ respectively. Moreso, BVMs have been considered on Hamiltonian problems, Volterra integro-differential problems, neutral pantograph equations and neutral multi-delay differential equations, differential algebraic equations in [24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 36, 37]. In fact, the BVMs gives room to obtain solution of (1.1) and (1.2) globally, unlike the LMMs in (1.3) which solution is obtained in a step-step fashion.

In this paper, the concept of the conventional BVMs in (1.4) is generalized through the use of the linear multi-block methods in [11] to introduce for the first time, multi-block boundary value methods (MB$_2$VMs). The linear multi-block methods of [11] is

$$Y_{n+k} = \sum_{j=0}^{k-1} A_j Y_{n+j} + h\sum_{j=0}^{k} B_j F_{n+j}; \qquad n = 0,1,\cdots, \quad k \geq 1 \tag{1.6}$$

where

$$A_j = \left[a_{u,v}^{(j)}\right]_{u,v=1(1)s}, \qquad Y_{n+j} = \left(y_{n+js}, y_{n+js+1}, y_{n+js+2},\cdots,y_{n+js+s-1}\right)^T,$$
$$B_j = \left[b_{u,v}^{(j)}\right]_{u,v=1(1)s}, \qquad F_{n+j} = \left(f_{n+js}, f_{n+js+1}, f_{n+js+2},\cdots,f_{n+js+s-1}\right)^T$$

The multi-block methods in [11] have been introduced to take advantage of parallelism which arise when B_k is diagonal or lower triangular. Fatunla [7] has extended the multi-block methods of [11] to second order IVPs in ODEs of (1.1). The new approach herein is to use the multi-block formulas in (1.6) as MB$_2$VMs for a multi-block of solution output of (1.1) and (1.2). We provide herein a detailed theoretical approach on how it is achievable. We start by introducing some results on the matrix difference equations having initial values and boundary values, along with Wiener-Hopf matrix factorization of a matrix polynomial to determined the root distribution of the stability polynomial of the arising multi-block boundary value method. In [39] is a theory on parallelism of multi-block methods of [11] following [38]. An extension of [11] is the generalization to the multi-derivative μ-output multi-block method

$$Y_{n+k} = \sum_{j=0}^{k-1} A_j^{(q)}(\mu) Y_{n+j} + \sum_{l=1}^{q} h^l \left(\sum_{j=0}^{k} B_j^{(l)}(\mu) F_{n+j}^{(l-1)} \right); \qquad k \geq 1, \quad q \geq 1, \quad A_k = I_s, \tag{1.7}$$

given in [39]. Here

$$\begin{gathered} A_j^{(q)}(\mu) = \left[a_{u,v}^{(q,j)}(\mu) \right]_{u,v=1}^{s}, \quad Y_{n+j} = \left(y_{n+\mu j+c_1}, y_{n+\mu j+c_2}, \cdots, y_{n+\mu j+c_s} \right)^T, \\ B_j^{(l)}(\mu) = \left[b_{u,v}^{(l,j)}(\mu) \right]_{u,v=1}^{s}, \quad c = (c_1, c_2, \cdots, c_s)^T, \quad \mu = 1, 2, \cdots, \\ F_{n+j}^{(l-1)} = \begin{cases} F_{n+j} = \left(f_{n+\mu j+c_1}, f_{n+\mu j+c_2}, \cdots, f_{n+\mu j+c_s} \right)^T; & l = 0 \\ \left(f_{n+\mu j+c_1}^{(l-1)}, f_{n+\mu j+c_2}^{(l-1)}, \cdots, f_{n+\mu j+c_s}^{(l-1)} \right)^T; & l \geq 1 \end{cases} \end{gathered} \tag{1.8}$$

as in [39]. The new formulation in (1.7) is significant over Chu and Hamilton [11] in (1.6) because of its order and stability advantage for increasing block number k and derivative order q, and among other advantages because of the introduction of the parameter μ. Here the multi-block methods in (1.7) is a q-derivative, k-block, s-point, μ-output block method. The c in (1.8) provides the benefit to introduce solution at hybrid points in the time variable t of (1.1) in the blocks of (1.7), while the μ denotes the number of component overlaps in the consecutive block solution of Y_{n+j} and Y_{n+j-1} and function values $F_{n+j}^{(l-1)}$ and $F_{n+j-1}^{(l-1)}$ respectively. If we consider Y_{n+j} and $F_{n+j}^{(l-1)}$ as sets of its components, then the overlap imply that,

$$Y_{n+j} \cap Y_{n+j-1} = \begin{cases} \left\{ y_{n+\mu j+c_{s-2}+\mu}, \cdots, y_{n+\mu j+c_{s-1}} \right\}; & \mu = 1(1)s-1 \\ \phi(empty); & \mu \geq s \end{cases}$$

Correspondingly,

$$F_{n+j}^{(l-1)} \cap F_{n+j-1}^{(l-1)} = \begin{cases} \left\{ f_{n+\mu j+c_{s-2}+\mu}^{(l-1)}, \cdots, f_{n+\mu j+c_{s-1}}^{(l-1)} \right\}; & \mu = 1(1)s-1 \\ \phi(empty); & \mu \geq s, \quad l = 1(1)q \end{cases}$$

and the number No.$(\cdot)$ of overlapping components in consecutive blocks is

$$No.\left(Y_{n+j} \cap Y_{n+j-1} \right) = No.\left(F_{n+j}^{(l-1)} \cap F_{n+j-1}^{(l-1)} \right) = s - \mu; \quad j = 1(1)k, \quad l = 1(1)q.$$

The different value of μ corresponds to different block formalism, see [7, 9, 10]. If $\mu = s$, then (1.7) corresponds to non-overlapping in the block definition in (1.8) and for $\mu = 1(1)s-1$ and $\mu \geq s+1$, the arising multi-block method in (1.7) is not amendable to multi-block boundary value method implementation. For convenience, the discrete problem generated by a k-block multi-derivative methods in (1.7) with k initial block conditions is written in matrix form by introducing the $(N-k+1)\,s$ by $(N-k+1)\,s$ block matrices,

$$A^{(q)}(\mu) = \begin{pmatrix} A_k^{(q)}(\mu) & & & & \\ \vdots & \ddots & & & \\ A_0^{(q)}(\mu) & & \ddots & & \\ & \ddots & & \ddots & \\ & & A_0^{(q)}(\mu) & \cdots & A_k^{(q)}(\mu) \end{pmatrix};$$

$$B^{(l)}(\mu) = \begin{pmatrix} B_k^{(l)}(\mu) & & & & \\ \vdots & \ddots & & & \\ B_0^{(l)}(\mu) & & \ddots & & \\ & \ddots & & \ddots & \\ & & B_0^{(l)}(\mu) & \cdots & B_k^{(l)}(\mu) \end{pmatrix}; \tag{1.9}$$

with $l = 1(1)q$ and the block vectors are define as

$$Y = (Y_{n+k}, Y_{n+k+1}, \cdots, Y_{n+N})^T, \qquad F^{(l-1)} = \left(F_{n+k}^{(l-1)}, F_{n+k+1}^{(l-1)}, \cdots, F_{n+N}^{(l-1)}\right)^T.$$

Then one has,

$$A^{(q)}(\mu)Y - \sum_{l=1}^{q} h^l B^{(l)}(\mu)F^{(l-1)} = - \begin{pmatrix} \sum_{j=0}^{k-1} A_j^{(q)}(\mu)Y_{n+j} - \sum_{l=1}^{q} h^l \left(\sum_{j=0}^{k-1} B_j^{(l)}(\mu)F_{n+j}^{(l-1)}\right) \\ \vdots \\ A_0^{(q)}(\mu)Y_n - \sum_{l=1}^{q} h^l B_0^{(l)}(\mu)F_n^{(l-1)} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \end{pmatrix} \tag{1.10}$$

where the block matrices $A^{(q)}(\mu)$ and $B^{(l)}(\mu)$, $l = 1(1)q$ are lower triangular Toeplitz-block matrices. These methods in (1.10) are multi-block q-derivative initial value methods , since they generate discrete initial value problems of (1.1) and (1.2). However, we shall consider the non-overlapping block of solutions in the MB$_2$VMs to be introduced based on (1.7). Herein, the one-block or block BVMs of [28, 37] is extended to the MB$_2$VMs by employing the initial value multi-block methods of [11]. The results in the following section then holds for the order conditions of multi-block methods in (1.7).

1.1. **The local truncation error and order conditions of multi-block methods in** (1.7)**.**

Following Ikhile and Muka [39], the local truncation error operator for the k-block s-point method, $q \geq 1$ in (1.7) is given as,

$$\begin{aligned} L\left[Y_n(t_n);h\right] &= Y_{n+k}(t_n) - \sum_{j=0}^{k-1} A_j^{(q)}(\mu) Y_{n+j}(t_n) \\ &\quad - \sum_{l=1}^{q} h^l \left(\sum_{j=0}^{k} B_j^{(l)}(\mu) F^{(l-1)}\left(Y_{n+j}(t_n)\right)\right); \quad A_k = I_s, \quad \mu \geq 1 \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} Y_{n+j}(t_n) &= \left(y\left(t_{n+js}\right), y\left(t_{n+js+1}\right), y\left(t_{n+js+2}\right), \cdots, y\left(t_{n+js+s-1}\right)\right)^T \\ F^{(l-1)}\left(Y_{n+j}(t_n)\right) &= \Big(f^{(l-1)}\left(t_{n+js}, y\left(t_{n+js}\right)\right), f^{(l-1)}\left(t_{n+js+1}, y\left(t_{n+js+1}\right)\right), \\ &\quad f^{(l-1)}\left(t_{n+js+2}, y\left(t_{n+js+2}\right)\right), \cdots, f^{(l-1)}\left(t_{n+js+s-1}, y\left(t_{n+js+s-1}\right)\right)\Big)^T \end{aligned}$$

The Taylor series about t_n in (1.11) gives

$$L\left[Y_n(t_n);h\right] = \sum_{j=0}^{\infty} \frac{C_j h^j}{j!} Y_n^{(j)}(t_n); \quad Y_n^{(j)}(t_n) = \underbrace{\left(y^{(j)}(t_n), y^{(j)}(t_n), \cdots, y^{(j)}(t_n)\right)^T}_{s}. \tag{1.12}$$

The next result shows the order p of convergence of the linear multi-block methods (LMBM) in (1.7).

Theorem 1.1. *(cf:*[39]*)*
Let $e = (1, \cdots, 1)^T$*. Then the vector coefficients* $\{C_t\}_{t=0}$ *in* (1.12) *are given by*

$$C_t = \begin{cases} e - \sum_{j=0}^{k-1} A_j^{(q)}(\mu) e; & t = 0 \\ c - \sum_{j=0}^{k-1} A_j^{(q)}(\mu)(c + \mu j e) - \sum_{j=0}^{k} B_j^{(1)}(\mu) e; & t = 1 \\ c^2 - \sum_{j=0}^{k-1} A_j^{(q)}(\mu)(c+\mu j e)^2 - 2\sum_{j=0}^{k} B_j^{(1)}(\mu)(c+\mu j e) & \\ \quad -2\sum_{j=0}^{k} B_j^{(2)}(\mu) e; & t = 2 \\ \vdots & \\ c^t - \sum_{j=0}^{k-1} A_j^{(q)}(\mu)(c+\mu j e)^t & \\ -\sum_{l=1}^{q}\left[\left(\prod_{j=1}^{l}(t-j+1)\right)\sum_{j=0}^{k} B_j^{(l)}(\mu)(c+\mu j e)^{t-1}\right]; & t = 3,4,5,\cdots \end{cases},$$

where $c = (c_1, c_2, \cdots, c_s)^T$.

Vector powers are component-wise powers.

Proof. The proof is in Ikhile and Muka [39], but our interest is with when $q = 1$, $s = \mu$, $c = (0, 1, 2, \cdots, s-1)^T$. $\square$

The multi-block methods in (1.7) is of order p if $C_j = 0$, $j = 1(1)p$ and $C_{p+1} \neq 0$. We also give the following definitions:

Definition 1.1. *The method in* (1.7) *is said to be pre-consistent if* $C_0 = 0$.

Definition 1.2. *The method in* (1.7) *is said to be consistent if it is pre-consistent and* $C_1 = 0$.

Thus, when

$$L[Y_{n+k}(t_n); h] = \frac{C_{p+1}}{(p+1)!} h^{p+1} Y_{n+k+1}^{(p+1)}(t_n) + O(h^{p+2})e; \quad e = (1, \cdots, 1)^T,$$

the multi-block methods in (1.7) is said to have a uniform order p. The interest is on MB$_2$VMs of uniform order of its constituent linear multistep formulas (LMFs). However, if the LMF components of the multi-block methods in (1.7) are of various order $\{p_j\}_{j=1(1)s}$ then the order of the method in (1.7) is atleast $p = \min_{1\leq j \leq s}\{p_j\}$. Here $\bar{C}_{p+1} = \frac{C_{p+1}}{(p+1)!}$ is the local truncating error (l.t.e) constant of the method in (1.7). The normalized local truncation error constant (EC) is given as

$$_{MBM}E_{p+1} = \frac{C_{p+1}}{(p+1)! \sum_{j=0}^{k} B_j^{(l)} e}; \quad l = 1.$$

The $_{MBM}E_{p+1}$ enables the comparison of the error constant of the method in (1.7) with that $_{LMM}E_{p+1}$ of the LMM (1.3),

$$_{LMM}E_{p+1} = \left(0, 0, \cdots, 0, \frac{d_{p+1}}{(p+1)!\sigma_{LMM}(1)}\right)^T, \quad \sigma_{LMM}(r) = \sum_{j=0}^{k} \beta_j r^j,$$

in one-block formalism where $\frac{d_{p+1}}{(p+1)!}$ is the error constant of the LMM in (1.3).

The paper is organized as follows. In Section 2, we discuss the matrix finite difference equation with initial value and boundary value conditions. Section 3 is devoted to the formulation of multi-block boundary value methods, where the location of zeros and Wiener-Hopf factorization of a matrix polynomial are fully discussed. In Section 4 is the construction of multi-block generalized backward differentiation formulas (MBGBDFs). Section 5 is on the effect of using additional block methods in place of exact block of solution, on the stability of MB$_2$VMs. Numerical results are reported in Section 6 and the conclusion follows in Section 7.

2. THE SOLUTION OF MATRIX FINITE DIFFERENCE EQUATION

To understand the formulation of the multi-block boundary value methods to be proposed in the later sections, it is necessary to present a theory on the solution of the multi-block finite difference equation. Consider the k-block difference equation defined by

$$\sum_{j=0}^{k} A_j Y_{n+j} = 0; \quad n = 0, 1, \cdots, \qquad \underbrace{Y_0, Y_1, \cdots, Y_{k-1}}_{\text{intial multi-block of solution values to be provided}}, \tag{2.1}$$

where

$$A_j = \left[a_{i,l}^{(j)}\right]_{i,l=1(1)s}, \quad Y_{n+j} = (y_{n+s\cdot j}, y_{n+s\cdot j+1}, \cdots, y_{n+s\cdot j+s-1})^T, \quad j = 0(1)k$$

are known real constant matrices and block vectors to determine the block solution Y_{n+k}. The block vectors in $\{Y_0, Y_1, \cdots, Y_{k-1}\}$ are the known block initial values to determine the block solution Y_k . The associated characteristic matrix polynomial to (2.1) is

$$\widehat{\rho}(R) = \sum_{j=0}^{k} A_j R^j; \quad E^j Y_n = Y_{n+j}, \quad Y_{n+j} \in R^s. \tag{2.2}$$

One can readily show that there are multi-block solution of (2.1) of the form $Y_n = R_j^n \mathrm{G}, n \geq 0$ provided that R_j is from a complete set of solvents for the characteristics matrix polynomial in (2.2) and G is a vector of dimension compatible with R_j. Refer to [40] on how solvents can be computed. Define the block vectors

$$A = (A_0, \cdots, A_k)^T, \quad Y_n^* = (Y_n, \cdots, Y_{n+k})^T, \quad D(R) = (I_s, I_s R, \cdots, I_s R^k)^T.$$

Thus (2.1) and (2.2) are transformed to

$$A^T Y_n^* = 0,$$
$$\widehat{\rho}(R) = A^T D(R),$$

respectively. We are assuming all through the case where all the roots of det $(\widehat{\rho}(R))$ in (2.2) are all simple. Consider the block vectors C and matrix η,

$$C = (I_s, \cdots, I_s) \in R^{s \cdot k}, \qquad \eta = diag(R_1, R_2, \cdots, R_k), \quad R_j \in R^{s \times s}, \tag{2.3}$$

respectively. The set $\{R_1, R_2, \cdots, R_k\}$ is a complete set of right solvents for the matrix polynomial $\widehat{\rho}(R)$ in (2.2) which means equivalently that $\{R_1, R_2, \cdots, R_k\}$ are the matrix roots of the matrix polynomial $\widehat{\rho}(R)$ in (2.2) that is,

$$A^T D(R_j) = 0; \quad j = 1(1)k.$$

The following theorem establishes the relation of the roots of the matrix polynomial (2.2) to the solution of the multi-block finite difference matrix equations (2.1) comprising of initial value conditions determined from the initial blocks of solution $\{Y_0, Y_1, \cdots, Y_{k-1}\}$.

Theorem 2.1. *If all the matrix roots (solvents) $R_1, R_2, \cdots, R_k$ of the characteristic matrix polynomial* (2.2) *are simple, then the generated multi-block solution Y_n from* (2.1) *is given as*

$$Y_n = C\eta^n G; \qquad n = 0, 1, \cdots, \tag{2.4}$$

where,

$$G = V^{-1}(Y_0, Y_1, \cdots, Y_{k-1})^T;$$

$$V = \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_1 & R_2 & \cdots & R_k \\ \vdots & \vdots & \vdots & \vdots \\ R_1^{k-1} & R_2^{k-1} & \cdots & R_k^{k-1} \end{pmatrix},$$

and $G \in C^{k \cdot s}$ is a block vector determined by the conditions defined by the initial multi-block solutions $Y_0, Y_1, \cdots, Y_{k-1}$ $(n = 0)$ of the multi-block difference Eq. (2.1).

Proof. The discrete vector function (2.4) is a multi-block solution of (2.1) since

$$A^T Y_n^* = (A_0, \cdots, A_k) \begin{pmatrix} Y_n \\ Y_{n+1} \\ \vdots \\ Y_{n+k} \end{pmatrix} = A^T \begin{pmatrix} C\eta^n \\ C\eta^{n+1} \\ \vdots \\ C\eta^{n+k} \end{pmatrix} G = A^T \begin{pmatrix} C\eta^0 \\ C\eta^1 \\ \vdots \\ C\eta^k \end{pmatrix} \eta^n G,$$

with C and η given in (2.3). Thus

$$A^T \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_1 & R_2 & \cdots & R_k \\ \vdots & \vdots & \vdots & \vdots \\ R_1^k & R_2^k & \cdots & R_k^k \end{pmatrix} \eta^n G = \left(A^T D(R_1), A^T D(R_2), \cdots, A^T D(R_k)\right) \eta^n G.$$

To show (2.4) is the general multi-block solution of (2.1), assume the initial block conditions

$$Y_0 = C\eta^0 G, \quad Y_1 = C\eta^1 G, \quad \cdots, \quad Y_{k-1} = C\eta^{k-1} G,$$

provided by the initial multi-block solution vectors $Y_0, Y_1, \cdots, Y_{k-1}$ $(n = 0)$ from the past. These transforms to

$$(Y_0, Y_1, \cdots, Y_{k-1})^T = VG,$$

where V is the Vandermonde block matrix (see V in (2.4)) with rows $C\eta^j; j = 0, 1, \cdots, k-1$. The V is non-singular when R_j are distinct or equivalently independent and the eigenvectors are independent so that the vector G is then uniquely determined by

$$G = V^{-1}(Y_0, Y_1, \cdots, Y_{k-1})^T.$$

Thus, the general multi-block solution of the matrix difference equation (2.1) is given by

$$Y_n = C\eta^n V^{-1}(Y_0, Y_1, \cdots, Y_{k-1})^T.$$

□

It is to be noted that the eigenvalues of the matrix roots or the solvents $\{R_1, R_2, \cdots, R_k\}$ of (2.2) are the roots of $\det(\widehat{\rho}(r))$. It is now clear, that for the convergence of the solution in (2.4) of the matrix equation in (2.1), the eigenvalues of the solvents are required to be in the unit circle, but those on the unit circle need be simple.

In what follows, we consider the solution of the boundary value matrix difference equation

$$\begin{gathered} \sum_{j=0}^{k} A_j Y_{n+j} = 0; \quad n = 0, 1, \cdots, \quad k_1 + k_2 = k, \quad N \geq k, \\ \underbrace{Y_0, \cdots, Y_{k_1-1}}_{\text{(a)}} \quad \underbrace{Y_{k_1}, \cdots, Y_{N-k_2}}_{\text{(c)}} \quad \underbrace{Y_{N-k_2+1}, \cdots, Y_N}_{\text{(b)}} \end{gathered} \tag{2.5}$$

(a): initial multi-block boundary condition (given)

(b): final multi-block boundary condition (given)

The initial multi-block and final multi-block solution values in (a) and (b) respectively are boundary values of the solution to be provided, while in (c) is the multi-block of solution values to be generated from the matrix recurrence relation in (2.5). The MB$_2$VMs in (2.5) are a novel approach at developing very large scale integration methods (VLSIM) in the numerical solution of differential equations.

Theorem 2.2. *Suppose that the characteristic matrix polynomial in* (2.2) *(with* $k = k_1 + k_2$*) is such that, the eigenvalues of* $\{R_1, R_2, \cdots, R_{k_1}\}$ *are strictly in the unit circle, while the eigenvalues of* $\{R_{k_1+1}, R_{k_1+2}, \cdots, R_k\}$ *are strictly outside the unit circle. Then the generated multi-block solution* Y_{n+k} *from* (2.1) *is given as*

$$\begin{aligned} Y_n &= C\eta^n G; \quad n = 0, 1, \cdots, \\ G &= V^{-1}(Y_0, Y_1, \cdots, Y_{k_1-1}, Y_{N-k_2+1}, \cdots, Y_N)^T, \end{aligned} \tag{2.6}$$

where $G \in C^{k \cdot s}$ *is a block vector determined by the conditions defined by the initial multi-block solutions* $Y_0, Y_1, \cdots, Y_{k_1-1}$ *and the final multi-block solutions* $Y_{N-k_2+1}, \cdots, Y_N$ *of the of the multi-block difference equation in* (2.1)

Proof. The proof is similar to theorem 2.1, since (2.2) is now a multi-block boundary value problem from (2.1), where one assign the initial block values $Y_0, Y_1, \cdots, Y_{k_1-1}$ $(n = 0)$ and final block values ,$Y_{N-k_2+1}, \cdots, Y_N$ $(n = 0)$ for $N \geq k_1$ and $k_1 + k_2 = k$. One has,

$$\begin{aligned} &Y_0 = C\eta^0 G, \quad Y_1 = C\eta^1 G, \cdots, \quad Y_{k_1-1} = C\eta^{k_1-1} G, \\ &Y_{N-k_2+1} = C\eta^{N-k_2+1} G, \quad \cdots \quad, Y_N = C\eta^N G. \end{aligned} \tag{2.7}$$

Then, the corresponding matrix system of equations to Vandermonde in (2.4) is given as

$$V^{(k_1,k_2)} G = (Y_0, Y_1, \cdots, Y_{k_1-1}, Y_{N-k_2+1}, \cdots, Y_N)^T,$$

where

$$V^{(k_1,k_2)} = \begin{pmatrix} I_s & \cdots & I_s & I_s & \cdots & I_s \\ R_1 & \cdots & R_{k_1} & R_{k_1+1} & \cdots & R_k \\ \vdots & & \vdots & & \vdots & \vdots \\ R_1^{k_1-1} & \cdots & R_{k_1}^{k_1-1} & R_{k_1+1}^{k_1-1} & \cdots & R_k^{k_1-1} \\ & & & & & \\ R_1^{N-k_2+1} & \cdots & R_{k_1}^{N-k_2+1} & R_{k_1+1}^{N-k_2+1} & \cdots & R_k^{N-k_2+1} \\ \vdots & & \vdots & \vdots & \vdots & \\ R_1^N & \cdots & R_{k_1}^N & R_{k_1+1}^N & \cdots & R_k^N \end{pmatrix}. \tag{2.8}$$

The block matrix is the mosaic Vandermode block matrix. If $V^{(k_1,k_2)}$ is non-singular then the multi-block boundary value problem (2.1) with the initial and final block conditions in (2.7) will have a unique multi-block solution. Similarly one has

$$Y_n = CD^n \left(V^{(k_1,k_2)}\right)^{-1} (Y_0, Y_1, \cdots, Y_{k_1-1}, Y_{N-k_2+1}, \cdots, Y_N)^T, \quad n = 0, 1, 2, \cdots.$$

□

The above theorem establishes the general form of the solution of the boundary valued finite difference equation from (2.1).

Theorem 2.3. *Suppose that k_1 is the number of the initial blocks of conditions and $R_1, R_1, \cdots, R_k$ be the complete set of (right) solvents of* (2.2) *and let*

$$R_{j+1} = Q_{j+1}^{-1} L_{j+1} Q_{j+1}; \quad j = 0(1)k-1$$

are the diagonalizable matrix roots of the characteristic matrix polynomial (2.2) *associated with the matrix difference equation* (2.1). *Here*

$$L_{j+1} = diag(r_{s\cdot j+1}, r_{s\cdot j+2}, \cdots, r_{s\cdot j+s}); \quad j = 0(1)k-1,$$

such that

$$\| L_{j+1} \|_\infty = \max_j \{| r_{s\cdot j+1} |, | r_{s\cdot j+2} |, \cdots, | r_{s\cdot j+s} |\}; \quad j = 0(1)k-1, \quad s \geq 2, \tag{2.9}$$

where $r_1, r_2, \cdots, r_{s\cdot k}$ are also the distinct characteristic roots of the polynomial det $(\widehat{\rho}(r))$ with the assumption that these eigenvalues are arranged in absolute magnitude as

$$\begin{aligned} | r_1 | < | r_2 | < \cdots < | r_{s-1} | < | r_s | < | r_{s+1} | < \cdots < | r_{2s-1} | < | r_{2s} | < \cdots < | r_{s\cdot k_1} | \\ < | r_{s\cdot k_1+1} | < \cdots < | r_{s\cdot k_1 + s\cdot k_2 - 1} | < | r_{s(k_1+k_2)} |; \quad k_1 + k_2 = k, \end{aligned} \tag{2.10}$$

Let

$$\| L_{k_1} \|_\infty \leq 1 < \| L_{k_1+1} \|_\infty, \tag{2.11}$$

then, there exist an integer δ_0 for all $N \geq \delta_0$, such that the matrix (2.8) *is non-singular.*

Proof. The proof follows the approach of theorem 2.2.2 in chapter two, page 19 of Brugnano and Trigiante [19] for matrix difference equation (2.1). By (2.10) or (2.11), the solvents R_{j+1}; $j = 0(1)k-1$ are diagonalisable and

$$R_{j+1} = Q_{j+1}^{-1} L_{j+1} Q_{j+1}; \quad j = 0(1)k-1,$$

for some invertible matrices Q_{j+1}. By defining the following block matrices

$$V_1^{(u)} = \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_1 & R_2 & \cdots & R_{k_1} \\ \vdots & \vdots & \vdots & \vdots \\ R_1^{u-1} & R_2^{u-1} & \cdots & R_{k_1}^{u-1} \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_{k_1} \end{pmatrix}$$

$$V_2^{(u)} = \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_{k_1+1} & R_{k_1+2} & \cdots & R_k \\ \vdots & \vdots & \vdots & \vdots \\ R_{k_1+1}^{u-1} & R_{k-1+2}^{u-1} & \cdots & R_k^{u-1} \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} R_{k_1+1} & & \\ & \ddots & \\ & & R_k \end{pmatrix},$$

then

$$V^{(k_1,k_2)} = \begin{pmatrix} V_1^{(k_1)} & V_2^{(k_1)} \\ V_1^{(k_2)}\eta_1^N & V_2^{(k_2)}\eta_2^N \end{pmatrix} = \begin{pmatrix} V_1^{(k_1)} & \mathbf{O} \\ V_1^{(k_2)}\eta_1^N & \sigma_n \end{pmatrix} \begin{pmatrix} I_{k_1} & \left(V_2^{(k_1)}\right)^{-1} \\ \mathbf{O} & I_{k_1} \end{pmatrix},$$

where

$$\sigma_n = V_2^{(k_2)}\eta_2^N - V_1^{(k_2)}\eta_1^N \left(V_1^{(k_1)}\right)^{-1} V_2^{k_1}.$$

Thus $V^{(k_1,k_2)}$ is non singular if and only if σ_n is non singular. Now, because of (2.10) and (2.11)

$$\eta_1^N = I_{k_1} O\left(\| L_{k_1} \|_\infty^N\right), \quad \eta_2^{-N} = I_{k_2} O\left(\| L_{k_2} \|_\infty^{-N}\right),$$

and

$$\begin{aligned} \sigma_n &= \left(V_2^{(k_2)} - V_1^{(k_2)}\eta_1^N \left(V_1^{(k_1)}\right)^{-1} V_2^{(k_1)}\eta_2^{-N}\right)\eta_2^N \\ &= \left(V_2^{(k_2)} + O\left(\| L_{k_1+1}^{-1} L_{k_1} \|_\infty^N\right)\right)\eta_2^N. \end{aligned}$$

Since $V_2^{(k_2)}$ is also non-singular and $\| L_{k_1+1}^{-1} L_{k_1} \|_\infty < 1$ is bounded there exist $\delta_0 > 0$ such that σ_n is non-singular for all $N \geq \delta_0$. The minimum value of vector column of δ_0 is observed to be of the order $p = s \cdot k$ (s is the dimension of the coefficient matrices of the method in (2.1)). □

The above theorem establishes the existence of the solution of the boundary value finite difference equation from (2.2) by showing the invertibility of the mosaic block Vandermonde matrix $V^{(k_1,k_2)}$.

Theorem 2.4. *Suppose that the matrix roots of the characteristics matrix polynomial in* (2.2) *associated with* (2.1) *are such that,*

$$\| L_{k_1-1} \|_\infty < \| L_{k_1} \|_\infty < \| L_{k_1+1} \|_\infty, \qquad \| L_{k_1} \|_\infty \leq 1.$$

Then the multi-block solution of the boundary value finite difference equation associated with (2.1) *having* k_1 *number of initial block conditions and* k_2 *number of final block conditions in* (2.7) *has a solution for* n *and* $N-n$ *sufficiently large. In fact, the multi-block solution of* (2.1) *subject to* (2.7) *behaves asymptotically as*

$$Y_n = R_{k_1}^n \left(\alpha + O\left(l_1^n\right) + O\left(l_2^{N-n}+\right) + O\left(l_3^{-N}\right)\right) + O\left(l_3^{N-n}\right); \quad n = 0, 1, \cdots, \tag{2.12}$$

where the vector α *depends on* $Y_0, Y_1, \cdots, Y_{k_1-1}$ $(n=0)$ *and*

$$\left\{\begin{array}{r} l_1 = \| L_{k_1}^{-1} L_{k_1-1} \|_\infty < 1, \quad k_1 + k_2 = k \\ l_3 = min\left\{| r_{s\cdot k_1+1} |, | r_{s\cdot k_1+2} |, \cdots, | r_{s\cdot k_1+s} |\right\} > 1, \\ l_2 = \| L_{k_1+1}^{-1} L_{k_1} \|_\infty < 1, \quad l_4 = \| L_{k_1-1} \|_\infty < 1, \\ L_{j+1} = diag\left(r_{s\cdot j+1}, r_{s\cdot j+2}, \cdots, r_{s\cdot j+s}\right), \quad j = 0(1)k-1 \\ \| L_{k_1} \|_\infty = \max\{| r_{s(k_1-1)+1} |, | r_{s(k_1-1)+2} |, \cdots, | r_{s\cdot k_1} |\} \leq 1; \\ | r_{s\cdot k_1} | = 1, \quad 1 < | r_{s\cdot k_1+r} |, \quad r = 1(1)s \end{array}\right. . \tag{2.13}$$

Proof. This is obtained by extending the augment in the proof of theorem 2.6.1 in chapter two, page 39 in [19]. The (2.1) is a more general case than that considered in [19], page 39. Thus from (2.4), write that

$$Y_n = C_I \eta_I^n G_I + G_{k_1} R_{k_1}^n + C_F \eta_F^n G_F; \quad n = 0, 1, 2, \cdots, \tag{2.14}$$

where

$$\begin{aligned} C_I = \left(I_s, I_s, \cdots, I_s\right), \quad C_F = \left(I_s, I_s, \cdots, I_s\right), \qquad \eta_I = diag\left(R_1, R_2, \cdots, R_{k_1-1}, R_{k_1}\right), \\ \eta_F = diag\left(R_{k_1+1}, R_{k_2+1}, \cdots, R_k\right), \quad Y_n = \left(y_{s\cdot n}, y_{s\cdot n+1}, \cdots, y_{s\cdot n+s-1}\right)^T, \\ Y_I = \left(y_{s\cdot i}, y_{s\cdot i+1}, \cdots, y_{s\cdot i+s-1}\right)^T, \quad Y_F = \left(y_{s\cdot f}, y_{s\cdot f+1}, \cdots, y_{s\cdot f+s-1}\right)^T, \\ L_{i+1} = diag\left(r_{s\cdot i}, r_{s\cdot i+1}, \cdots, r_{s\cdot i+s-1}\right), \quad i = 0(1)k_1-1, \quad f = 0(1)k-k_1 \quad s \geq 2. \end{aligned}$$

The entries of the block vectors G_I and G_F symbolises the initial and final multi-block boundary conditions of the boundary value difference equations in (2.1). In a well defined compact form for (2.14),

$$\begin{pmatrix} I_s & C_I & C_F \\ V_{k_1-1} R_{k_1} & H_{k_1-1}\eta_I & Q_{k_1-1}\eta_F \\ V_{k_2} R_{k_2}^N & H_{k_2}\eta_I^N & Q_{k_2}\eta_F^N \end{pmatrix} \begin{pmatrix} G_{k_1} \\ G_I \\ G_F \end{pmatrix} = \begin{pmatrix} Y_0 \\ Y_I \\ Y_F \end{pmatrix},$$

where

$$H_j = \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_1 & R_2 & \cdots & R_{k_1-1} \\ \vdots & \vdots & & \vdots \\ R_1^{j-1} & R_2^{j-1} & \cdots & R_{k_1-1}^{j-1} \end{pmatrix}, \quad Q_j = \begin{pmatrix} I_s & I_s & \cdots & I_s \\ R_{k_1+1} & R_{k_1+2} & \cdots & R_k \\ \vdots & \vdots & & \vdots \\ R_{k_1+1}^{j-1} & R_{k_1+2}^{j-1} & \cdots & R_k^{j-1} \end{pmatrix},$$

with $V_j = \left(I_s, R_{k_1}, \cdots, R_{k_1}^{j-1}\right)^T$. Set,

$$Z = \begin{pmatrix} I_s & C_I & C_F \\ V_{k_1-1} R_{k_1} & H_{k_1-1}\eta_I & Q_{k_1-1}\eta_F \\ V_{k_2} R_{k_1}^N & H_{k_2}\eta_I^N & Q_{k_2}\eta_F^N \end{pmatrix}.$$

Next is show that the inverse of Z exist through the asymptotic sense. Decomposing Z into the form,

$$Z=\left(\begin{array}{ccc} I_s & \mathbf{O} & \mathbf{O} \\ V_{k_1-1}R_{k_1} & I_{C_I} & \mathbf{O} \\ V_{k_2}R_{k_1}^N & w & I_{C_F} \end{array}\right)\left(\begin{array}{ccc} I_s & C_I & C_F \\ \mathbf{O} & P_1 & P_2 \\ \mathbf{O} & \mathbf{O} & \psi \end{array}\right),$$

where

$$\begin{aligned} P_1 &= H_{k_1-1}\eta_I - V_{k_1-1}R_{k_1}C_I, \quad P_2 = Q_{k_1-1}\eta_F - V_{k_1-1}R_{k_1}C_F, \\ w &= \left(H_{k_2}\eta_I^N - V_{k_2}R_{k_1}^N C_I\right)P_1^{-1} = -R_{k_1}^N\left(V_{k_2}C_I + O\left(\| L_{k_1}^{-1}L_{k_1-1}\|_\infty^N\right)\right)P_1^{-1} \\ &= O\left(\| L_{k_1}\|_\infty^N\right), \quad \| L_{k_1}\|_\infty < 1 \qquad , \\ \psi &= Q_{k_2}\eta_F^N - V_{k_2}R_{k_1}^N C_F - wP_2 = \left(Q_{k_2} + O\left(\| L_{k_1+1}^{-1}L_{k_1}\|_\infty^N\right)\right)\eta_F^N. \end{aligned}$$

From the above,

$$Z^{-1}=\left(\begin{array}{ccc} I_s + C_I P_1^{-1}V_{k_1}R_{k_1}r^T\psi^{-1}\theta & r^T\psi^{-1}w - C_I P_1^{-1} & -r^T\psi^{-1} \\ P_1^{-1}\left(P_2\psi^{-1}\theta - V_{k_1-1}R_{k_1}\right) & P_1^{-1}\left(I_I P_2\psi^{-1}w\right) & P_1^{-1}P_2\psi^{-1} \\ \psi^{-1}\theta & -\psi^{-1}w & \psi^{-1} \end{array}\right),$$

where

$$r^T = C_I - C_I P_1^{-1}P_2, \quad \theta = V_{k_2}R_{k_1}^N - V_{k_1-1}R_k w = O\left(\| L_{k_1}\|_\infty^N\right).$$

Then one obtains

$$\begin{aligned} G_{k_1} &= \left(I_s + C_I P_1^{-1}V_{k_1-1}R_{k_1} + O\left((l_2)^N\right)\right)Y_0 \\ &\quad - \left(C_I P_1^{-1} + O\left((l_2)^N\right)\right)Y_I + O\left((l_3)^{-N}\right) = \alpha + O\left((l_2)^N\right) + O\left((l_3)^{-N}\right), \end{aligned}$$

$$\begin{aligned} G_I &= \left(-P_1^{-1}V_{k_1-1}R_k + O\left((l_2)^N\right)\right)Y_0 \\ &\quad - \left(I_I + O\left((l_2)^N\right)\right)Y_I + O\left((l_3)^{-N}\right) = \vartheta + O\left((l_2)^N\right) + O\left((l_3)^{-N}\right), \end{aligned}$$

and

$$\begin{aligned} G_F &= \eta_F^{-N}\left(Q_{k_2}^{-1} + O\left((l_2)^N\right)\right)\left(Y_F + O\left((l_4)^N\right)\right) \\ &= \eta_F^{-N}\left(\phi + O\left((l_2)^N\right) + O\left((l_4)^N\right)\right). \end{aligned}$$

where $l_q; q = 1(1)4$ as defined in (2.13). The block vectors $\vartheta = P_1^{-1}\left(Y_I - V_{k_1-1}R_{k_1}Y_0\right)$ and $\phi = Q_{k_2}^{-1}Y_F$ are constants independent of N, and

$$\alpha = Y_0 + C_I P_1^{-1}\left(Y_0 R_{k_1}V_{k_1-1} - Y_I\right),$$

depends only on the initial block conditions in (2.7) . Therefore,

$$Y_n = R_{k_1}^n\left(\alpha + O\left((l_1)^n\right) + O\left((l_2)^{N-n}\right) + O\left((l_3)^{-N}\right)\right) + O\left((l_3)^{N-n}\right). \tag{2.15}$$

Here R_{k_1} is the generating matrix root (solvent) of the associated characteristics matrix polynomial of (2.5) . $\square$

The solution (2.6) as restructured in (2.14) and the analysis of its convergence of the boundary value matrix difference equations (2.5) is by no means trivial, as can be seen in the proofs of the theorems 2.1– 2.4. For the convergence of the solution in (2.6) as restructured in (2.14) and with the asymptotic solution given in (2.15) of the boundary value matrix difference equations (2.5), the eigenvalues of the k_1 number of the solvents and k_2 number of the solvents are required to be in the unit circle and outside the unit circle respectively. However, those eigenvalues on the unit circle need be simple. It is now established through theorem 2.1– 2.4 that the matrix finite difference equation (2.1) subject to the initial block conditions and final block conditions in (2.7) has bounded block vector solutions (2.6) with respect to the conditions of the theorems. It is on the bases of this that a MB$_2$VMs can be formulated using (1.7) and (2.7) in the sense of the classic BVMs in (1.4).

3. Multi-block Boundary Value Methods (MB_2VMs)

Consider the linear multi-block formulas (LMBFs) of [11] in (1.6) of the form

$$\sum_{j=0}^{k} A_j Y_{n+j} = h \sum_{j=0}^{k} B_j F_{n+j}; \qquad n = 0, 1, \cdots : \quad k \geq 1. \tag{3.1}$$

obtained from (1.7) when $q = 1$, $\mu = s$, where

$$A_j = \left[a_{i,l}^{(j)}\right]_{i,l=1(1)s}, \quad B_j = \left[b_{i,l}^{(j)}\right]_{i,l=1(1)s},$$
$$Y_{n+j} = (y_{n+s\cdot j}, y_{n+s\cdot j+1}, \cdots, y_{n+s\cdot j+s-1})^T, \quad j = 0(1)k$$
$$F_{n+j} = (f_{n+s\cdot j}, f_{n+s\cdot j+1}, \cdots, f_{n+s\cdot j+s-1})^T.$$

The $\{Y_{n+j}\}_{j=0(1)k}$ are the multi-block of non-overlapping solution values and $\{F_{n+j}\}_{j=0(1)k}$ denotes the corresponding multi-block of non-overlapping function values of (3.1). The formula (3.1) is a k-block, s-point block formula. Here, the block shift operator E is defined as $E^j Y_n = Y_{n+j}$ see (2.2). Define the first and second characteristics matrix polynomial of (3.1) as

$$\widehat{\rho}(R) = \sum_{j=0}^{k} A_j R^j, \qquad \widehat{\sigma}(R) = \sum_{j=0}^{k} B_j R^j, \tag{3.2}$$

respectively. The first and second characteristic stability polynomial of (3.1) are

$$\rho(r) = det\left(\widehat{\rho}(r)\right) = det\left(\sum_{j=0}^{k} A_j r^j\right), \quad \sigma(r) = det\left(\sum_{j=0}^{k} B_j r^j\right). \tag{3.3}$$

The stability matrix polynomial of (3.1) on application on the scalar test equation $y' = \lambda y$, $Re(\lambda) < 0$ is

$$\widehat{\prod}(R,z) = \widehat{\rho}(R) - z\widehat{\sigma}(R); \quad z = \lambda h \tag{3.4}$$

The corresponding stability polynomial associated with (3.1) is thus,

$$\begin{gathered}\prod(r,z) = det\left(\widehat{\prod}(r,z)\right) = det\left(\widehat{\rho}(r) - z\widehat{\sigma}(r)\right); \\ r = e^{j\theta}, \quad 0 < \theta \leq 2\pi, \quad z = \lambda h, \quad Re(z) < 0\end{gathered}. \tag{3.5}$$

Implementing (3.1) as a MB$_2$VMs, we shall have

$$\begin{gathered}\sum_{j=-k_1}^{k_2} A_{j+k_1} Y_{n+j} = h \sum_{j=-k_1}^{k_2} B_{j+k_1} F_{n+j}; \quad n = 0(1)(N-k), \quad k > 1, \\ \underbrace{Y_0, \cdots, Y_{k_1-1}}_{\text{(a)}} \qquad \underbrace{Y_{k_1}, \cdots, Y_{N-k_2}}_{\text{multi-block of solution values to be generated by the MB}_2\text{VMs}} \qquad \underbrace{Y_{N-k_2+1}, \cdots, Y_N}_{\text{(b)}},\end{gathered} \tag{3.6}$$

as the main block formula while the initial multi-block formulas (a) and final multi-block formulas (b) in (3.6) are to be provided or replaced by multi-block multistep formula in (3.1). The coefficients $\{A_j, B_j\}$ are determined by imposing a $O\left(h^{2s\cdot k+1}\right)$ truncation error. By this the constituent linear multistep formulas (LMFs) of the linear block multistep formulas (LBMFs) of (3.6) can have maximum order $p = 2s \cdot k$. This, no doubt is an order advantage of MB$_2$VMs over the conventional BVMs in (1.4) . Here $q_1 = s \cdot k_1$ is the number of roots lying inside the unit circle and $q_2 = s \cdot k_2$ is the number of roots lying outside the unit circle of the stability polynomial in (3.5) of the main methods in (3.6). The discrete problem generated by a MB$_2$VMs (3.6) with (k_1, k_2)-block boundary conditions is written in the compact form

$$AY - hBF = -\begin{pmatrix} \sum_{j=0}^{k_1-1}\left(A_j Y_{n+j} - hB_j F_{n+j}\right) \\ \vdots \\ A_0 Y_{n+k_1-1} - hB_0 F_{n+k_1-1} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \\ A_k Y_{n+N-k_2+1} - hB_k F_{n+N-k_2+1} \\ \vdots \\ \sum_{j=1}^{k_2} A_{k_1+j} Y_{n+N-k_2+j} - h\beta_{k_1+j} f_{n+N-k_2+j} \end{pmatrix}, \tag{3.7}$$

where

$$Y = \left(Y_{n+k_1}, \cdots, Y_{n+N-k_2}\right)^T, \quad F = \left(F_{n+k_1}, \cdots, F_{n+N-k_2}\right)^T, \tag{3.8}$$

as the multi-block solution and function vectors of (3.7). The A and B are the multi-block Toeplitz matrices obtained from the main formula (3.6) without the initial multi-block formulas and final multi-block formulas. The arising MB$_2$VMs in (3.6) is thus A_{k_1,k_2}-stable. The multi-block Toeplitz matrix A is of the form

$$A = \begin{pmatrix} A_{k_1} & A_{k_1+1} & \cdots & A_k & \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \vdots & \ddots & & & & & & & \vdots \\ A_1 & & \ddots & & & & & & \vdots \\ A_0 & & & & & & & & \vdots \\ \mathbf{O} & \ddots & & & \ddots & & & & \vdots \\ \mathbf{O} & & \ddots & & & \ddots & & & A_k \\ \vdots & & & \ddots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & & & \ddots & A_{k_1+1} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} & \mathbf{O} & A_0 & A_1 & \cdots & A_{k_1} \end{pmatrix}_{(N-k)s\times(N-k)s} ; k_1 + k_2 = k, \tag{3.9}$$

and B is of a similar form, but with the $B_j's$ instead of the $A_j's$. The coefficient block matrices are banded Toeplitz-block matrices having lower band k_1 (equal to the number of block initial conditions) and upper band k_2 (equal to the number of block final conditions). It is trivial that the class of MB$_2$VMs (3.6) contains the conventional multi-block methods as introduced by Chu and Hamilton [11] for the case $k_1 = k$ and $k_2 = 0$ as in (1.9). The implementation of (3.6) is amenable to parallelism on a larger scale than the block method of (1.6) and the conventional BVMs in (1.4). An advantage of the MB$_2VMs$ in (3.6) or equivalently in (3.7) is that it outputs the multi-block of solution $\{Y_{n+k_1}, \cdots, Y_{n+N-k_2}\}$, unlike the conventional linear multistep methods in (1.3) which output a solution y_{n+k} at a point or the conventional boundary value methods in (1.4) which output the block of solution $\{y_{n+k_1}, \cdots, y_{n+N-k_2}\}$ and multi-block methods [11] in (1.6) which output a block of solution Y_{n+k} per step. The continuous problem (1.1) provides only the initial value y_0, whereas the k_1 extra initial blocks $Y_0, \cdots, Y_{k_1-1}$ $(n = 0)$, of solution values in (3.7) are to be given by the initial block formula,

$$\sum_{j=0}^{k} A_j^{(i)} Y_j = h \sum_{j=0}^{k} B_j^{(i)} F_j; \quad n = 0, \quad i = 0(1)k_1 - 1, \tag{3.10}$$

and the k_2 extra final blocks $Y_N, \cdots, Y_{N+k_2-1}$ $(n = 0)$ of solution values are similarly given by the final block formula,

$$\sum_{j=0}^{k} A_{N-k+j}^{(i)} Y_{N-k+j} = h \sum_{j=0}^{k} B_{N-k+j}^{(i)} F_{N-k+j}; \quad n = 0, \quad i = (N - k_2 + 1)(1)N. \tag{3.11}$$

Moreover, it is important for the composite matrix scheme, (3.6), (3.10) and (3.11) which is a MB$_2$VMs to have uniform order p or atleast $p = \min_{1\leq j\leq s}\{p_j\}$ where, p_j; $j = 1(1)s$ are the order of the respective constituent LMFs. Thus the composition is written as

$$A_N Y - h B_N F = \mathbf{O}, \quad \mathbf{O} = (\mathbf{O}, \cdots, \mathbf{O})^T. \tag{3.12}$$

Here the multi-block of solutions and functions are given as

$$\begin{aligned} Y &= \left(Y_n, \cdots, Y_{n+k_1-1}, Y_{n+k}, \cdots, Y_{n+N-k_2}, Y_{n+N-k_2+1}, \cdots, Y_{n+N}\right)^T, \\ F &= \left(F_n, \cdots, F_{n+k_1-1}, F_{n+k}, \cdots, F_{n+N-k_2}, F_{n+N-k_2+1}, \cdots, F_{n+N}\right)^T, \end{aligned} \tag{3.13}$$

and $A_N = \left[a \mid \bar{A}_N\right] \in R^{Ns\times(N+1)s}$ is

$$A_N = \left(\begin{array}{c|cccccc} A_0^{(0)} & A_1^{(0)} & \cdots & A_k^{(0)} & & & \\ A_0^{(1)} & A_1^{(1)} & \cdots & A_k^{(1)} & & & \\ \vdots & \vdots & \ddots & \vdots & & & \\ A_0^{(k_1-1)} & A_1^{(k_1-1)} & \cdots & A_k^{(k_1-1)} & & & \\ A_0 & A_1 & \cdots & A_k & & & \\ & A_0 & A_1 & \cdots & A_k & & \\ & & \ddots & \ddots & \cdots & \ddots & \\ & & & A_0 & A_1 & \cdots & A_k \\ & & & A_0^{(N-k_2+1)} & A_1^{(N-k_2+1)} & \cdots & A_k^{(N-k_2+1)} \\ & & & \vdots & \vdots & \cdots & \vdots \\ & & & A_0^{(N)} & A_1^{(N)} & \cdots & A_k^{(N)} \end{array}\right), \tag{3.14}$$

and $B_N = \left[b \mid \bar{B}_N\right] \in R^{Ns\times(N+1)s}$ is of similar form, but with $B_j's$ instead of $A_j's$. The matrix $A_N - zB_N$, has a multi-block quasi-Toeplitz structure [41, 42, 43] as a result of the additional multi-block formulas from (3.10) and (3.11). The (3.12) is equivalent to the one-block method

$$\bar{A}_N \bar{Y}_{n+1} + \bar{A}_0 \bar{Y}_n = h\left(\bar{B}_N \bar{F}_{n+1} + \bar{B}_0 \bar{F}_n\right), \quad n = 0, 1, \cdots, \tag{3.15}$$

in higher dimensional block with multi-block of solution output. Here the mult-block of solution and function values are given as

$$\begin{aligned} \bar{Y}_{n+1} &= \left(Y_{n+1}, \cdots, Y_{n+k_1-1}, Y_{n+k}, \cdots, Y_{n+N-k_2}, Y_{n+N-k_2+1}, \cdots, Y_{n+N}\right)^T, \\ \bar{F}_{n+1} &= \left(F_{n+1}, \cdots, F_{n+k_1-1}, F_{n+k}, \cdots, F_{n+N-k_2}, F_{n+N-k_2+1}, \cdots, F_{n+N}\right)^T, \end{aligned} \tag{3.16}$$

$$\bar{A}_0 = \left[\mathbf{O}_{(N-1)s} \times Ns \mid a\right] = \left(\begin{array}{c|c} & A_0^{(0)} \\ & A_0^{(1)} \\ & \vdots \\ & A_0^{(k_1-1)} \\ & A_0 \\ \mathbf{O}_{(N-1)s} \times Ns & \mathbf{O} \\ & \vdots \\ & \mathbf{O} \end{array}\right), \tag{3.17}$$

and

$$\bar{B}_0 = \left[\mathbf{O}_{(N-1)s} \times Ns \mid b\right] = \left(\begin{array}{c|c} & B_0^{(0)} \\ & B_0^{(1)} \\ & \vdots \\ & B_0^{(k_1-1)} \\ & B_0 \\ \mathbf{O}_{(N-1)s} \times Ns & \mathbf{O} \\ & \vdots \\ & \mathbf{O} \end{array}\right). \tag{3.18}$$

The MB$_2$VMs in (3.15) can be implemented through the use of Newton-Raphson method. Thus the multi-block solution $\bar{Y}_{n+1} = \bar{Y}_{n+1}^{[q]}$, $q > 0$ in (3.16) is iteratively obtained from,

$$\begin{aligned} \bar{Y}_{n+1}^{[i+1]} = \bar{Y}_{n+1}^{[i]} - (\bar{A}_N - h\bar{B}_N \frac{\partial M_{n+1}}{\partial Y_{n+1}})^{-1} (\bar{A}_N \bar{Y}_{n+1}^{[i]} + \bar{A}_0 \bar{Y}_n \\ -h\bar{B}_0 \bar{F}_n - h\bar{B}_N \bar{F}_{n+1}^{[i]}); \qquad i = 0(1)q \quad q > 1, \end{aligned} \tag{3.19}$$

where

$$\frac{\partial M(Y_{n+1})}{\partial Y_{n+1}} = \frac{\partial\left(f_{n+1}, \cdots, f_{n+N\cdot s}\right)}{\partial\left(y_{n+1}, \cdots, y_{n+N\cdot s}\right)} = \begin{pmatrix} \frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{\partial f_{n+1}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+1}}{\partial y_{n+N\cdot s}} \\ \frac{\partial f_{n+2}}{\partial y_{n+1}} & \frac{\delta f_{n+2}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+2}}{\partial y_{n+N\cdot s}} \\ \vdots & & & \\ \frac{\partial f_{n+s}}{\partial y_{n+1}} & \frac{\partial f_{n+s}}{\partial y_{n+2}} & \cdots & \frac{\partial f_{n+N\cdot s}}{\partial y_{n+N\cdot s}} \end{pmatrix}. \tag{3.20}$$

$$M(Y_{n+1}) = \bar{A}_N \bar{Y}_{n+1}^{[i]} + \bar{A}_0 \bar{Y}_n - h\bar{B}_0 \bar{F}_n - h\bar{B}_N \bar{F}_{n+1}^{[i]} = 0 \tag{3.21}$$

A modified Newton-Raphson method which uses a fixed Jacobian $J = \frac{\partial M}{\partial Y}$ from the ODEs in (1.1) and (1.2) when feasible can also be considered. The method in (3.15) are implemented with minimum block size using the Newton-Raphson method in (3.19), see Section 5. Next, is the consideration of the root distribution of the stability polynomial in (3.5) in order to determine the stability of the MB$_2$VMs in (3.6).

3.1. **Location of zeros of a polynomial and Wiener-Hopf Factorization of a matrix polynomial.**

This section considers location of zeros or root distribution of a polynomial arising from the determinant of a Wiener-Hopf factorization of a matrix polynomial in (2.2).

3.1.1. *Location of zeros of a polynomial.*

The stability of the $\mathrm{MB_2VMs}$ in $(3.7)$for the solution of ODEs in (1.1) and (1.2) rely on the distribution of the roots of the stability polynomial (3.5) with respect to the unit disk. We shall discuss schur criterion for a matrix polynomial in general for the case of initial and boundary value methods in (1.6) and (3.6) respectively. The characteristic polynomial we seek its root distribution is in general arising from first equation in (3.3) and (3.5). Thus, we defined the following polynomial

$$\rho(r) = det\left(\widehat{\rho}(r)\right) = \sum_{j=0}^{q} a_j r^j, \tag{3.22}$$

here $q = s \cdot k$ is the degree of the polynomial $\rho(r)$ and k is the degree of matrix polynomial $\widehat{\rho}(R)$ in (3.2). Suppose

$$d_1 = \{r \in \mathbb{C} :\mid r \mid< 1\}, \quad d_2 = \{r \in \mathbb{C} :\mid r \mid= 1\}, \quad d_3 = \{r \in \mathbb{C} :\mid r \mid> 1\},$$

and let q_1, q_2, q_3, q be four non-negative integers such that

$$q_1 + q_2 + q_3 = s \cdot k = q.$$

We make the following definitions;

Definition 3.1.

(a) *The polynomial $\rho(r)$ in (3.22) is said to be of root distribution type (q_1, q_2, q_3) if has q_1 number of zeros inside d_1, q_2 number of zeros on d_2 and q_3 number of zeros inside d_3.*

(a*) *The matrix polynomial $\widehat{\rho}(R)$ in (2.2) is said to be of block root (solvent) distribution type (k_1, k_2, k_3) if equivalently, the associated matrix polynomial $\rho(r) = det(\widehat{\rho}(r))$ is of root distribution type (q_1, q_2, q_3) when $q_i = s \cdot k_i$; $i = 1(1)3$.*

(b) *The matrix polynomial $\widehat{\rho}(R)$ in (2.2) is called Schur matrix polynomial if the polynomial $\rho(r)$ in (3.22) is of the type $(q, 0, 0)$, that is $q_1 = q$, $q_2 = q_3 = 0$.*

(c) *The matrix polynomial $\widehat{\rho}(R)$ in (2.2) is called Von-Neumann matrix polynomial if the polynomial $\rho(r)$ in (3.22) is of the type $(q_1, q - q_1), 0)$ with simple zeros on the unit circle.*

(d) *The matrix polynomial $\widehat{\rho}(R)$ in (2.2) is said to be self inversive, if the polynomial $\rho(r)$ in (3.22) and*

$$\rho^*(r) = \sum_{j=0}^{q} \bar{a}_j r^{-j} = r^q \bar{\rho}\left(r^{-1}\right); q = s \cdot k,$$

have the same set of roots, is the reverse polynomial of (3.22) and $\bar{a}_j$ is the conjugate of a_j. Since $\rho(r)$ in definition $(3.1a)$ is of the type (q_1, q_2, q_3), then then $\rho^*(r)$ is equivalent to the type (q_3, q_2, q_1).

However, the polynomial $\rho(r)$ of type $(q_1, 0, q-q_1))$ will be used in the subsequent sections for characterizing the stability of the multi-block boundary value methods in (3.6). An equivalent of definition (3.1a*) is that the matrix polynomial $\widehat{\rho}(R)$ in (2.2) of solvent distribution type (k_1, k_2, k_3) implies that k_1 number of the solvents have eigenvalues inside the unit circle, k_2 number of the solvents have eigenvalues on the unit circle and k_3 number of the solvents have eigenvalues outside the unit circle.

Theorem 3.1.
A matrix polynomial $\widehat{\rho}(R)$ in (2.2) is self inversive if and only if the polynomial $\rho(r)$ in (3.22) satisfy,

$$\rho^*(0)\rho(r) = \rho(0)\rho^*(r); \quad r \in \mathbb{C}.$$

Proof. The proof follows from definition 3.1d and [19], page 53. □

Theorem 3.2.
Suppose that $\mid \rho(0) \mid \neq \mid \rho^*(0) \mid$. Then $\widehat{\rho}(R)$ in (2.2) is of type (k_1, k_2, k_3) matrix polynomial if and only if the polynomial $\rho(r)$ of degree $q = s \cdot k$ in (3.22) satisfy the following conditions:

(a) $\rho_1(r)$ *is of the type* $(s \cdot k_1, s \cdot k_2, s \cdot k_3)$*, when* $\mid \rho^*(0) \mid > \mid \rho(0) \mid$
(b) $\rho_1(r)$ *is of the type* $(s \cdot k_3, s \cdot k_2, s \cdot k_1)$*, when* $\mid \rho^*(0) \mid < \mid \rho(0) \mid$,

where

$$\rho_1(r) = \frac{\rho^*(0)\rho(r) - \rho(0)\rho^*(r)}{r}; \quad \mid \rho^*(0) \mid \neq \mid \rho(0) \mid,$$

is of degree $s \cdot k - 1$.

Proof. This follows from [19], page 54. □

Example 3.1: Consider a matrix polynomial

$$\widehat{\rho}(R) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R^4 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R^3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.23)$$

This polynomial is self inversive matrix polynomial if

$$\rho(r) = det\,(\widehat{\rho}(r)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r^4 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r^3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = r^8 - 2r^7 + 3r^6 + r^4 - 3r^2 + 2r + 1,$$

is a self inversive polynomial. Let $(s \cdot k_1, s \cdot k_2, s \cdot k_3)$ be the unknown type of $\rho(r)$. Through iterative application of Theorem 3.2, we obtain the polynomial $\rho_8(r)$ of the type $(1, 0, 1)$. It implies that the matrix polynomial $\widehat{\rho}(R)$ in (3.23) is of the block root or solvent distribution type $(2, 0, 2)$ and $\rho(r)$ is equivalently of root distribution type $(4, 0, 4)$, see definition $3.1d$).

Theorem 3.3. *The matrix polynomial $\widehat{\rho}(R)$ in (2.2) is a Schur matrix polynomial if the associated polynomial $\rho(r)$ in (3.22) satisfy the following conditions, $\mid \rho^*(0) \mid > \mid \rho(0) \mid$ and $\rho_1(r)$ is a Schur polynomial.*

Proof. Following [19], page 55, suppose that the matrix polynomial (2.2) is a Schur matrix polynomial then the polynomial $\rho(r)$ in (3.22) of degree $q = s \cdot k$ is of the type $(s \cdot k, 0, 0)$ and thus,$\mid \rho^*(0) \mid > \mid \rho(0) \mid$. From theorem 3.2, it shows that $\rho_1(r)$ must be of type $(s \cdot k - 1, 0, 0)$. $\square$

The application of definition $(3.1a, b)$ is considered in the next example.

Example 3.2: Consider the matrix polynomial

$$\widehat{\rho}(R, z) = \begin{pmatrix} \frac{1}{2} & -2 \\ -\frac{1}{3} & \frac{3}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} & 0 \\ -3 & \frac{11}{6} \end{pmatrix} R - z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R, \quad z \in \mathbb{C}. \tag{3.24}$$

The associated polynomial $\rho(r)$ in (3.22) is given as

$$\rho(r) = det\left(\widehat{\rho}(r, 0)\right) = \frac{1}{12} - \frac{17}{6}r + \frac{11}{4}r^2, \tag{3.25}$$

with $\mid \rho^*(0) \mid > \mid \rho(0) \mid$ and $\rho_1(r) = \frac{68}{9}(r-1)$ which is a Von Neumann polynomial. It follows that the matrix polynomial in (3.24) is a Von Neumann matrix polynomial. Here $\rho(r) = (r-1)\left(\frac{33}{12}r - \frac{1}{12}\right)$ in (3.25).

Similarly, the polynomial $\rho(r)$ in (3.24) is given as

$$\rho(r, z) = det\left(\widehat{\rho}(r, z)\right) = \frac{1}{12} - \frac{17}{6}r + \frac{11}{4}r^2 - 2rz - \frac{10}{3}r^2 z + r^2 z^2,$$

$$\rho(r, 6) = \frac{1}{12} - \frac{89}{6}r + \frac{75}{4}r^2, \quad z = 6. \tag{3.26}$$

Here $\mid \rho^*(0) \mid > \mid \rho(0) \mid$ and $\rho_1(r) = \frac{-2492}{9} + \frac{3164}{9}r$ is a Shur polynomial. Thus the matrix polynomial in (3.24) is a Schur matrix polynomial since roots are $r_1 = 0.00565845$ and $r_2 = 0.785453$ in (3.26).

For this reason, the generalization of multi-block IVMs in definition (3.1) to multi-block BVMs is given in the following definition.

Definition 3.2. *A matrix polynomial $\widehat{\rho}(R)$ of degree $k = k_1 + k_2$ in (3.22) is an S_{k_1,k_2}-matrix polynomial, if the roots $\{r_j\}_{j=1}^{q}$ of the polynomial $\rho(r)$ are such that*

$$\mid r_1 \mid \leq \cdots \leq \mid r_{q_1} \mid < 1 < \mid r_{q_1+1} \mid \leq \cdots \mid r_q \mid, \quad q_1 + q_2 = q = s \cdot k.$$

Definition 3.3. *A matrix polynomial $\widehat{\rho}(R)$ of degree $k = k_1 + k_2$ in (2.2) is an N_{k_1,k_2}-matrix polynomial, if the roots $\{r_j\}_{j=1}^{q}$ of the polynomial $\rho(r)$ in (3.22) are such that*

$$\mid r_1 \mid \leq \cdots \leq \mid r_{q_1} \mid \leq 1 < \mid r_{q_1+1} \mid \leq \cdots \mid r_q \mid, \quad q_1 + q_2 = q = s \cdot k.$$

Definition 3.4. *The MB$_2$VM (3.7) with (k_1, k_2)- block boundary conditions where $k = k_1 + k_2$ is ;*

(a) *O_{k_1,k_2}-stable if the corresponding first characteristics matrix polynomial $\widehat{\rho}(R)$ in (3.2) is a N_{k_1,k_2}- matrix polynomial with $q_1 = s \cdot k_1$ and $q_2 = s \cdot k_2$.*

(b) (k_1, k_2)- *absolutely stable for a given* $z \in \mathbb{C}$, *if the corresponding matrix polynomial* $\widehat{\prod}(R, z)$ *in* (3.4) *is a* S_{k_1,k_2}- *matrix polynomial.*

(c) *The region* $D_{k_1,k_2} = \{z \in \mathbb{C} : \widehat{\prod}(R, z)$ *in* (3.4) *is a* S_{k_1,k_2}- *matrix polynomial* $\}$ *is said to be the region of* (k_1, k_2)*-absolute stability.*

(d) A_{k_1,k_2}*-stable if* $\bar{\mathbb{C}} \subseteq D_{k_1,k_2}$.

The A_{k_1,k_2}-stability define the stability of the MB$_2$VMs in terms of the block number k which is the degree of the stability matrix polynomial (3.4). It can as well be referred to as A_{k_1,k_2}-block stability.

3.1.2. *The Wiener-Hopf factorization of a matrix polynomial.*

Following [44, 45], recall the matrix polynomial

$$\widehat{\rho}(R) = \sum_{j=0}^{k} A_j R^j = A_0 + A_1 R + \cdots + A_k R^k$$

from (2.2) which may be with $A_j \in \mathbb{C}^{s\times s}$ such that $A_0 \neq \mathbf{O}$ and $A_k \neq \mathbf{O}$, where $\det(\widehat{\rho}(R)) \neq 0$. To obtain a Wiener-Hopf factorization of this matrix polynomial $\widehat{\rho}(R)$, we seek the root distribution of $det(\widehat{\rho}(R))$ in the factorization

$$\widehat{\rho}(R) = F(R)U(R); \qquad det(\widehat{\rho}(R)) = det(F(R))det(U(R)) \tag{3.27}$$

relative to the interior and exterior of the unit circle $\mid r \mid= 1$, where

$$F(R) = F_0 + \cdots + F_{k_1} R^{k_1} \text{ and } U(R) = U_0 + U_1 R + \cdots + U_{k_2} R^{k_2}.$$

such that the roots of $\{r_j\}_{j=1}^{s\cdot k}$ are of the roots of $\det(\widehat{\rho}(r))$ and

$$\left.\begin{array}{rll} det\left(r^{-k_1}F(r)\right): & \mid r_j \mid> 1: & j = 1(1)s\cdot k_2 \\ det\left(U(r)\right): & \mid r_j \mid\leq 1: & j = 1(1)s\cdot k_1 \end{array}\right\} \; s\cdot k = s\left(k_1 + k_2\right) \ . \tag{3.28}$$

respectively. In this sense, we assume the matrix polynomial $F(R)$ is monic by fixing $F_{k_1} = I_{k_1}$, since F_{k_1} is required to be invertible. Thus the matrix polynomial $\widehat{\rho}(R) = F(R)U(R)$ is said to be canonical right factorization of $\widehat{\rho}(R)$, if

$$F(R) = F_0 + \cdots + F_{k_1-1}R^{k_1-1} + I_{k_1}R^{k_1}, \quad U(R) = U_0 + U_1 R + \cdots + U_{k_2}R^{k_2},$$

and the roots are distributed relative to the interior and exterior of the unit circle as below

$$\left.\begin{array}{rll} det\left(F(r)\right) \neq 0: & \mid r_j \mid\leq 1 & j = 1(1)s\cdot k_1 \\ det\left(U(r)\right) \neq 0: & \mid r_j \mid> 1 & j = 1(1)s\cdot k_2 \end{array}\right\} \; s\cdot k = s\left(k_1 + k_2\right) \tag{3.29}$$

respectively. From (3.28) to (3.29), a right canonical factorization is assumed to exist, thus

$$G(R) = R^{-k_1}\widehat{\rho}(R) = \sum_{j=-k_1}^{k_2} A_j R^j, \quad k_1 + k_2 = k, \tag{3.30}$$

has a matrix factorization,

$$G(R) = G_-(R) \bigwedge G_+(R), \qquad \bigwedge = diag\left(R^{g_1}, \cdots, R^{g_s}\right), \tag{3.31}$$

with

$$G_-(R) = R^{-k_1}F(R) = I + F_{k_1-1}R^{-1} + \cdots + F_0 R^{-k_1}, \tag{3.32}$$

and

$$G_+(R) = U_0 + U_1 R + \cdots + U_{k_2} R^{k_2}. \tag{3.33}$$

The matrix $G_-(R)$ and $G_+(R)$ in (3.32) and (3.33) respectively are said to be invertible for $\mid r_j \mid > 1$ and $\mid r_j(z) \mid < 1$. The matrix function in (3.31) is called the Wiener-Hopf factorization of $\widehat{\rho}(R)$ or as a right Wiener-Hopf Factorization. From (3.30) , we thus have,

$$G(R) = G_-(R) diag\left(R^{g_1}, \cdots, R^{g_s}\right) G_+(R), \tag{3.34}$$

with $g_1, \cdots, g_s \in R$, then $G(R)$ be is said to invertible for $\mid r_j(z) \mid \geq 1$ and $\mid r_j(z) \mid \leq 1$ with respect to G_- and G_+ respectively, see [44]. In (3.32) and (3.33), $g_1, \cdots, g_s$ are the right partial indices of $G(R)$ and are unique up to their order. The factorization in (3.34) is said to be canonical, if all partial indices are such that $g_1 = \cdots = g_s = 0$. Thus,

$$G(R) = G_-(R)G_+(R). \tag{3.35}$$

The matrix polynomial factor representation in (3.35) provides a block upper and lower (UL) triangular factorization see [46],

$$G = G_- G_+,$$

of the block Toeplitz matrix A in (3.9), that is

$$G = A = \begin{pmatrix} A_{k_1} & A_{k_1+1} & \cdots & A_k & \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \vdots & \ddots & & & & & & & \vdots \\ A_1 & & \ddots & & & & & & \vdots \\ A_0 & & & & & & & & \vdots \\ \mathbf{O} & \ddots & & & \ddots & & & & \vdots \\ \mathbf{O} & & \ddots & & & \ddots & & & A_k \\ \vdots & & & \ddots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & & & \ddots & A_{k_1+1} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} & \mathbf{O} & A_0 & A_1 & \cdots & A_{k_1} \end{pmatrix}_{(N-k)s \times (N-k)s} ; k_1 + k_2 = k,$$

$$G_- = \begin{pmatrix} F_{k_1} & F_{k_1+1} & \cdots & I_s & \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \mathbf{O} & \ddots & & & & & & & \vdots \\ \vdots & & \ddots & & & & & & \vdots \\ \mathbf{O} & & & & & & & & \vdots \\ \mathbf{O} & \ddots & & & \ddots & & & & \mathbf{O} \\ \mathbf{O} & & \ddots & & & \ddots & & & I_s \\ \vdots & & & \ddots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & & & \ddots & F_{k_1+1} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & F_{k_1} \end{pmatrix}_{(N-k_2)s\times(N-k_2)s},$$

$$G_+ = \begin{pmatrix} U_{k_1} & 0 & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \vdots & \ddots & & & & & & & \vdots \\ U_1 & & \ddots & & & & & & \vdots \\ U_0 & & & & & & & & \vdots \\ \mathbf{O} & \ddots & & & \ddots & & & & \vdots \\ \mathbf{O} & & \ddots & & & \ddots & & & \mathbf{O} \\ \vdots & & & \ddots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & & & \ddots & \mathbf{O} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} & \mathbf{O} & U_0 & U_1 & \cdots & U_{k_1} \end{pmatrix}_{(N-k_1)s\times(N-k_1)s}.$$

The Wiener-Hopf matrix factorization in (3.35), if it exist suggest the MB$_2$VMs in (3.6) which stability polynomial in (3.5) has $s \cdot k_1$, number of roots in the unit circle and $s \cdot k_2$ number outside it. This is equivalent to the band structure of G_+ and G_- respectively for all $z \in \mathbf{C}$ preferably in the left half of the complex plane that include the stable region of the MB$_2$VMs in (3.6). The stable region of methods have been obtained herein by boundary locus plots of the corresponding stability polynomial $\prod(r, z)$ in (3.5), see Fig. 1 and others. In general, for the existence of the Wiener-Hopf matrix factorization in (3.35) of the matrix polynomial $\widehat{\rho}(R)$ in (3.2) which has the associated stability polynomial det$(\widehat{\rho}(r))$ in (3.3) having the root distribution $(q_1, 0, q_2)$, then the dimension of the coefficient matrices of the MB$_2$VM in (3.6) must be a multiplicative factor of q_1 and q_2. This apply to polynomial in (3.4) and (3.5) for a fixed $z \in \mathbb{C}$. The existence of such Wiener-Hopf factorization gives rise to a stable multi-block boundary value method and its band structure.

Remark 3.1. *The root distribution of the stability polynomial in* (3.5) *gives the appropriate* (k_1, k_2)*-block boundary conditions of the respective method from* (3.6)*. The correct implementation is obtained when the first characteristics polynomial* $\rho(r)$ *in* (3.3) *has the same root distribution with that of* (3.5) *for all values of* z *in the stability region in definition* (3.4).

Consider now the Newton-Raphson approach by Albrecht and Martin [44] to obtain the matrix factors $F(R)$ and $U(R)$ when they exist. Recall again the matrix polynomial $\widehat{\rho}(R) = \sum_{j=0}^{k} A_j R^j = A_0 + A_1 R + \cdots + A_k R^k$ and assume $\widehat{\rho}(R)$ has a right canonical factorization (3.35) with $k_1 \geq k_2$. The case $k_2 \geq k_1$ can be worked out similarly. The equation in (3.27) can be expressed as the non- linear system of the form

$$\begin{pmatrix} \psi_0 \\ \hline \psi_1 \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \hline \phi_1 \end{pmatrix} U, \tag{3.36}$$

in search of the unknown matrices $U_0, \cdots, U_{k_2}$ and $F_0, \cdots, F_{k_1-1}$, with

$$\phi_0 = \begin{pmatrix} F_0 & \mathbf{O} & & \cdots & \mathbf{O} \\ F_1 & F_0 & \mathbf{O} & \cdots & \\ \vdots & & \ddots & \mathbf{O} & \vdots \\ F_{k_1-1} & \cdots & \cdots & F_0 & \mathbf{O} \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} A_{k_1} \\ \vdots \\ A_k \end{pmatrix}, \quad U = \begin{pmatrix} U_0 \\ \vdots \\ U_{k_2} \end{pmatrix},$$

$$\psi_0 = \begin{pmatrix} A_0 \\ \vdots \\ A_{k_1-1} \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} I & F_{k_1-1} & \cdots & F_0 & \mathbf{O} \\ & I & F_{k_1-1} & \cdots & F_0 \\ & & \ddots & & \\ & & & & I \end{pmatrix}.$$

System (3.36) is equivalent to the two equations $\psi_0 = \phi_0 U$ and $\psi_1 = \phi_1 U$. Thus

$$E(F) = \psi_0 - \phi_0 \left(\phi_1\right)^{-1} \psi_1 = \mathbf{O} \tag{3.37}$$

To illustrate this with an example consider the matrix polynomial,

$$\widehat{\rho}(R) = A_2 R^2 + A_1 R + A_0 - z B_1 R; \quad z = \lambda h, \tag{3.38}$$

where the matrix coefficients are given as

$$A_0 = \begin{pmatrix} \frac{1}{12} & -\frac{2}{3} \\ \frac{1}{10} & -\frac{3}{4} \end{pmatrix}; A_1 = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{2} \end{pmatrix}; A_2 = \begin{pmatrix} -\frac{1}{12} & 0 \\ 0 & -\frac{1}{60} \end{pmatrix}; B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

which correspond to a MB$_2$VMs that is $A_{1,1}$-stability. Here the case of $z = 0$ in (3.38) gives rise to the first characteristics matrix polynomial $\widehat{\rho}_2(R) = F(R)U(R)$, which

$$\begin{pmatrix} A_0 \\ \hline A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{-2}{3} \\ \frac{1}{10} & \frac{-3}{4} \\ \hline 0 & \frac{2}{3} \\ \frac{1}{6} & \frac{-1}{2} \\ \frac{-1}{12} & 0 \\ 0 & \frac{-1}{60} \end{pmatrix} = \begin{pmatrix} f_{01} & f_{02} & 0 & 0 \\ f_{03} & f_{04} & 0 & 0 \\ \hline 1 & 0 & f_{01} & f_{02} \\ 0 & 1 & f_{03} & f_{04} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{01} & u_{02} \\ u_{03} & u_{04} \\ u_{11} & u_{12} \\ u_{13} & u_{14} \end{pmatrix}. \tag{3.39}$$

Using Netwon-Raphson approach as in [44] to resolve this non-linear equation in (3.37), we have

$$\begin{aligned} f_{01} &= 0.18873316096819862, f_{02} = -34.05633419515901, \\ f_{03} &= -2.02831719316661, f_{04} = -28.858414034166945, \\ u_{01} &= -0.01572776341401655, u_{02} = 0.09906109674734993, \\ u_{03} &= -0.0023597660972 1752, u_{04} = 0.01902643276388418, \\ u_{11} &= -0.08333333333333334, u_{12} = 0, u_{13} = 0, u_{14} = -0.016666666666666666. \end{aligned} \tag{3.40}$$

By this,

$$F(R) = \begin{pmatrix} 0.1887331 & -34.0563341 \\ -2.0283171 & -28.8584140 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R, \tag{3.41}$$

where the roots of the det $(F(r))$ gives two real outside the unit circle;
$r_1 = -2.04642$ and $r_2 = 31.0936$. From (3.40),

$$U(R) = \begin{pmatrix} -0.015727 & 0.0990610 \\ -0.0023597 & 0.0190264 \end{pmatrix} + \begin{pmatrix} -0.0833333 & 0 \\ 0 & -0.0166666 \end{pmatrix} R. \tag{3.42}$$

Similarly, the roots of the det $(U(r))$ gives two real roots inside the unit circle;
$r_3 = -0.0471472$ and $r_4 = 1$. From (3.38), arise the 2-block s-point formula

$$A_2Y_{n+2} + A_1Y_{n+1} + A_0Y_n = hB_1F_{n+1}; \quad n = 0, 1, \cdots . \tag{3.43}$$

which is found to be $O_{1,1}$-stable and $A_{1,1}$-stable of order $p = 4$. The result is the 2-block BVM

$$AY - hBF = \begin{pmatrix} -A_0Y_n + hB_0F_n \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \\ \mathbf{O} \\ -A_2Y_{n+N} + hB_2F_{n+N} \end{pmatrix}, \tag{3.44}$$

defined by the band structured block Toeplitz matrix

$$A = \begin{pmatrix} A_1 & A_2 & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ A_0 & A_1 & A_2 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & A_0 & A_1 & A_2 & \cdots & \mathbf{O} \\ & & & \ddots & & \\ \mathbf{O} & \cdots & \mathbf{O} & A_0 & A_1 & A_2 \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & A_0 & A_1 \end{pmatrix}; \quad B = \begin{pmatrix} \mathbf{O} & B_2 & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & B_2 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & B_2 & \cdots & \mathbf{O} \\ & & & & \ddots & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & B_2 \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} \end{pmatrix}, \tag{3.45}$$

from (3.38). The matrix A has a Wiener-Hopf factorization G_-G_+ given by (3.35). The remark 3.1 holds with respect to (3.44), see (3.39) to (3.42). The MB$_2$VMs in (3.6) is appropriately implemented if its stability nature is correctly determined through the Wiener-Hopf factorization of its first characteristics stability matrix polynomial $\widehat{\rho}(r)$ and its stability matrix polynomial $\widehat{\prod}(R, z)$ to have the same root distribution. The following definition emphasizes the essence of this.

Definition 3.5. *A consistent MB$_2$VMs in* (3.6) *is correctly used with* $z \in \mathbb{C}^-$, *where its stability matrix polynomial* $\widehat{\prod}\,(R, z)$ *in form of* (3.4) *is of the root distribution type* $(k_1, 0, k_2)$ *when* k_1 *number of block conditions are imposed at the initial block of points and* k_2 *number of block conditions imposed at the end of the interval of interest of the integration.*

By definition 3.5, the polynomial det$(\widehat{\rho}(r))$ must be of the same root distribution as det$(\widehat{\prod}\,(r, z))$, $z \in \mathbb{C}^-$. More so, for the MB$_2$VMs in (3.6) with (k_1, k_2) $-$block-boundary conditions, the corresponding family of block matrices $T_N^{(k)} = A - zB$ from (3.7) are well conditioned when $z \in D_{k_1,k_2}$. That is, the condition numbers of the block matrix $T_N^{(k)}$ are uniformly bounded with respect to increasing N and k, see Table 2.

4. CONSTUCTION OF THE MULTI-BLOCK GENERALIZED BACKWARD DIFFERENTIATION FORMULAS (MBGBDFS)

A particular family of MB$_2$VMs in (3.6) to be proposed, is the multi-block GBDFs (MBGBDFs),

$$\begin{gathered}\sum_{j=0}^{k} A_j Y_{n+j} = hB_i F\left(Y_{n+i}\right); \quad i = 1(1)k, \quad n = 0, 1, \cdots; \quad k = k_1 + k_2, \\ \underbrace{Y_0, \cdots, Y_{i-1}}_{\text{(a)}} \qquad \underbrace{Y_i, \cdots, Y_{N-k+i}}_{\text{multi-block solution values to be generated by the MBGBDFs}} \qquad \underbrace{Y_{N-k+i+1}, \cdots, Y_N}_{\text{(b)}}\end{gathered} \tag{4.1}$$

as the main formula, while the block solutions input (a) and (b) are to be provided or replaced by multi-block multistep formulas (MBMFs) as equations. The $i = k$ in (4.1) is the conventional initial value methods of multi-block backward differentiation formulas (MB$_2$DFs). For example, the one-block 2-point BDF is A-stable, 2-block 2-point BDF is A(63^o)-stable, while the case of 3-block 2-point BDF is unstable. In general, the A-stability property of MB$_2$DFs is limited as the block number k and the dimension s increases. The BVMs overcome the Dahlquist order-stability barrier [1, 3, 15] which has been extended to LMMs that employs second derivative or higher order derivatives in the Daniel-Moore order-stability barrier conjecture [1]. However, this setback in the stability of the initial value multi-block methods (1.6) from [11] is corrected through its implementation as MB$_2$VMs as will be seen in what are to follow. Using (4.1) as MB$_2$VMs with $i \neq k$, we gain the freedom of choosing the appropriate values of i that provides methods having the best stability properties for all block number $k \geq 2$ and block size $s \geq 2$. The stability polynomial of (4.1) is

$$\begin{aligned}\prod(r, z) &= det\left(\widehat{\rho}(r) - z\widehat{\sigma}(r)\right) = det\left(A_j r^j - zB_i r^i\right), \quad i = 0(1)k \\ &= \sum_{j=0}^{q} a_j r^j - z\sum_{i=1}^{q_1} b_i r^i, \quad q = s \cdot k, \quad q_1 = s \cdot k_1,\end{aligned} \tag{4.2}$$

with

$$\rho\left(r\right) = \sum_{j=0}^{q} a_j r^j, \quad \sigma\left(r\right) = \sum_{i=1}^{q_1} b_i r^i,$$

as the first and second stability polynomials associated with (4.1) respectively. Note that for a stable method, stability region is the exterior of the simple closed curve defined by the boundary locus of the stability polynomial in (3.5) or (4.2) in all the stability plots of the methods, see Fig. 1, 2, and others. The MBGBDFs in (4.1) has $k+1$ blocks which allow the method to attain maximum order $p = q(= s \cdot k)$. However, B_i can be choosen as I_s, the identity matrix or

$$B_i = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \\ \vdots & \cdots & \vdots & 1 & \vdots & \cdots & \vdots \\ \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.3}$$

The $B_i = I$ is preferred when solving the DAEs problems in (1.2), in order to avoid singularity in the Jacobian of the method. The B_i in (4.3) is suitable for solving stiff problems (see, problem 1 in (6.5)).

Theorem 4.1. *Suppose the MBGBDF in* (4.1) *is* A_{k_1,k_2}*-stable. Then* $k_1 = i$ *and* $k_2 = k - i$.

Proof. The stability polynomial of an A_{k_1,k_2}-stable MBGBDF in (4.1) is given in (4.2) which does not change its type as z varies in $\mathbb{C}^-$, since the boundary plot belong to $\mathbb{C} \setminus \mathbb{C}^-$ and as $z \to \infty$. The stability polynomial $\prod(r, z)$ in (4.2) approaches the polynomial $\sum_{i=0}^{q_1} b_i r^i$ if $z \to \infty$ which has matrix polynomial of the type $(i, 0, m-i)$. The $s \cdot i = q_1$ is the number of zeros inside the unit circle and $s(m-i) = q - q_1$ is the number of roots outside the unit circle of the stability polynomial in (4.2) of the method in (4.1) □

Here $i = u$ is choosen as

$$u = \begin{cases} \frac{k+2m}{2}; & k \quad even \quad \frac{-k}{2} \le m \le \frac{k}{2} \\ \frac{k+1+2m}{2}; & k \quad odd \quad \frac{-k+1}{2} \le m \le \frac{k-1}{2} \end{cases} \tag{4.4}$$

The method in (4.1) are $A_{u,k-u}$-stable thus implemented with $(u, k-u)$-block boundary conditions. The following set of initial additional block formulas

$$\sum_{j=0}^{k} A_j^{(i)} Y_j = h B_i^{(i)} F_i; \quad n = 0, \quad i = 0, \cdots, u-1, \tag{4.5}$$

and final additional block formulas,

$$\sum_{j=0}^{k} A_j^{(i)} Y_{N-k+j} = h B_i^{(i)} F_i; \quad n = 0, \quad i = N-m+1, \cdots, N, \tag{4.6}$$

are needed to implement a MBGBDFs in (4.1). The coefficients of (4.5) and (4.6) are uniquely determined to attain the same order as the MB$_2$VMs in (4.1). For simplicity, we fix $s = 2$, with $B_u =_a B_u$ in (4.1) given as

$$
{}_aB_u = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & \text{u is even}, \quad a = 1 \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & \text{u is odd}, \quad a = 2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for any u}, \quad a = 3 \end{cases}, \tag{4.7}
$$

to derive a family of methods in (4.1) for an increasing block number k. The method in (4.1) has the stability polynomial (4.2) of type $(s \cdot k_1, 0, s \cdot k_2)$ and possesses O_{k_1,k_2}-stability and A_{k_1,k_2}-stability properties. Here the third option $a = 3$ in (4.7) is preferred for ${}_aB_u$ to avoid singularity in the blending of the method (4.1) during implementation (see, [47]). The first characteristics stability polynomial in (4.2) is of the type $(s \cdot k_1 - 1, 1, s \cdot k_2)$. The boundary loci are shown in Fig. 1a for m odd and Fig. 1b for m even. The MBGBDF in (4.1) have been derived for various dimension of block size of solution Y_{n+j} and block number k. The Fig. 1 has the stability plots of MBGBDF in (4.1) of order $p = 2k$, $s = 2$, $k = 2(1)15$ for k odd or even with aid of MATHEMATICA version 11.1 [51]. See Fig. 2 for MBGBDF in (4.1) of order $p = 3k$, $s = 3$, $k = 2(1)10$ for k odd or even, and Fig. 3 and 4 for block dimension $s = 2(1)7$ and $s = 8(1)10$ respectively. The Fig. 8 contains the boundary loci of MBGBDF in (4.1) of order $p = 2k$, $s = 2$, $k = 2(2)16$.

Let Y_n be block discrete solution obtained by using the k-block GBDF (4.1) on the test equation $y' = \lambda y$. Suppose the roots associated with the corresponding stability polynomial (4.2), for Re$(z) < 0$ is such that,

$$
\| L_1 \|_\infty \le \cdots \le \| L_{k_1} \|_\infty \le 1 < \| L_{k_1+1} \|_\infty \le \cdots \le \| L_k \|_\infty,
$$

from theorem (2.3). By theorem (2.4), it shows that R_{k_1} is the solvent block vector generating root and then

$$
Y_n \approx R_{k_1}^n Y_0.
$$

Also from theorem (4.1), it follows as $z \to -\infty$ then that $R_{k_1}^n \to 0$ as n increases. Therefore the multi-block GBDFs is $L_{u,k-u}$-stable for diagonal matrix ${}_aB_u$. To fix idea, using MATHEMATICA version 11.1 [50, 51], the matrix coefficients of the fourth order 2-block, s-point MBGBDFs,

$$
A_2 Y_{n+2} + A_1 Y_{n+1} + A_0 Y_n = h B_1 F_{n+1}; \quad n = 0(1)(N-2), \qquad B_1 = {}_3B_u, \tag{4.8}
$$

in (4.1) as main method have been derived with $s \cdot k_1 = q_1$, $s \cdot k_2 = q_2$ and $s = 2$, where $m = 0$ in (4.4) and

$$
\begin{gathered}
A_0 = \begin{pmatrix} \frac{1}{200} & -\frac{1}{15} \\ \frac{-1}{75} & \frac{3}{40} \end{pmatrix}; A_1 = \begin{pmatrix} 0 & \frac{1}{15} \\ \frac{-1}{5} & \frac{2}{15} \end{pmatrix}; A_2 = \begin{pmatrix} -\frac{1}{120} & 0 \\ 0 & -\frac{1}{200} \end{pmatrix}; \\
B_1 = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \qquad \bar{C}_5 = \begin{pmatrix} -\frac{1}{300} \\ -\frac{1}{100} \end{pmatrix}.
\end{gathered} \tag{4.9}
$$

In general, the entries of the matrix coefficients of the MB$_2$VMs presented herein are normalized to avoid round off amplification. It is $A_{1,1}$-stable and can be used with one initial block formulas in (4.5) having the matrix coefficients,

$$A_0^{(0)} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{40} & -\frac{1}{12} \end{pmatrix}; A_1^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{3}{20} & -\frac{1}{20} \end{pmatrix}; A_2^{(0)} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{120} & 0 \end{pmatrix};$$
$$B_0^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \qquad \bar{C}_5 = \begin{pmatrix} 0 \\ -\frac{1}{300} \end{pmatrix}, \tag{4.10}$$

and one final additional block formulas in (4.6) with matrix coefficient,

$$A_0^{(N)} = \begin{pmatrix} \frac{1}{40} & -\frac{2}{15} \\ \frac{2}{25} & -\frac{3}{8} \end{pmatrix}; A_1^{(N)} = \begin{pmatrix} \frac{3}{10} & -\frac{2}{5} \\ \frac{2}{3} & \frac{-1}{2} \end{pmatrix}; A_2^{(N)} = \begin{pmatrix} \frac{5}{24} & 0 \\ 0 & \frac{77}{600} \end{pmatrix};$$
$$B_2^{(N)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \qquad \bar{C}_5 = \begin{pmatrix} -\frac{1}{50} \\ -\frac{1}{10} \end{pmatrix}. \tag{4.11}$$

In one-block form in (3.15), then (4.8) is,

$$\bar{A}_N = \begin{pmatrix} A_1^{(0)} & A_2^{(0)} & \mathbf{O} & \cdots & \cdots & & \mathbf{O} \\ A_0 & A_1 & A_2 & \mathbf{O} & & & \mathbf{O} \\ \mathbf{O} & A_0 & A_1 & A_2 & \mathbf{O} & & \mathbf{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \mathbf{O} \\ \vdots & & \mathbf{O} & A_0 & A_1 & A_2 & \mathbf{O} \\ \vdots & & \mathbf{O} & \mathbf{O} & A_0 & A_1 & A_2 \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & A_0^{(N)} & A_1^{(N)} & A_2^{(N)} \end{pmatrix}_{Ns\times Ns},$$

$$\bar{B}_N = \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} \\ \mathbf{O} & B_1 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & B_1 & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{O} \\ \vdots & & \mathbf{O} & \mathbf{O} & B_1 & \mathbf{O} \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} & B_2^{(N)} \end{pmatrix}_{Ns\times Ns}, \bar{B}_0 = \left(\begin{array}{c|c} & B_0^{(0)} \\ & \mathbf{O} \\ \mathbf{O}_{(N-1)s\times N\cdot s} & \mathbf{O} \\ & \mathbf{O} \\ & \vdots \\ & \mathbf{O} \end{array}\right),$$

and

$$\bar{A}_0 = \left(\begin{array}{c|c} & A_0^{(0)} \\ & \mathbf{O} \\ \mathbf{O}_{(N-1)s\times N\cdot s} & \mathbf{O} \\ & \mathbf{O} \\ & \vdots \\ & \mathbf{O} \end{array}\right)$$

For $k = 3$, one obtains the matrix coefficients of a sixth order 3-block, 2-point GBDF in (4.1) with $s \cdot k_1 = 4$, $s \cdot k_2 = 2$,$s = 2$, $B_2 = \frac{1}{10} I_s$, $m = 0$ as

$$\begin{aligned} A_0 &= \begin{pmatrix} \frac{1}{600} & -\frac{2}{150} \\ -\frac{1}{1050} & \frac{1}{120} \end{pmatrix}; A_1 = \begin{pmatrix} \frac{1}{20} & -\frac{4}{30} \\ -\frac{1}{30} & \frac{5}{60} \end{pmatrix}; A_2 = \begin{pmatrix} \frac{7}{120} & \frac{2}{50} \\ -\frac{1}{6} & \frac{47}{600} \end{pmatrix}; \\ A_3 &= \begin{pmatrix} -\frac{1}{300} & 0 \\ \frac{1}{30} & -\frac{1}{420} \end{pmatrix}; B_2 = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{1050} \\ -\frac{1}{1680} \end{pmatrix}. \end{aligned} \tag{4.12}$$

It is $A_{2,1}$-stable, which can be used with two additional initial block equations in (4.5) defined by the coefficient matrices

$$\begin{aligned} A_0^{(0)} &= \begin{pmatrix} 0 & 0 \\ -\frac{1}{60} & -\frac{77}{600} \end{pmatrix}; A_1^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & -\frac{1}{6} \end{pmatrix}; A_2^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{12} & -\frac{1}{40} \end{pmatrix}; \\ A_3^{(0)} &= \begin{pmatrix} 0 & 0 \\ \frac{1}{300} & 0 \end{pmatrix}; B_0^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_7 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{420} \end{pmatrix}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} A_0^{(1)} &= \begin{pmatrix} \frac{1}{300} & -\frac{1}{25} \\ -\frac{1}{1050} & \frac{1}{100} \end{pmatrix}; A_1^{(1)} = \begin{pmatrix} -\frac{7}{120} & \frac{2}{15} \\ -\frac{3}{50} & -\frac{1}{40} \end{pmatrix}; A_2^{(1)} = \begin{pmatrix} -\frac{1}{20} & \frac{1}{75} \\ \frac{1}{10} & -\frac{3}{100} \end{pmatrix}; \\ A_3^{(1)} &= \begin{pmatrix} -\frac{1}{600} & 0 \\ \frac{1}{150} & -\frac{1}{1400} \end{pmatrix}; B_2^{(1)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{1050} \\ \frac{1}{560} \end{pmatrix}. \end{aligned} \tag{4.14}$$

The one final additional block equation in (4.6) with the matrix coefficients,

$$\begin{aligned} A_0^{(N)} &= \begin{pmatrix} \frac{1}{60} & -\frac{3}{25} \\ -\frac{1}{70} & \frac{7}{60} \end{pmatrix}; A_1^{(N)} = \begin{pmatrix} \frac{3}{8} & -\frac{2}{3} \\ -\frac{21}{50} & \frac{7}{8} \end{pmatrix}; A_2^{(N)} = \begin{pmatrix} \frac{3}{4} & -\frac{3}{5} \\ -\frac{7}{6} & \frac{21}{20} \end{pmatrix}; \\ A_3^{(N)} &= \begin{pmatrix} \frac{49}{200} & 0 \\ -\frac{7}{10} & \frac{363}{1400} \end{pmatrix}; B_3^{(N)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{70} \\ -\frac{1}{80} \end{pmatrix}. \end{aligned} \tag{4.15}$$

Similarly, for $k = 3$, one obtains the matrix coefficients of a sixth order multi-block GBDF in (4.1) with $s \cdot k_1 = 4$, $s \cdot k_2 = 2$, $s = 2$, $B_i = {}_2B_u$, $u = 2$, $m = 0$

$$\begin{aligned} A_0 &= \begin{pmatrix} -\frac{1}{300} & \frac{1}{40} \\ -\frac{1}{75} & \frac{1}{24} \end{pmatrix}; A_1 = \begin{pmatrix} -\frac{1}{12} & \frac{1}{6} \\ -\frac{2}{15} & \frac{1}{4} \end{pmatrix}; A_2 = \begin{pmatrix} -\frac{1}{4} & \frac{77}{600} \\ -\frac{1}{3} & \frac{107}{600} \end{pmatrix}; \\ A_3 &= \begin{pmatrix} \frac{1}{60} & 0 \\ 0 & \frac{1}{420} \end{pmatrix}; B_2 = \begin{pmatrix} 0 & \frac{1}{10} \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_7 \end{pmatrix} = \begin{pmatrix} \frac{1}{420} \\ \frac{1}{210} \end{pmatrix}. \end{aligned} \tag{4.16}$$

Since it is $A_{2,1}$-stable, it can be implemented with the same coefficient matrices given in (4.13), (4.14) and (4.15) respectively or having additional methods of the same order with (4.16) as given below; two additional initial block equations in (4.5) with matrix coefficient given from (4.13) and

$$\begin{aligned} A_0^{(1)} &= \begin{pmatrix} \frac{1}{300} & -\frac{1}{25} \\ -\frac{1}{525} & \frac{1}{60} \end{pmatrix}; A_1^{(1)} = \begin{pmatrix} -\frac{7}{120} & \frac{2}{15} \\ -\frac{2}{25} & \frac{1}{120} \end{pmatrix}; A_2^{(1)} = \begin{pmatrix} -\frac{1}{20} & \frac{1}{75} \\ \frac{1}{15} & -\frac{1}{100} \end{pmatrix}; \\ A_3^{(1)} &= \begin{pmatrix} -\frac{1}{600} & 0 \\ 0 & \frac{1}{420} \end{pmatrix}; B_2^{(1)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{1050} \\ \frac{1}{1050} \end{pmatrix}, \end{aligned}$$

and one final additional block equation from (4.6), with the coefficient matrices,

$$A_0^{(N)} = \begin{pmatrix} \frac{1}{60} & -\frac{3}{25} \\ \frac{3}{35} & -\frac{7}{12} \end{pmatrix}; A_1^{(N)} = \begin{pmatrix} \frac{3}{8} & -\frac{2}{3} \\ \frac{42}{25} & -\frac{21}{8} \end{pmatrix}; A_2^{(N)} = \begin{pmatrix} \frac{3}{4} & -\frac{3}{5} \\ \frac{7}{3} & -\frac{21}{20} \end{pmatrix};$$

$$A_3^{(N)} = \begin{pmatrix} \frac{49}{200} & 0 \\ 0 & \frac{223}{1400} \end{pmatrix}; B_3^{(N)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_7 \\ \bar{C}_7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{70} \\ -\frac{1}{10} \end{pmatrix}.$$

Also for $k = 3$, $u = 1$, $m = -1$ in (4.1), the resultant method is not a boundary value method since the root distribution of the corresponding stability polyomials $\rho(r)$ and $\prod(r, z)$ are not the same. In this circumstance such method can not be implemented as a multi-block boundary value method. In general when the stability polynomial $\rho(r)$ from (3.3) and $\prod(r, z)$ in (3.5) do not have the same root distribution, the arising method defies the correct use notion in definition 3.5 and therefore cannot be used as a MB$_2$VM.

Another approach of implementing the method in (4.1) without need for initial and final additional block equations in $(4.5, 4.6)$ is to transform (4.1) into a multi-block GBDFs of the form in (3.7). The case $k = 2$ is the $O_{1,1}$, $A_{1,1}$-stable methods of (4.8) given in (3.43) (3.44) and (3.45) with $Y = (Y_{n+1}, Y_{n+2}, \cdots, Y_{n+N-1})^T$, $\quad F = (F_{n+1}, F_{n+2}, \cdots, F_{n+N-1})^T$. In similar fashion, further examples of the k-block s-point block boundary value method in (4.1) with $m = 0$, $B_i = I_s$, $s = 3$ are listed below. The matrix coefficients for 2-block 3-point block boundary value method in (4.1) with $m = 0$, $B_1 = \frac{1}{10} I_s$ are given by,

$$A_0 = \begin{pmatrix} -\frac{1}{600} & \frac{3}{200} & -\frac{3}{40} \\ \frac{3}{1400} & -\frac{1}{60} & \frac{3}{50} \\ -\frac{3}{400} & \frac{3}{56} & -\frac{1}{6} \end{pmatrix}; A_1 = \begin{pmatrix} 0 & \frac{3}{40} & -\frac{3}{200} \\ -\frac{3}{20} & \frac{3}{40} & \frac{3}{100} \\ \frac{3}{10} & -\frac{3}{8} & \frac{39}{200} \end{pmatrix};$$

$$A_2 = \begin{pmatrix} \frac{1}{600} & 0 & 0 \\ 0 & -\frac{1}{2100} & 0 \\ 0 & 0 & \frac{1}{1680} \end{pmatrix}; B_1 = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}; \quad \bar{C}_7 = \begin{pmatrix} \frac{1}{1400} \\ -\frac{1}{700} \\ \frac{1}{140} \end{pmatrix};$$

For 3-block 3-point block boundary value method in (4.1) with $m = 0$, $B_2 = \frac{1}{10} I_s$, one obtains the matrix coefficients

$$A_0 = \begin{pmatrix} \frac{1}{5040} & -\frac{3}{1400} & \frac{3}{280} \\ -\frac{1}{8400} & \frac{1}{720} & -\frac{3}{400} \\ \frac{1}{13200} & -\frac{1}{1050} & \frac{1}{180} \end{pmatrix}; A_1 = \begin{pmatrix} -\frac{1}{30} & \frac{3}{40} & -\frac{3}{20} \\ \frac{1}{40} & -\frac{7}{120} & \frac{21}{200} \\ -\frac{1}{50} & \frac{1}{20} & -\frac{7}{75} \end{pmatrix};$$

$$A_2 = \begin{pmatrix} \frac{37}{600} & \frac{3}{70} & -\frac{3}{560} \\ -\frac{7}{40} & \frac{319}{4200} & -\frac{3}{80} \\ \frac{7}{50} & -\frac{1}{5} & \frac{743}{8400} \end{pmatrix}; A_3 = \begin{pmatrix} \frac{1}{2520} & 0 & 0 \\ -\frac{1}{240} & \frac{1}{3600} & 0 \\ \frac{1}{30} & -\frac{1}{300} & \frac{1}{4950} \end{pmatrix};$$

$$B_2 = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}; \quad \begin{pmatrix} \bar{C}_{10} \\ \bar{C}_{11} \\ \bar{C}_{12} \end{pmatrix} = \begin{pmatrix} -\frac{1}{8400} \\ \frac{1}{13200} \\ -\frac{1}{19800} \end{pmatrix};$$

The matrix coefficients for 4-block 3-point block boundary value method in (4.1) with $m = 0$, $B_2 = \frac{1}{10} I_s$ are,

$$A_0 = \begin{pmatrix} -\frac{1}{55440} & -\frac{1}{3850} & \frac{1}{560} \\ -\frac{1}{50050} & \frac{1}{3600} & -\frac{1}{550} \\ \frac{1}{30800} & -\frac{2}{4550} & \frac{1}{360} \end{pmatrix}; A_1 = \begin{pmatrix} -\frac{5}{630} & \frac{15}{560} & -\frac{6}{70} \\ \frac{3}{400} & -\frac{2}{90} & \frac{21}{400} \\ -\frac{6}{550} & \frac{3}{100} & -\frac{28}{50} \end{pmatrix};$$

$$A_3 = \begin{pmatrix} \frac{5}{630} & -\frac{1}{560} & \frac{1}{3850} \\ -\frac{1}{80} & \frac{1}{450} & -\frac{1}{4400} \\ \frac{2}{50} & -\frac{1}{200} & \frac{2}{4950} \end{pmatrix}; A_4 = \begin{pmatrix} -\frac{1}{55440} & 0 & 0 \\ 0 & \frac{1}{514800} & 0 \\ 0 & 0 & -\frac{1}{1801800} \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & \frac{6}{70} & -\frac{15}{560} \\ -\frac{6}{50} & \frac{12}{350} & -\frac{3}{50} \\ \frac{21}{200} & -\frac{12}{70} & \frac{201}{2800} \end{pmatrix}; B_2 = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}; \quad \bar{C}_{13} = \begin{pmatrix} -\frac{1}{120120} \\ \frac{1}{85800} \\ -\frac{1}{42900} \end{pmatrix}.$$

Other MB_2VMs from (3.6) can be derived similarly.The Fig. 9 has the plot of absolute error constant against block number for the MGBDF in (4.1) in comparison with SDGEBDF [23], SDGBDF [48] and GBDF [19]. The new multi-block boundary value methods MBGBDF in (4.1) have the advantage of smaller absolute error constant than the cited methods as can being seen in Fig. 9 with an enlargement in Fig. 10 for block dimension of $s = 2$, 3.

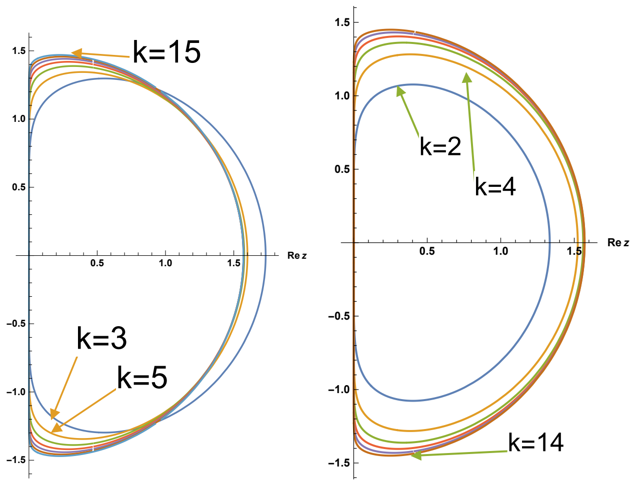
5. Effects of Additional multiblock methods on the stability of MB_2VMs and minimum block size implementation of the MB_2VMs

From the results in the section 3, we conclude that the main formula (3.6) which is A_{k_1,k_2}-stable is implemented with (k_1, k_2)-block boundary conditions without need for initial and final block formulas. However providing the exact k_1 block of initial values (a) and k_2 block final values (b) in the implementation of (3.6) on (1.1) and (1.2) may be demanding, especially with a high block step number m of the main method in (3.6). The way, is to replace them by an equivalent number of block equations formed by LBMFs as in (3.10) and (3.11) preferably of the same order as in the example of (4.8) to (4.11). The introduction of the initial and final block formulas into a MB_2VMs implementation in (3.15) affects its stability. This stability is reduced to $A(\alpha)-$stability when these values are instead replaced by initial block and final block methods. We now shall study the role of this additional block methods related to the whole composite scheme in (3.15), which make such a choice appealing. To study this effect of the initial methods (3.10) and the final methods (3.11) on the whole composite schemes of the MB_2VMs (3.15), we obtain the following discrete problem

$$\left(\bar{A}_N - z\bar{B}_N\right)\bar{Y}_{n+1} = -(a - zb)Y_n, \tag{5.1}$$

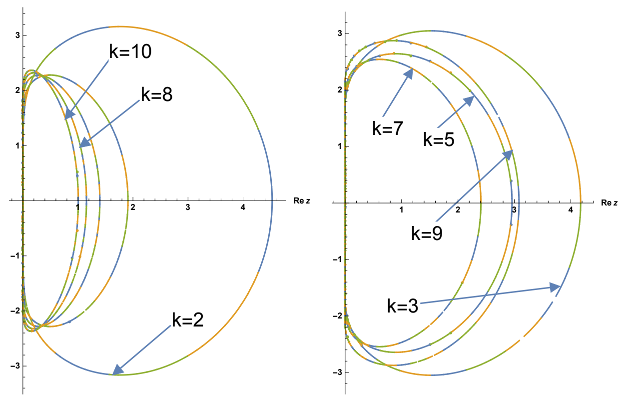
from (3.15) (see,(3.13), (3.16), (3.17) and (3.18)). The method in (5.1) will have a multi-block solution for all $Re(z) < 0$, if the eigenvalues of the matrix pencil

$$M_P(\mu) = \left(\bar{A}_N - \mu\bar{B}_N\right), \tag{5.2}$$



(A) (B)

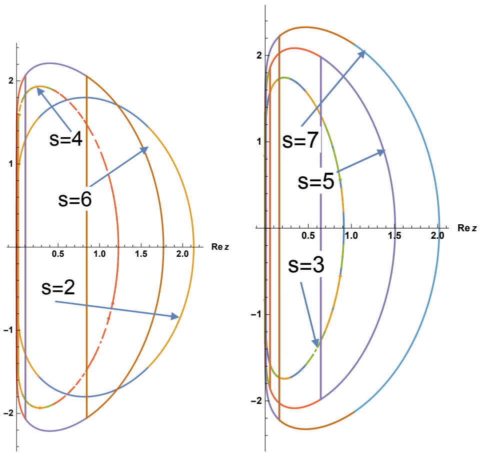
Figure 1: Boundary loci of the MBGBDFs (4.1) of order $p = 2k$, $s = 2$, (A) $k = 3(2)15$ $m = 0$ and (B) $k = 2(2)14$, $m = 1$. The exterior of the closed curves is the stable region of the method.



(A) (B)

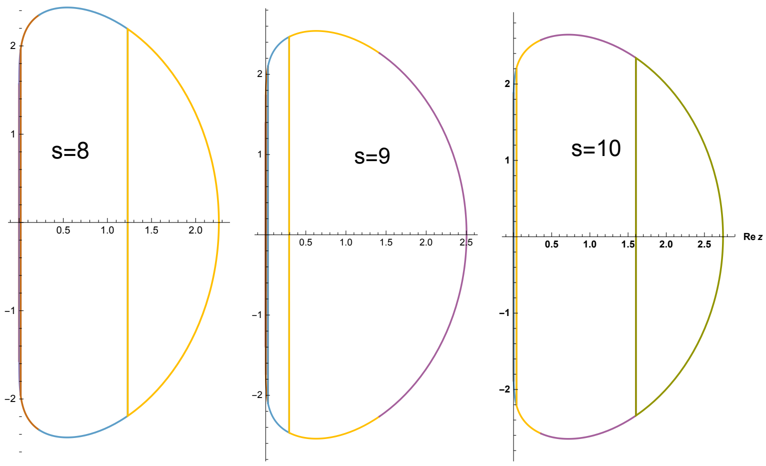
Figure 2: Boundary loci of the MBGBDFs (4.1) of order $p = 3k$, $s = 3$, $m = 0$, (A) $k = 2(2)10$ and (B) $k = 3(2)9$. The exterior of the closed curves is the stable region of the method.

have positive real part for all values of the block size $N \cdot s$. In fact, a multi-block method in (3.15) is said to be pre-stable if the spectrum of the corresponding block matrix pencil $M_P(\mu)$ in (5.2) is contained in $\mathbb{C}^+$. Note that,



(A) (B)

Figure 3: Boundary loci of the MBGBDFs (4.1) of order $p = 4s$, $k = 2$, $m = 0$, (A) $s = 2(2)6$ and (B) $s = 3(2)7$. The exterior of the closed curves is the stable region of the method.



(A) (B) (C)

Figure 4: Boundary loci of the MBGBDFs (4.1) of order $p = 4s$, $k = 2$, $m = 0$, (A) $s = 8$, (B) $s = 9$ and (C) $s = 10$. The exterior of the closed curves is the stable region of the method.

$$-(\bar{A}_N)^{-1}a = \begin{pmatrix} I_s \\ \vdots \\ I_s \end{pmatrix} \in R^{N\cdot s}.$$

However, the introduction of extra row at the top and bottom respectively, when implementing the methods in (3.15) as seen from (3.14) allows some eigenvalues of the matrix pencil in (5.2) to enter $\mathbb{C}^-$. Thus, the method becomes $A_{k_1,k_2}(\alpha)$-stable see [23]. For example, this is the case with MBGBDF in (4.1) when $k \geq 5$, when the additional initial (4.5) and final block formulas (4.6) are used during the implementation of the composite methods in (3.15) see Fig. 5, 6 and 7. The method (3.15) can be safely used for all the block number of k in (3.6) for any particular problem provided that $\mu = z = h\lambda$ is not closed to imaginary axis. Nevertheless, to have all the eigenvalues of (5.2) for the case of MBGBDF (4.1) for $k \geq 5$ in $\mathbb{C}^-$, we adopt the suggestion of Brugnano and Trigiante [19] to consider taking the output solution inside each block Y in (3.8) not equally spaced. For example, let

$$\begin{aligned} &T_0, T_1, \cdots, T_r, T_{r+1}, \cdots, T_{N-r-1}, T_{N-r}, \cdots, T_{N-1}, T_N; \\ &T_j = \left(t_{j\cdot s}, t_{js+1}, t_{j\cdot s+2}, \cdots, t_{j\cdot s+s-1}\right)^T; \qquad j = 0(1)N \end{aligned} \tag{5.3}$$

be the grid of block points inside the multi-block vector Y and let the following block output points

$$T_0, T_r, \cdots, T_{N-r}, T_N,$$

be equally spaced with steps size h. The remaining points in (5.3) are computed from

$$T_i = T_{i-1} + V^{r+1-i}h, \quad i = 1, \cdots, r, \tag{5.4}$$

$$T_{N-i} = T_{N+i-1} - V^{r+1-i} I_s h, \quad i = 1, \cdots, r, \tag{5.5}$$

where

$$V = diag(\nu_{(i-1)s+1}, \nu_{(i-1)s+2}, \cdots, \nu_{(i-1)s+s})^T; \quad i = 1(1)r.$$

For simplicity, we set $V = \nu I_s$, that means fixing $\nu = \nu_j = \nu_i$, where $i, j = 1(1)s$. Then when $r = 1$, $\nu = 1$; when $r = 2$, $\nu = \frac{-1}{2} + \frac{\sqrt{5}}{2}$ and when $r = 3$, $\nu = 0.543689$. Thus, for the case $r \geq 4$, ν is determined from

$$\sum_{i=i}^{r} \nu^i - 1 = 0; \quad 0 < \nu \leq 1.$$

This suggest the introduction of hybrid solution in the block of solution vectors Y_{n+j}, $j = 1(1)k$. The points (5.4) and (5.5) are referred to as the initial and final block of auxiliary points, respectively. In Table 1, we have the minimum values of r required to have all the eigenvalues of the corresponding matrix pencil (5.2) to have positive real part for all chosen values of $N \cdot s$. Where N is the multi-block size of the method (3.15) and the matrix in (5.2). The minimum value $(N \cdot s)^*$ $((N \cdot s)^* \leq N \cdot s)$ to have an $A-$stable method depend on the block number k of the main method of interest and on the number r of the auxiliary grid points used. Thus, we have

$$\bar{Y}_N = -E_N^T \left(\bar{A}_N - z\bar{B}_N\right)^{-1} (a - zb)\, Y_0 \equiv \phi(z) Y_0; \quad n = 0,$$

from (5.1), where $E_{N\cdot s}^T = (\mathbf{O}, \mathbf{O}, ..., \mathbf{O}, I_s) \in R^{(N\cdot s)}$. The function $\phi\,(z)$ is analytic for $z \in \mathbb{C}^-$, since r auxiliary grid points are introduced, the matrix pencil in (5.2) contains all the eigenvalues with positive real part. Therefore,such methods are $A-$stable provided that

$$| \phi(it) | \leq 1, \quad \forall t \in R \quad i = \sqrt{-1}.$$

Table 1, shows the values of $(N \cdot s)^*$ corresponding to different values of the number of r of auxiliary grid points used.

TABLE 1. Minimum blocksize $(N \cdot s)^*$ for $A-$stability of BVM

Boundary value methods												
GBDF [20]	k	1	2	3		4		5	6		7	
(order $p = k$)	r	1	1	1	2	1	2	2	2	3	1	2
$s = 1$	$(N \cdot s)^*$	1	2	7	4	19	6	6	8	7	12	7
SDGEBDFs [23]	k	1	2	3	4	5	6					
(order $p = 2k-1$)	r	1	1	2	4	4	8					
$s = 1$	$(N \cdot s)^*$	2	3	5	7	9	11					
Multi-block boundary value methods (3.6)												
MBGBDFs	k	2	3	4	5	6	7	8	9			
(order $p = s \cdot k$)	r	1	1	1	2	2	2	4	4			
$s = 2$	$(N \cdot s)^*$	4	6	8	10	12	14	16	18			
MBGBDFs	k	2	3	4	5							
(order $p = s \cdot k$)	r	1	1	2	2							
$s = 3$	$(N \cdot s)^*$	6	9	12	15							

TABLE 2. Condition number of the matrix $A - zB$ associated with $A_{k_1,k_2}-$stable MBGBDFs (3.6); $h = 1$; $z = h\lambda$, $s = 2$

λ		-1		
$N \cdot s \setminus k$	2	3	4	5
50	1.712	2.918	6.297	1.164
100	1.714	2.925	6.328	1.165
500	1.714	2.926	6.339	1.165
1000	1.714	2.926	6.339	1.165
5000	1.714	2.926	6.339	1.165

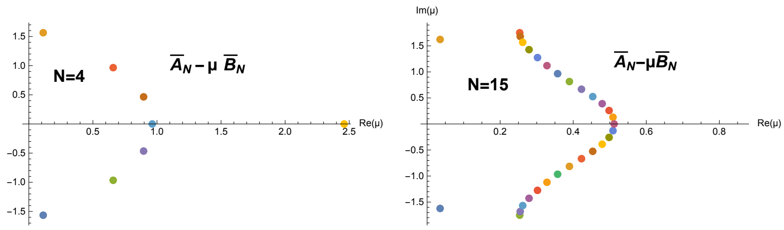


Figure 5: Eigenvalues of the pencil (5.2) corresponding to the eight order MBGBDF (4.1) (with block additional conditions, see (3.15)) $(k = 4,\ s = 2)$ for $N \cdot s = 8, 30$.

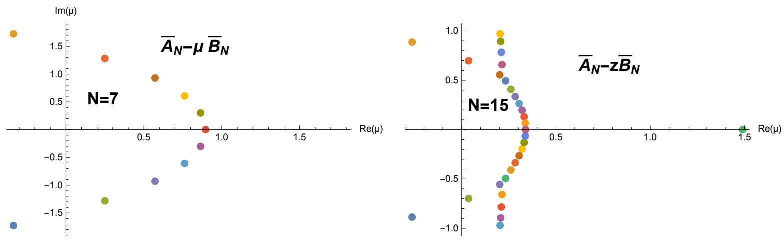


Figure 6: Eigenvalues of the pencil (5.2) corresponding to the order $p = 14$ MBGBDF (4.1) (with block additional conditions, see (3.15)) $(k = 7,\ s = 2)$ for $N \cdot s = 14, 30$.

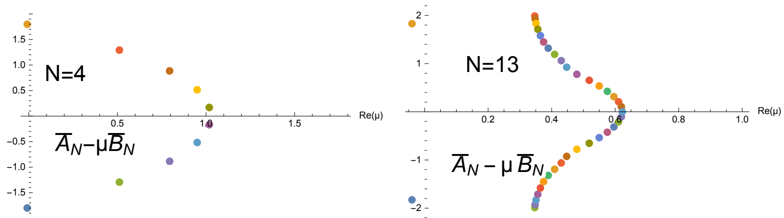


Figure 7: Eigenvalues of the pencil (5.2) corresponding to the order $p = 12$ MBGBDF (4.1) (with block additional conditions, see (3.15)) $(k = 4,\ s = 3)$ for $N \cdot s = 12, 39$.

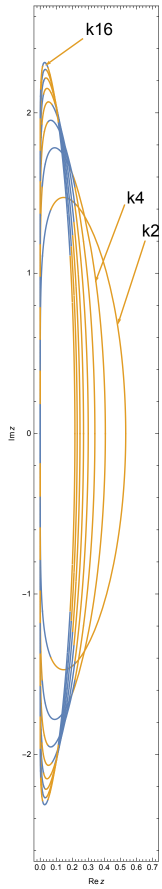
6. The MBGBDFs (4.1) on differential algebraic equations (DAEs) and numerical results

In this section, we apply the MBGBDF (4.1) to solving the DAEs problem (1.2) of the form

$$My' = f(t, y), \tag{6.1}$$

where $M \in R^{m \times m}$ may be singular, such problem arise in modelling applications which include, electric circuit design, optimal control, chemical reaction process control, chemical kinetics, mechanics and robotics. For convenience, Eq. (6.1) can be a linear autonomous case as

$$M(t)y'(t) + Gy(t) = f(t),$$



(A)

Figure 8: Boundary loci of the MBGBDFs (4.1) of order $p = 2k$, $s = 2$, $k = 2(2)16$, $m = 0$. The exterior of the closed curves is the stable region of the method.

and is easier in the equivalent form

$$M(t)y' + Gy(t) - M'(t)y = f(t). \tag{6.2}$$

Here M and G are a constant $v \times v$ matrix and $f(t)$ is a vector valued function. However, the problem in (6.2) will have a solution if and only if the pencil

$$\mu M + G,$$

is regular. That is the pencil is said to be regular, if the polynomial $\Omega(\mu) = \det(\mu M + G)$ is not identically zeros, as a function μ. Since the pencil (M, G) is regular, it can be transformed into the Weierstaβ-Kronicker canonical form, see [1, 2]. Details on DAEs can be found in book of Hairer [1, 2] . This leads to the following definition.

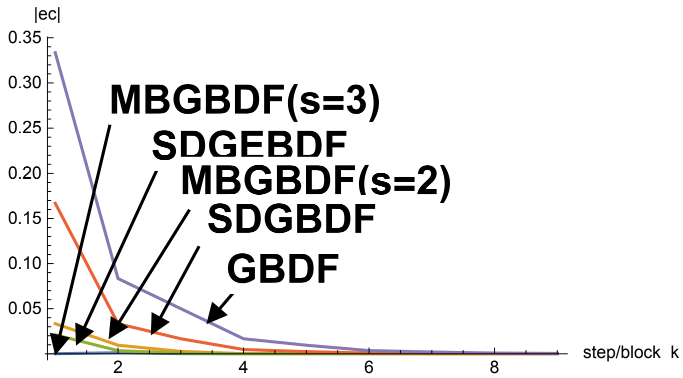


Figure 9: The plot of absolute value of error constant (ec),$\mid \bar{C}_{p+1} \mid = \mid \frac{C_{p+1}}{(p+1)!} \mid$ against step number k of the GBDF in (1.3), SDGBDF in [48], SDGEBDF in (2.4) and block number k of MGBDF in (4.1)

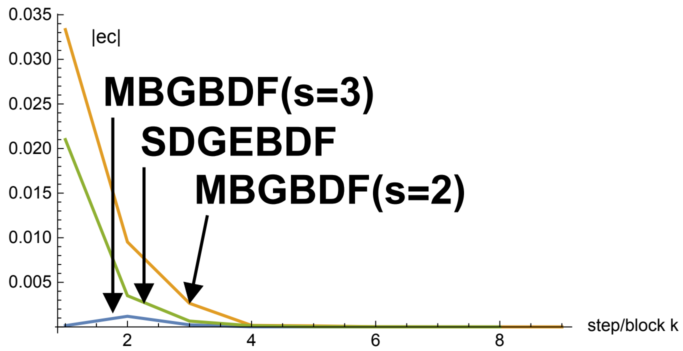


Figure 10: This is an enlargement of Fig. 9 about its origin.

Definition 6.1. *cf:* [1] *The non-linear DAEs*

$$f\left(y'(t), y(t), t\right) = 0,$$

has index μ, if μ is the minimal number of differentiation,

$$f\left(y'(t), y(t), t\right) = 0, \frac{df\left(y'(t), y(t), t\right)}{dt} = 0, \cdots, \frac{d^{\mu} f\left(y'(t), y(t), t\right)}{dt^{\mu}} = 0, \qquad (6.3)$$

where (6.3) gives room to extract an explicit system of ordinary differential equations $g'(t) = \varphi(y(t), t)$.

A regular matrix pencil for the linear DAEs in definition (6.1) will have a differentiation index μ if the kronecker index is μ see [1, 2, 19]. The numerical solution of DAEs in (1.2) becomes more difficult with increasing index μ. The next subsection considers the application of MBGBDF (4.1) on the ODEs in (1.1) and DAEs in (1.2).

6.1. **Numerical results from the implementation of the MBGBDFs** (4.1).

This subsection presents as well some numerical results of experiments to illustrate the performance of the MBGBDFs in (4.1). of the boundary value methods comprising of (4.12), (4.13), (4.14)and (4.15) on some stiff ODEs and DAEs problems. The MBGBDFs (4.1) for $k = 3$, $p = 6$ is in (4.12) is implemented as main method in one-block formalism in (3.15) along with two initial block formulas and one final block formula in (4.13), (4.14)and (4.15) respectively. Here the one-block form

$$\bar{A}_N\bar{Y}_{n+1} + \bar{A}_0\bar{Y}_n = h\left(\bar{B}_N\bar{F}_{n+1} + \bar{B}_0\bar{F}_n\right): \quad MBGBDFs3(a), \tag{6.4}$$

with option that $a = 2, 3$ indicating the choice of ${}_aB_u$ in (4.7); where the block matrices $\{\bar{A}_N, \bar{A}_0, \bar{B}_N, \bar{B}_0\}$ as defined in (3.15) and are determine from the main method (4.12), initial block formualas (4.13), (4.14) and final block formula (4.15). Denote the method in (6.4) by MBGBDFs3(a), with $N \cdot s = (N \cdot s)^* = 6$ being the minimum multi-block size that preserves A$-$stability of the composite boundary value method comprising of (4.12), (4.13), (4.14) and (4.15). All computations are carried out using the MATLAB 2010a on a Dell laptop with configuration of Intel (R) Core (TM) $i5$ with processor speed of $2.5GH$ and RAM of $8GB$. The minimum block dimension leads to minimial block method and is employed in the implementation as shown in the Table 3, 4, 5 and 6. The numerical solution from ODE15s in MATLAB is used as our reference solution. Consider the numerical solution of the following stiff ODEs and DAEs.

Problem 1: Consider the linear problem in [19, 23]

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \tag{6.5}$$

$$y(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} + e^{-40t}\left(cos(40t) + sin(40t)\right) \\ e^{-2t} - e^{-40t}\left(cos(40t) + sin(40t)\right) \\ 2e^{-40t}\left(cos(40t) - sin(40t)\right) \end{pmatrix}$$

Table 3 shows the numerical test containing the maximum relative error $\max\left(\frac{|y-y(t)|}{1+|y(t)|}\right)$ in the interval $0 \le t \le 1$. The results from MBGBDFs3 are compare with GBDFs6 of order 6 in Brugnano and Trigiante [19], and SDGEBDFs3(a) in [23] of order 6. The case of the method MBGBDFs(a=2) compares in accuracy with other methods in the Table 3. In Table 3, it seen that the MBGBDFs3(a=3) performs better than GBDFs6 in accuracy and compares with the accuracy from SDGEBDFs3. The *(rate)* is the numerical order of convergence. In all, the *rate* is in agreement with the order p of the respective multi-block boundary value methods in Table 3.

Problem 2: A non-autonomous DAEs of index 2 from [19],

$$\begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix} y' + \begin{pmatrix} 1 & \eta t \\ 0 & 1+\eta \end{pmatrix} y = \begin{pmatrix} e^t \\ t^2 \end{pmatrix}; \qquad t \in [-0.5, 0.5]\,,$$

$$y(-0.5) = \begin{pmatrix} -0.356530659712633 \\ 0.606530659712633 \end{pmatrix}, \quad y(t) = \begin{pmatrix} e^t \\ t^2 - e^t \end{pmatrix}$$

This problem 2 is conveniently written in an equivalent form as in (6.2) before being solved for $\eta = 0, 1, -1$. In Table 4, are results of the numerical errors and the numerical order *(rate)* of convergences at the output $T = 0.5$. The numerical accuracy of MBGBDFs3(a=3) are in comparison with that from GBDFs6 in [19], with MBGBDFs3(a=3) converging with appropriate numerical order *rate* which close to the real order $p = 6$. The order of convergence for MBGBDFs3 is $sk - \mu + 2$.

TABLE 3. Numerical solution of problem 1 in the interval $0 < t \leq 1$ with $Ns = (N \cdot s)^* = 6$

	MBGBDFs3(a=2)	MBGBDFs3(a=3)	GBDFs6	SDGBEDFs3
$(N \cdot s)^*$	6	6	6	5
p	6	6	6	6
h	error (rate)	error (rate)	error (rate)	error (rate)
$1.0e-2$	$2.56e-4$ $(-)$	$1.64e-4$ $(-)$	$2.51e-3$ $(-)$	$3.50e-5$ $(-)$
$5.0e-3$	$1.77e-5$ (3.84)	$8.94e-6$ (4.18)	$9.096e-5$ (4.79)	$2.21e-6$ (4.00)
$2.5e-3$	$2.40e-7$ (6.20)	$1.10e-7$ (6.34)	$1.81e-6$ (5.65)	$2.77e-8$ (6.60)
$1.25e-3$	$1.99e-9$ (6.90)	$9.60e-10$ (6.84)	$3.04e-8$ (5.90)	$2.29e-9$ (6.91)
$6.25e-4$	$2.04e-11$ (6.60)	$9.59e-12$ (6.65)	$4.85e-10$ (5.97)	$2.29e-12$ (6.64)

The maximum relative error from Ode15s at $t = 1$ is $3.660087954199254e-5$

TABLE 4. Numerical solution of problem 2 at the output $T = 0.5$ with $N = 3$

	MBGBDFs3(a=3)	GBDFs6
$(N \cdot s)^*$	6	6
p	6	6
h	error (rate)	error (rate)
$1.250e-1$	$1.87e-7$ $(-)$	$6.50e-7$ $(-)$
$6.250e-2$	$9.89e-10$ (7.56)	$1.19e-8$ (5.80)
$3.125e-2$	$1.42e-11$ (6.12)	$2.02e-10$ (5.90)

Problem 3: The problem consider is of index 3, [19]

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} y' + y = \begin{pmatrix} cost \\ 0 \\ 0 \end{pmatrix}; \quad t \in [-0.5, 0.5],$$

$$y(t) = \begin{pmatrix} cost \\ sint \\ -cost \end{pmatrix}, \quad y(-0.5) = \begin{pmatrix} 0.877582561890373 \\ -0.479425538604203 \\ -0.877582561890373 \end{pmatrix}$$

The results of solving problem 3 are in Table5. In Table 5, are the absolute errors , along with the *rate* of convergence of the implemented method MBGBDFs3($a = 3$). The order of convergences of the MBGBDFs3($a = 3$) is such that $s \cdot k - \mu + 2 = s \cdot k - 1$. It is clear from the numerical results in Table 5 that the MBGBDFs3($a = 3$) and GBDFs6 are of comparable accuracy. The choice of $(a = 2)$ as noted already in MBGBDFs3($a = 2$) is not amenable to the numerical solution of DAEs because it leads to singularity of the Jacobian in the method.
Problem 4a: Robertson's equation of index 1 in [1]

$$y_1' = -0.04y_1 + 10^4 y_2 y_3, \quad y_2' = 0.04y_1 - 10^4 y_2 y_3 - 3 \times 10^7 y_2^2,$$
$$0 = y_1 + y_2 + y_3 - 1; \quad y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0,$$

From definition (6.1), the DAE of index one in problem 4 can be written in ODEs in (1.1) as,
Problem 4b: Robertson's equation, [1]

$$y_1' = -0.04y_1 + 10^4 y_2 y_3, \quad y_2' = 0.04y_1 - 10^4 y_2 y_3 - 3 \times 10^7 y_2^2,$$
$$y_3' = 3 \times 10^7 y_2^2; \quad y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0.$$

TABLE 5. Numerical solution of problem 3 at the output $T = 0.5$ with $N = 3$

	MBGBDFs3(a=3)	GBDFs6
$(N \cdot s)^*$	6	6
p	6	6
h	error (rate)	error (rate)
$1.250e-1$	$2.68e-5$ $(-)$	$1.06e-5$ $(-)$
$6.25e-2$	$6.53e-7$ (5.35)	$3.78e-7$ (4.80)
$3.125e-2$	$1.72e-8$ (5.24)	$1.25e-8$ (4.90)

Table 6, contains the absolute error which is given as the modulus of the ODE15s in MATLAB [49] minus the numerical solution of the MBGBDFs3(a=3).
The stepsize $h = 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1$ have been used to compute the $\log\left(\max\left(\frac{|y-y(t)|}{1+|y(t)|}\right)\right)$ for all the problems considered, the plot is given in Fig. 11. We observe that the number of Jacobian calls can be estimated by Integer$\left(\frac{step}{(N\cdot s)^*}\right)$ in Table 7.

7. CONCLUSION, FURTHER AND FUTURE INVESTIGATION

This paper has presented for the first time the multi-block boundary value methods (MB$_2$VMs). The MB$_2$VMs introduced herein are a novel approach at developing very large scale integration methods (VLSIM) in the numerical solution of differential equations. The derivation of the

TABLE 6. Errors from problem 4a, 4b using $Erry_i = |\ y_i(3.15) - ODE15s(y_i)\ |, i = 1(1)3, h = 0.0001$

	Problem 4a			problem 4b		
t	$Erry_1$	$Erry_2$	$Erry_3$	$Erry_1$	$Erry_2$	$Erry_3$
1	$1.48e-6$	$2.35e-10$	$-1.48e-6$	$-4.41e-7$	$-7.03e-11$	$4.41e-7$
5	$7.76e-6$	$8.23e-10$	$7.76e-6$	$-2.89e-6$	$7.11e-10$	$-2.89e-6$
10	$2.27e-5$	$1.72e-9$	$2.27e-5$	$1.09e-5$	$1.26e-9$	$1.09e-5$

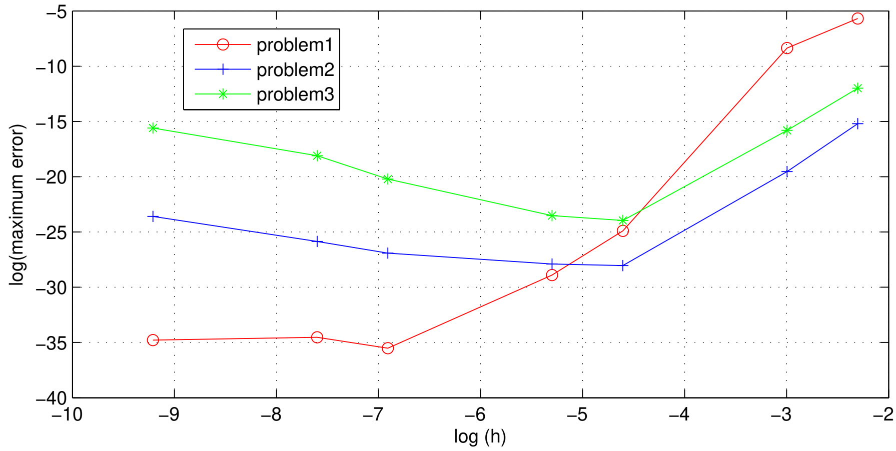


Figure 11: Log of maximum absolute error from MBGBDFs3(a=3) against log h

proposed families of MB_2VMs have been done in a unified framework, based on multi-block methods of [11]. The theoretical properties of the methods with respect to consistency, order and stability along with other practical aspect of implementation have also been presented. The Weiner-Hopf matrix factorization of the characteristics matrix polynomial of the main method along with the root distribution of the arising stability polynomial have been used to determine the structure of the arising multi-block boundary value method in (3.6). Examples are in (4.12) amongst others. More so, a particular family of MB_2VMs called MBGBDFs in (4.1) has been proposed. Finally, the numerical results presented in Tables 3, 4, 5 and 6 , shows that MBGBDFs compare in accuracy with methods from [19] and [23] on some considered ODEs and DAEs in Section 6. The errors of the three problems are plotted against stepsize h on a log-log scale in Fig. 11.

TABLE 7. The number of function calls, Jacobian calls, number of LU decomposition and response time in seconds at the output point $t = 1.0$ for problem 1 and $T = 0.5$ for Problem 2 and 3, on implementing the method MBGBDFs3(a=3) with minimum block size $(N \cdot s)^*$

Problem	$(N \cdot s)^*$	steps	Function calls	Jacobian calls	LU	Time (seconds)
2	6	105	1260	15	90	0.21
1 and 3	6	105	1890	15	90	0.25

On a conclusive note, further research will focus on deriving the multi-block boundary value methods

$$\sum_{j=-k_1}^{k_2} A_j^{(q)}(s) Y_{n+j} = \sum_{l=1}^{q} h^l \left(\sum_{j=-k_1}^{k_2} B_j^{(l)}(s) F_{n+j}^{(l-1)} \right); \quad \begin{array}{cc} k_1 + k_2 & = k \geq 2 \\ q & \geq 1 \end{array}, \quad n = 0, 1, \cdots$$
$$\underbrace{Y_0, Y_1, \cdots, Y_{k_1-1}}_{\text{(a1)}} \qquad \underbrace{Y_{k_1}, \cdots, Y_{N-k_2}}_{\text{multi-block of solution values to be generated by the MB}_2\text{VM}} \qquad \underbrace{Y_{N-k_2+1}, \cdots, Y_N}_{\text{(a2)}} \tag{7.1}$$

based on (1.7) with block definitions in (1.8) which may employ derivative evaluations. The case of $k = 1$ gives rise to initial value multi-block methods. The multi-block boundary value methods to be presented in future works are along the line of thoughts of the well-known boundary value methods of TOM, GAM and ETR$_2$s in [19], SDGEBDF in [23] and GSDLMM in [21] based on Enright [22]. We will also consider the use of pre-conditioning (see [41]) in (3.19) considered instead as a block of system of linear equations,

$$(\bar{A}_N - h\bar{B}_N J)\triangle_{n+1}^{[i]} = (\bar{A}_N \bar{Y}_{n+1}^{[i]} + \bar{A}_0 \bar{Y}_n - h\bar{B}_0 \bar{F}_n - h\bar{B}_N \bar{F}_{n+1}^{[i]}); \tag{7.2}$$

$$\bar{Y}_{n+1}^{[i+1]} = \bar{Y}_{n+1}^{[i]} + \triangle_{n+1}^{[i]}; \quad i = 0(1)q, \quad q > 1, \quad J = \frac{\partial M}{\partial Y} \tag{7.3}$$

while also taking advantage of parallelism in the efficient implementation of these new multi-block boundary value methods. The MB$_2$VMs in (3.6) and in (7.1) output multi-block of solutions on a single application on ODEs in (1.1) and (1.2), unlike the conventional linear multistep methods in (1.3) which output a solution at a point or the conventional boundary value methods (1.4) and multi-block methods [11] which output a block of solution per step. This Multi-block of solution output of the MB$_2$VMs in (3.6) and in (7.1) is a considered advantage over existing numerical methods for the solution of the ODEs in (1.1) and the DAEs in (1.2). The large-scale linear algebra implementation in (7.2) of the novel approach of the methods in (3.6) on a high dimensional system of ODEs in (1.1), will benefit from the massive parallelism offered by the modern system of heterogeneous CPU-GPU parallel architecture for high performance computing (HPC) [52, 53] at extreme/exa-scale. This will be an open worthwhile investigation in the future.

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