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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF PERIODIC SOLUTIONS TO A FRACTIONAL CHEMOTAXIS SYSTEM ON THE WEAKLY COMPETITIVE CASE

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Abstract. In this paper, we consider a two-species parabolic-parabolic-elliptic chemotaxis system with weak competition and a fractional diffusion of order $s \in (0, 2)$. It is proved that for $s > 2p_0$, where p_0 is a nonnegative constant depending on the system's parameters, there admits a global classical solution. Apart from this, under the circumstance of small chemotactic strengths, we arrive at the global asymptotic stability of the coexistence steady state.

1. Introduction

In this paper, we investigate the following chemotaxis system with Lotka-Volterra type competition and a fractional diffusion on two dimensional periodic torus $\mathbb{T}^2 = [-\pi, \pi]^2$:

$$(1.1)\qquad \begin{cases} u_t = -\Lambda^s u - \chi_1 \nabla \cdot (u \nabla w) + r_1 u(1 - u - a_1 v),\ x \in \mathbb{T}^2,\ t > 0, \\ v_t = -\Lambda^s v - \chi_2 \nabla \cdot (v \nabla w) + r_2 v(1 - v - a_2 u),\ x \in \mathbb{T}^2,\ t > 0, \\ 0 = \Delta w + \alpha u + \beta v - \gamma w,\ x \in \mathbb{T}^2,\ t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x),\ x \in \mathbb{T}^2, \end{cases}$$

where u and v denote the densities of two species which interact with each other, and w is the concentration of a chemoattractant. Furthermore, $r_i > 0$, $\chi_i \geq 0$, $a_i \in [0, 1)$, $(i \in 1, 2)$ are used to denote the strengths of growth kinetics, chemotaxis and competition for each species, respectively. In the meantime, α, $\beta > 0$ denote the production of the chemoattractant by each species and $\gamma > 0$ represents chemoattractant's decay. Here we write $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ with

$$\widehat{\Lambda^s u(\xi)} = |\xi|^s \hat{u}(\xi),$$

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where $\hat{}$ denotes the usual Fourier transform, and the differential operator Λ^s has the following kernel representation:

$$\Lambda^s f(x) = C_{s,\,d}\,\text{P.V.}\int_{\mathbb{T}^d}\frac{f(x)-f(y)}{|x-y|^{d+s}}\mathrm{d}y + C_{s,\,d}\sum_{k\in\mathbb{Z}^d,\,k\neq 0}\int_{\mathbb{T}^d}\frac{f(x)-f(y)}{|x-y+2k\pi|^{d+s}}\mathrm{d}y,$$

where $C_{s,\,d} = \frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{\frac{d}{2}}\left|\Gamma(-\frac{s}{2})\right|} > 0$ is a normalization constant.

In the following of this introduction, we will discuss the inspirations and reasons for studying this problem, and at last, the main conclusions are given.

1.1. **Classical Keller-Segel system.** For one single species, the original chemotaxis model was proposed by Keller and Segel (see [27, 28]). Tello and Winkler [40] considered the following parabolic-elliptic Keller-Segel system with a logistic source:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u),\ x\in\Omega,\ t>0, \\ 0 = \Delta v - v + u,\ x\in\Omega,\ t>0. \end{cases} \tag{1.2}$$

In biology, u is the bacterial density, v is the concentration of a chemoattractant, and χ denotes the chemotactic sensitivity coefficient. Meanwhile, the bacteria u has the ability to move in the direction of higher concentration of chemoattractant v. $f(u) = ru(1-u)$ is the logistic source in literature and $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary. In the last decades, scientists have done extensive research for the system (1.2) and its generalizations. What concern scientists most is the existence of global solutions or the occurrence of a blowup at a finite time, and the asymptotic behavior of global solutions was also considered. It is proved in [40] that for dimensions $n\le 2$ or $r > \frac{n-2}{n}\chi$, there admits a global bounded classical solution which is unique. That is, for a sufficient large r, the logistic term might suppress the occurrence of a blowup in some way. The correlative literatures are rich, readers can refer to the surveys [26, 30, 34, 44, 45] for a detail study.

1.2. **Derivation of our problem.** Notice that there are two differences between our system and the parabolic-elliptic Keller-Segel model (1.2): one is the fractional diffusion and another is the bispecies competition term. Hence we are going to introduce the motivations for these two differences. At the same time, we will state some previous results that inspired our interests of the problem (1.1).

1.2.1 *Motivation for the fractional diffusion.* Since the 1990's, with the development of research, it is found that the behaviors of many creatures in nature are not completely suitable to be described by the classical chemotactic model. For example, it is proved by Carfinkel [15] that the movement of mesenchymal cells due to the attraction of a certain chemical substance does not conform to the classical chemotactic model. For a stronger theoretical and more empirical evidence, Escudero [18] improved the classical chemotaxis model and replaced the classical Laplace diffusion with a fractional one in Keller-Segel system:

Λ^s, $0 < s < 2$. (When $s = 1$, $\Lambda^1 = (-\Delta)^{\frac{1}{2}}$, particularly, $-\Delta = \Lambda^2$.) That is, using levy flight instead of Brownian motion to describe the spread of cells. By this way, a large number of phenomena in nature are better modelled, the interested readers can refer to [2, 9, 11, 25, 38, 42, 43, 49] for a more exhaustive discussion.

Let us mainly recall the fractional parabolic-elliptic Keller-Segel system with a logistic source:

$$\begin{cases} u_t = -\Lambda^s u - \chi\nabla\cdot(u\nabla v) + au - bu^2, \ x \in \mathbb{R}^n, \ t > 0, \\ 0 = \Delta v - v + u, \ x \in \mathbb{R}^n, \ t > 0. \end{cases} \tag{1.3}$$

Because $-\Lambda^s u$ provides a weaker dissipation than the standardized one for $s < 2$, it is more possible for the occurrence of a blowup, we can consult [3, 6, 7] by Biler and collaborators, as well as [33–36] by Li and Rodrigo for more results in the case of generic fractional parabolic-elliptic system. Recalling the suppression of a logistic source, it is more significant to consider the combined effect of fractional diffusion and logistic term incorporated in (1.3). For this question, Zhang systematically studied the global existence and the asymptotic behavior of classical solutions of the problem (1.3) in [48], it is proved that for positive initial data $u_0 \in C^b_{unif}(\mathbb{R}^n)$ and $s \in (\frac{1}{2}, 1)$, if $\chi \leq b$, there admits a unique global classical solution (u, v) which is bounded for $\chi < b$. Furthermore, for $b > 2\chi$:

$$u(t) \to \frac{a}{b} \text{ and } v(t) \to \frac{a}{b} \text{ as } t \to \infty.$$

Burczak and Granero-Belinchón came up with a series of results on periodic torus. [11] proved that for any initial data and any positive time, the solutions of parabolic-elliptic Keller-Segel system on $\mathbb{S}^1$ with critical fractional diffusion $(-\Delta)^{\frac{1}{2}}$ remain smooth. For one-dimensional case, [12] showed the global existence of classical solutions under the condition of $c_1 < s \leq 2$, where the nonnegative constant c_1 is a specific constant related to the coefficients of problem. When the spatial dimension $n = 2$, for $c_2 < s < 2$, where $c_2 \in (0, 2)$ depends on the parameters of system, the global existence of the regular solutions was obtained in [13]. For d-dimensional ($d = 1, 2$) case, [10] obtained the uniform in time boundedness of solutions for $s > d(1 - c_3)$, where c_3 is an explicit constant relying on the parameters of problem. Meanwhile, the stability of the homogeneous solutions was also considered by [14].

1.2.2 *Motivation for the bispecies competition terms.* The biological activities of individual species have been well modelled by Keller and Segel. But most species in nature interact with one another to survive. One of the simplest but most representative case is the interaction of two species and their surroundings. Similar to the single-species chemotactic model, for the multi-species model, the occurrence of a blowup and global existence of solutions are mainly concerned in [4,5,17,22,29,32]. The pure two-species chemotaxis model without competition term or logistic source was introduced by [46], and for the qualitative features of solutions, there are many colorful results in the bispecies case

[19–21]. Moreover, the long-term behavior of global solutions was also studied in [31, 39, 47].

The two-species chemotaxis model with Lotka-Volterra type competition and classical diffusion was first introduced by Tello and Winker in [41]:

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + r_1u(1-u-a_1v),\ x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + r_2v(1-v-a_2u),\ x\in\Omega,\ t>0,\\ 0 = d_3\Delta w + \alpha u + \beta v - \gamma w,\ x\in\Omega,\ t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = 0,\ x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), w(x,0)=w_0(x),\ x\in\Omega, \end{cases} \tag{1.4}$$

where the nonnegative constants d_1, d_2 and d_3 denote the diffusion rates of species and chemoattractant, respectively. In [41], it is showed the global existence and the asymptotic stability of solutions under the conditions that

$$2(\chi_1+\chi_2)+a_1r_2<r_1 \text{ and } 2(\chi_1+\chi_2)+a_2r_1<r_2.$$

Black, Lankeit and Mizukami improved the conditions and came to the same conclusion in [8]. Assume $q_1 := \frac{\chi_1}{r_1}$, $q_2 := \frac{\chi_2}{r_2}$ satisfy the conditions

$$\begin{aligned} &q_1\in[0,\frac{d_3}{2\alpha})\cap[0,\frac{a_1d_3}{\beta}),\ q_2\in[0,\frac{d_3}{2\beta})\cap[0,\frac{a_2d_3}{\alpha}) \text{ and}\\ &a_1a_2d_3^2<(d_3-2\alpha q_1)(d_3-2\beta q_2), \end{aligned} \tag{1.5}$$

then there exists a unique global classical positive solution (u, v, w) from nonnegative functions u_0, $v_0 \in C(\bar{\Omega})$ with the following asymptotic behaviour:

$$u(t)\to u^*,\ v(t)\to v^* \text{ and } w(t)\to\frac{\alpha u^*+\beta v^*}{\gamma} \text{ as } t\to\infty$$

uniformly in Ω, where

$$u^*=\frac{1-a_1}{1-a_1a_2} \text{ and } v^*=\frac{1-a_2}{1-a_1a_2}.$$

Inspired by [8] and [13], in this paper, we consider a fractional chemotaxis system with Lotka-Volterra type competition on two dimensional periodic torus $\mathbb{T}^2$. This system is a generalization of the model (1.4). We study the fractional Laplace diffusion of species, which is more consistent with biological behaviors than classical Laplace diffusion. The problem (1.1) describes the dynamical behavior of two competitive species attracted by the same chemical signal more precisely. What interested us are whether the problem (1.1) can achieve coexistence steady state and the required conditions. We will study the global existence of classical solutions for $s > 2p_0$, where p_0 is a nonnegative constant to be fixed later. Due to the influence of fractional diffusion, the global existence of the solution to the problem (1.1) is different from the classical proof in the model (1.4). We will use the properties of the fractional diffusion and the periodic domain to prove it with a different method under different conditions, the specific process has been given in Section 4. Furthermore, only by obtaining the global existence can we further consider the global asymptotic stability of the solution. Here are our main results.

Theorem 1.1. *Let* u_0, $v_0 \in H^4(\mathbb{T}^2)$ *be nonnegative initial data and* T *be any positive number. If*

$$2 > s > 2p_0, \tag{1.6}$$

where

$$p_0 = \max\left\{\frac{(\alpha\chi_1 - r_1)_+}{\alpha\chi_1}, \frac{(\beta\chi_1 - a_1 r_1)_+}{\beta\chi_1}, \frac{(\beta\chi_2 - r_2)_+}{\beta\chi_2}, \frac{(\alpha\chi_2 - a_2 r_2)_+}{\alpha\chi_2}\right\}, \tag{1.7}$$

then the problem (1.1) *admits a global nonnegative classical solution*

$$(u, v, w) \in C([0, T); H^4(\mathbb{T}^2)) \cap C^{2,1}(\mathbb{T}^2 \times [0, T)).$$

Remark 1. For the sake of simplicity, we only consider two-dimensional case. For d-dimensional, we have the same conclusion in $H^{3+[\frac{d}{2}]^+}(\mathbb{T}^2)$, where $[\cdot]^+$ denotes the integer operator.

Compared with [8], we do not need the parameters of the equation to meet the smallness condition on the relative chemotactic strength (1.5) (see [8]), but only add the condition (1.6) to the order s in the problem (1.1). Moreover, the range of parameters in the problem (1.1) are expanded. (1.7) gives the sufficient condition for the chemotaxis, logistic, competition and growth terms to prevent blowups from occurring. When the logistic terms r_1, r_2 and the competition terms a_1, a_2 are sufficiently large, while the chemotaxis terms χ_1, χ_2 and the production terms α, β are sufficiently small, the condition (1.6) is automatically satisfied. Under these assumptions, the global existence of the solution is also proved.

Theorem 1.2. *Assume that the parameters satisfy*

$$\begin{aligned} &\frac{\chi_1}{r_1} \in [0, \frac{1}{2\alpha}) \cap [0, \frac{a_1}{\beta}), \ \frac{\chi_2}{r_2} \in [0, \frac{1}{2\beta}) \cap [0, \frac{a_2}{\alpha}) \ \textit{and} \\ &a_1 a_2 < (1 - \frac{2\alpha\chi_1}{r_1})(1 - \frac{2\beta\chi_2}{r_2}). \end{aligned} \tag{1.8}$$

Then for any $0 < s < 2$, *the unique global classical solution* (u, v, w) *of* (1.1) *possesses the following asymptotic behavior*:

$$u(t) \to u^*,\ v(t) \to v^* \ \textit{and}\ w(t) \to \frac{\alpha u^* + \beta v^*}{\gamma} \ \textit{as}\ t \to \infty$$

uniformly in $\mathbb{T}^2$, *where*

$$u^* = \frac{1 - a_1}{1 - a_1 a_2} \ \textit{and}\ v^* = \frac{1 - a_2}{1 - a_1 a_2}.$$

Remark 2. Notice that when (1.8) in Theorem 1.2 is satisfied, p_0 defined in (1.7) is zero. Thus the problem (1.1) admits a global classical solution for any $0 < s < 2$.

The rest of this paper is formed as follows. In Section 2, we introduce some preliminary notations and function spaces as well several elementary lemmas which are required for the proof of our main conclusions. In Section 3, we give some auxiliary results and energy estimates. Taking advantage of these statements, we will prove Theorem 1.1 and Theorem 1.2 in Section 4 and Section 5, respectively.

2. Preliminary

In this section, we will give some symbols which are used extensively in this paper.

We write ∂^n for $n \in \mathbb{Z}^+$ by a generic derivative of order n, and define the fractional L^p-based Sobolev spaces $W^{s,p}(\mathbb{T}^d)$ as

$$W^{s,p}(\mathbb{T}^d)=\Big\{f\in L^p(\mathbb{T}^d)\mid \partial^{\lfloor s\rfloor}f\in L^p(\mathbb{T}^d),\ \frac{|\partial^{\lfloor s\rfloor}f(x)-\partial^{\lfloor s\rfloor}f(y)|}{|x-y|^{\frac{d}{p}+(s-\lfloor s\rfloor)}}\in L^p(\mathbb{T}^d\times\mathbb{T}^d)\Big\},$$

with the norm

$$||f||^p_{W^{s,p}} = ||f||^p_{L^p} + ||f||^p_{\dot{W}^{s,p}}$$

and

$$||f||^p_{\dot{W}^{s,p}} = ||\partial^{\lfloor s\rfloor} f||^p_{L^p} + \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{|\partial^{\lfloor s\rfloor}f(x)-\partial^{\lfloor s\rfloor}f(y)|^p}{|x-y|^{d+(s-\lfloor s\rfloor)p}}\mathrm{d}x\mathrm{d}y.$$

In the case $p = 2$, we denote $H^s(\mathbb{T}^d) \triangleq W^{s,2}(\mathbb{T}^d)$ for the standard non-homogeneous Sobolev space endowed with the norm

$$||f||^2_{H^s} = ||f||^2_{L^2} + ||f||^2_{\dot{H}^s} \text{ and } ||f||_{\dot{H}^s} = ||\Lambda^s f||_{L^2}.$$

Before giving the proof of our main results, we first introduce several elementary lemmas which will be needed later. Their proofs are classic, so we omit them here, those who are interested can consult to the relevant reference.

Lemma 2.1 ([13, 16, 37], Stroock-Varopoulos inequality)**.** *Let $m > 0$, $0 < s < 2$ and $d \geq 1$. Then for a sufficiently smooth $u \geq 0$ ($u \in L^\infty(\mathbb{T}^d) \cap H^s(\mathbb{T}^d)$ is enough) it holds*

$$\frac{4m}{(1+m)^2}\int_{\mathbb{T}^d} |\Lambda^{\frac{s}{2}}(u^{\frac{m+1}{2}})|^2 \leq \int_{\mathbb{T}^d} u^m \Lambda^s u. \tag{2.1}$$

Lemma 2.2 ([1, 12, 14, 23])**.** *Let $h \in C^2(\mathbb{T}^d)$ be a function. Assume that $h(x^*) := \max_x h(x) > 0$. Then there exist two constants $M_i(d,p,s)$, $i = 1, 2$ such that either*

$$M_1(d,p,s)\|h\|_{L^p} \geq h(x^*), \tag{2.2}$$

or

$$\Lambda^s h(x^*) \geq M_2(d,p,s)\frac{h(x^*)^{1+\frac{sp}{d}}}{\|h\|^{\frac{sp}{d}}_{L^p}}, \tag{2.3}$$

with

$$M_1(d,p,s) = \Big(\frac{\pi^{\frac{d}{2}}}{2^{1+p}} \int_0^\infty z^{\frac{d}{2}} e^{-z}\, dz\Big)^{\frac{1}{p}}$$

and

$$M_2(d,p,s) = \varepsilon_{d,s} \frac{\Big(\frac{\pi^{\frac{d}{2}}}{\int_0^\infty z^{\frac{d}{2}} e^{-z}\, dz}\Big)^{1+\frac{s}{d}}}{4 \cdot 2^{\frac{(p+1)s}{d}}}, \ \varepsilon_{d,s} = 2\Big(\int_{\mathbb{R}^d} \frac{4\sin^2(\frac{x_1}{2})}{|x|^{d+s}}\, dx\Big)^{-1}.$$

3. Auxiliary results

At first, we obtain the local existence of classical solutions with a continuation criterion for the problem (1.1). Then we give some needed energy estimates which are necessary later. Finally, a standard extension criterion provides convenience for the proof of Theorem 1.1.

Lemma 3.1 (Local existence)**.** *Assume* u_0*,* $v_0 \in H^4(\mathbb{T}^2)$ *are nonnegative initial data and* $0 < s < 2$*. Then there exists a time* $0 < T_{\max}(u_0, v_0) \le \infty$ *such that there admits a nonnegative solution* (u, v, w) *on* $\mathbb{T}^2 \times [0, T_{\max}(u_0, v_0))$ *which belongs to*

$$C([0, T_{\max}(u_0, v_0)); H^4(\mathbb{T}^2)) \cap C^{2,1}(\mathbb{T}^2 \times [0, T_{\max}(u_0, v_0))).$$

Moreover, if $T_{\max}(u_0, v_0) < +\infty$*, then*

$$||u(\cdot, t)||_{H^4} + ||v(\cdot, t)||_{H^4} \to \infty \ \textit{as} \ t \nearrow T_{\max}(u_0, v_0)$$

is fulfilled.

Proof. **Part 1. (a priori estimates)** Testing the first equation of (1.1) with u, we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} u u_t &= -\int_{\mathbb{T}^2} u \Lambda^s u - \chi_1 \int_{\mathbb{T}^2} u \nabla \cdot (u \nabla w) + r_1 \int_{\mathbb{T}^2} u^2 (1 - u - a_1 v) \\ &= -\int_{\mathbb{T}^2} |\Lambda^{\frac{s}{2}} u|^2 + \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{3.1}$$

Integrating by parts for the term I_1 yields

$$\mathrm{I}_1 = \chi_1 \int_{\mathbb{T}^2} u \nabla w \cdot \nabla u = \frac{\chi_1}{2} \int_{\mathbb{T}^2} \nabla w \cdot \nabla u^2 = -\frac{\chi_1}{2} \int_{\mathbb{T}^2} u^2 \Delta w, \tag{3.2}$$

using the Sobolev embedding $H^k(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$ for $k > 1$ and the third equation of (1.1), we get

$$\begin{aligned} \mathrm{I}_1 &\le C \|\Delta w\|_{L^\infty} \|u\|_{L^2}^2 \\ &\le C \|w\|_{H^4} \|u\|_{H^4}^2 \\ &\le C (\|u\|_{H^2} + \|v\|_{H^2}) \|u\|_{H^4}^2 \\ &\le C (\|u\|_{H^4} + \|v\|_{H^4}) \|u\|_{H^4}^2. \end{aligned} \tag{3.3}$$

Here and thereafter, we denote C by a general constant. And for the term I_2, we obviously have

$$\mathrm{I}_2 \le r_1(1+\|u\|_{L^\infty}+a_1\|v\|_{L^\infty})\|u\|_{L^2}^2 \le C(1+\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \le C(1+\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2. \tag{3.5}$$

Taking the fourth derivative with respect to x on both side of the first equation of (1.1) and testing it with $\partial_x^4 u$, we get

$$\begin{aligned}\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4 u_t =& -\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4\Lambda^s u-\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4\nabla\cdot(u\nabla w)\\ &+r_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4[u(1-u-a_1v)]\\ =& -\int_{\mathbb{T}^2}|\Lambda^{\frac{s}{2}}\partial_x^4 u|^2+\mathrm{I}_3+\mathrm{I}_4.\end{aligned} \tag{3.6}$$

Taking advantage of Leibniz' formula and the third equation of (1.1), we have

$$\begin{aligned}\mathrm{I}_3 =& -\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4(\nabla u\cdot\nabla w)-\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4(u\Delta w)\\ =& -\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4\nabla u\cdot\nabla w-4\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^3\nabla u\cdot\partial_x\nabla w\\ &-6\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^2\nabla u\cdot\partial_x^2\nabla w-4\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x\nabla u\cdot\partial_x^3\nabla w\\ &-\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\nabla u\cdot\partial_x^4\nabla w-\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4(uw)\\ &+\alpha\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4(u^2)+\beta\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4(uv)\\ :=& \mathrm{I}_3^1+\mathrm{I}_3^2+\mathrm{I}_3^3+\mathrm{I}_3^4+\mathrm{I}_3^5+\mathrm{I}_3^6+\mathrm{I}_3^7+\mathrm{I}_3^8.\end{aligned} \tag{3.7}$$

For the term I_3^1, integrating by parts yields

$$\mathrm{I}_3^1=-\frac{\chi_1}{2}\int_{\mathbb{T}^2}\nabla(\partial_x^4 u)^2\cdot\nabla w=\frac{\chi_1}{2}\int_{\mathbb{T}^2}(\partial_x^4 u)^2\Delta w, \tag{3.8}$$

by (3.3), we obtain

$$\mathrm{I}_3^1 \le \frac{\chi_1}{2}\|\Delta w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}^2 \le C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2. \tag{3.9}$$

By Hölder's inequality and the Sobolev embedding theorem, as well the third equation of (1.1), the terms I_3^2, I_3^3, I_3^4 and I_3^5 are controlled by

$$\mathrm{I}_3^2 \le 4\chi_1\|\Delta w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}^2 \le C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2, \tag{3.10}$$

$$
\begin{aligned}
\mathrm{I}_3^3 &\le 6\chi_1\|\partial_x\Delta w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^2\nabla u\|_{L^2}\\
&\le C(\|\partial_x w\|_{L^\infty}+\|\partial_x u\|_{L^\infty}+\|\partial_x v\|_{L^\infty})\|u\|_{H^4}\|u\|_{H^3}\\
&\le C(\|w\|_{H^4}+\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2\\
&\le C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2,
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\mathrm{I}_3^4 &\le 4\chi_1\|\partial_x\nabla u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^3\nabla w\|_{L^2}\\
&\le C\|u\|_{H^4}^2\|w\|_{H^4}\\
&\le C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
\mathrm{I}_3^5 &\le \chi_1\|\nabla u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^4\nabla w\|_{L^2}\\
&\le C\|u\|_{H^4}^2\|w\|_{H^5}\\
&\le C(\|u\|_{H^3}+\|v\|_{H^3})\|u\|_{H^4}^2\\
&\le C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2.
\end{aligned} \tag{3.13}
$$

Making use of Leibniz' formula, Hölder's inequality and the Sobolev embedding $H^4(\mathbb{T}^2)\subset W^{2,\infty}(\mathbb{T}^2)$, the terms I_3^6, I_3^7 and I_3^8 are estimated by

$$
\begin{aligned}
\mathrm{I}_3^6 = &-\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^4 u w-4\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^3 u\partial_x w\\
&-6\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^2 u\partial_x^2 w-4\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x u\partial_x^3 w\\
&-\gamma\chi_1\int_{\mathbb{T}^2}\partial_x^4 u u\partial_x^4 w\\
\le\ &\gamma\chi_1\|w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}^2+4\gamma\chi_1\|\partial_x w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^3 u\|_{L^2}\\
&+6\gamma\chi_1\|\partial_x^2 w\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^2 u\|_{L^2}\\
&+4\gamma\chi_1\|\partial_x u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^3 w\|_{L^2}\\
&+\gamma\chi_1\|u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^4 w\|_{L^2}\\
\le\ &C\|u\|_{H^4}^2\|w\|_{H^4}\\
\le\ &C(\|u\|_{H^4}+\|v\|_{H^4})\|u\|_{H^4}^2,
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\mathrm{I}_3^7 &= 2\alpha\chi_1\int_{\mathbb{T}^2}\partial_x^4 u u\partial_x^4 u+8\alpha\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x u\partial_x^3 u+6\alpha\chi_1\int_{\mathbb{T}^2}\partial_x^4 u\partial_x^2 u\partial_x^2 u\\
&\le 2\alpha\chi_1\|u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}^2+8\alpha\chi_1\|\partial_x u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^3 u\|_{L^2}\\
&\quad+6\alpha\chi_1\|\partial_x^2 u\|_{L^\infty}\|\partial_x^4 u\|_{L^2}\|\partial_x^2 u\|_{L^2}\\
&\le C\|u\|_{H^4}\|u\|_{H^4}^2
\end{aligned} \tag{3.15}
$$

and

$$\begin{aligned}\mathrm{I}_3^8 &= \beta\chi_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^4uv + 4\beta\chi_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^3u\partial_xv + 6\beta\chi_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^2u\partial_x^2v\\ &\quad + 4\beta\chi_1\int_{\mathbb{T}^2}\partial_x^4u\partial_xu\partial_x^3v + \beta\chi_1\int_{\mathbb{T}^2}\partial_x^4uu\partial_x^4v\\ &\le \beta\chi_1\|v\|_{L^\infty}\|\partial_x^4u\|_{L^2}^2 + 4\beta\chi_1\|\partial_xv\|_{L^\infty}\|\partial_x^4u\|_{L^2}\|\partial_x^3u\|_{L^2}\\ &\quad + 6\beta\chi_1\|\partial_x^2v\|_{L^\infty}\|\partial_x^4u\|_{L^2}\|\partial_x^2u\|_{L^2}\\ &\quad + 4\beta\chi_1\|\partial_xu\|_{L^\infty}\|\partial_x^4u\|_{L^2}\|\partial_x^3v\|_{L^2} + \beta\chi_1\|u\|_{L^\infty}\|\partial_x^4u\|_{L^2}\|\partial_x^4v\|_{L^2}\\ &\le C\|v\|_{H^4}\|u\|_{H^4}^2.\end{aligned} \tag{3.16}$$

Plugging (3.9)–(3.16) into (3.7), we obtain

$$\mathrm{I}_3 \le C(\|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}^2. \tag{3.17}$$

As for the term I_4, by (3.15) and (3.16), we have

$$\begin{aligned}\mathrm{I}_4 &= r_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^4u - r_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^4(u^2) - a_1r_1\int_{\mathbb{T}^2}\partial_x^4u\partial_x^4(uv)\\ &\le r_1\|\partial_x^4u\|_{L^2}^2 + C\|u\|_{H^4}\|u\|_{H^4}^2 + C\|v\|_{H^4}\|u\|_{H^4}^2\\ &\le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}^2.\end{aligned} \tag{3.18}$$

Inserting (3.17) and (3.18) into (3.6), we conclude

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^4u\|_{L^2}^2 \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}^2. \tag{3.19}$$

Similarly, we can obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^ku\|_{L^2}^2 \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}^2 \text{ for } k = 1,\ 2,\ 3. \tag{3.20}$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^4}^2 = \frac{1}{2}\frac{d}{dt}\Big(\sum_{k=0}^{4}\|\partial_x^ku\|_{L^2}^2\Big) \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}^2, \tag{3.21}$$

i.e.,

$$\frac{d}{dt}\|u\|_{H^4} \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|u\|_{H^4}. \tag{3.22}$$

As for the second equation in (1.1), in the same way, we can get

$$\frac{d}{dt}\|v\|_{H^4} \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})\|v\|_{H^4}. \tag{3.23}$$

Combining (3.22) with (3.23), we finally arrive at

$$\frac{d}{dt}(\|u\|_{H^4} + \|v\|_{H^4}) \le C(1 + \|u\|_{H^4} + \|v\|_{H^4})^2. \tag{3.24}$$

By a comparison theorem, we obtain

$$\|u\|_{H^4} + \|v\|_{H^4} + 1 \le \frac{\|u_0\|_{H^4} + \|v_0\|_{H^4} + 1}{1 - C(\|u_0\|_{H^4} + \|v_0\|_{H^4} + 1)t}. \tag{3.25}$$

Part 2. (Existence) As for the existence of smooth solutions to the problem (1.1), the proof is classic, we can refer to the proof of Theorem 1 of [1] and the proof of Theorem 1 of [24]. Let us consider a nonnegative function $\mathcal{J} \in C_0^\infty$, $\mathcal{J}(x) = \mathcal{J}(|x|)$ with $\int_{\mathbb{R}^2} \mathcal{J} = 1$. For $\varepsilon > 0$, we define the mollifiers $\mathcal{J}_\varepsilon = \frac{1}{\varepsilon^2}\mathcal{J}(\frac{x}{\varepsilon})$ and the regularized initial data $u^\varepsilon(x,0) = \mathcal{J}_\varepsilon * u_0(x) \ge 0$, $v^\varepsilon(x,0) = \mathcal{J}_\varepsilon * v_0(x) \ge 0$, $w^\varepsilon(x,0) = \mathcal{J}_\varepsilon * w_0(x) \ge 0$. By Tonelli's Theorem, we have $\|\mathcal{J}_\varepsilon * u_0(x)\|_{L^1} = \|u_0\|_{L^1}$. Polishing both side of the equations of the problem (1.1), we can get the regularized problem

$$\begin{cases} \partial_t u^\varepsilon = & -\mathcal{J}_\varepsilon * \Lambda^s(\mathcal{J}_\varepsilon * u^\varepsilon) - \chi_1 \mathcal{J}_\varepsilon * \nabla \cdot (\mathcal{J}_\varepsilon * u^\varepsilon \nabla(\mathcal{J}_\varepsilon * w^\varepsilon)) \\ & + r_1 u^\varepsilon(1 - u^\varepsilon - a_1 v^\varepsilon), \\ \partial_t v^\varepsilon = & -\mathcal{J}_\varepsilon * \Lambda^s(\mathcal{J}_\varepsilon * v^\varepsilon) - \chi_2 \mathcal{J}_\varepsilon * \nabla \cdot (\mathcal{J}_\varepsilon * v^\varepsilon \nabla(\mathcal{J}_\varepsilon * w^\varepsilon)) \\ & + r_2 v^\varepsilon(1 - v^\varepsilon - a_2 u^\varepsilon), \\ \quad 0 = \Delta w^\varepsilon + \alpha u^\varepsilon + \beta v^\varepsilon - \gamma w^\varepsilon. \end{cases} \tag{3.26}$$

In the approximate problem (3.26), we use Picard-Lindelöf Existence-Uniqueness Theorem in $H^4 \times H^4 \times H^6$. Define the set

$$\mathcal{O}_\nu^\mu = \{u, v \in H^4(\mathbb{T}^2), \|u\|_{H^4} + \|v\|_{H^4} < \mu, \|u\|_{W^{2,\infty}} + \|v\|_{W^{2,\infty}} < \nu\},$$

with $\|u_0\|_{H^4} + \|v_0\|_{H^4} < \mu$ and $\|u_0\|_{W^{2,\infty}} + \|v_0\|_{W^{2,\infty}} < \nu$. Due to the Sobolev embedding $H^4(\mathbb{T}^2) \subset W^{2,\infty}(\mathbb{T}^2)$ are continuous functionals, $\mathcal{O}_\nu^\mu$ is a non-empty open set in $H^4(\mathbb{T}^2)$, so $\|u\|_{H^4} + \|v\|_{H^4}$ is bounded in this set. By Picard-Lindelöf Existence-Uniqueness Theorem, we can prove that the regularized problem admits smooth solutions sequence $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$, and these solutions satisfy the same energy estimates in the last part, so the solutions exist in the same time interval $[0, T(u_0, v_0)]$. Furthermore, by the iterative method we know that the sequence $(u^\varepsilon, v^\varepsilon)$ is a Cauchy sequence in $C([0,T], H^s)$ for $0 \le s < 4$ with $T = T(u_0, v_0) \le \frac{1}{C(\|u_0\|_{H^4} + \|v_0\|_{H^4} + 1)}$. Thus, it admits a limit (u, v) which is a smooth solution to our problem. Furthermore we have u, $v \in H^4$, because u^ε, v^ε are uniformly bounded in H^4 which implies u^ε and v^ε are weakly convergent to u and v, respectively in H^4.

Part 3. (Uniqueness) We argue by contradiction in order to show the uniqueness of solutions to our problem. Assume that for the same initial data $(u_0, v_0, w_0) \in H^4 \times H^4 \times H^6$, we have two different solutions written by (u_1, v_1, w_1) and (u_2, v_2, w_2). Then the system for $(\bar{u}, \bar{v}, \bar{w}) = (u_1 - u_2, v_1 -$

$v_2, w_1 - w_2)$ is

$$\begin{cases} \partial_t \bar{u} = -\Lambda^s \bar{u} - \chi_1 \nabla \cdot (\bar{u} \nabla w_1 + u_2 \nabla \bar{w}) + r_1 \bar{u} \\ \qquad - r_1 \bar{u}(u_1 + u_2) - a_1 r_1 (\bar{u} v_1 + u_2 \bar{v}), \\ \partial_t \bar{v} = -\Lambda^s \bar{v} - \chi_2 \nabla \cdot (\bar{v} \nabla w_1 + v_2 \nabla \bar{w}) + r_2 \bar{v} \\ \qquad - r_2 \bar{v}(v_1 + v_2) - a_2 r_2 (\bar{u} v_1 + u_2 \bar{v}), \\ 0 = \Delta \bar{w} + \alpha \bar{u} + \beta \bar{v} - \gamma \bar{w}, \\ \bar{u}(x,0) = 0, \ \bar{v}(x,0) = 0, \ \bar{w}(x,0) = 0. \end{cases} \tag{3.27}$$

Testing the first equation of (3.27) with $\bar{u}$, we can obtain

$$\begin{aligned} \int_{\mathbb{T}^2} \bar{u} \bar{u}_t = & - \int_{\mathbb{T}^2} \bar{u} \Lambda^s \bar{u} - \chi_1 \int_{\mathbb{T}^2} \bar{u} \nabla \cdot (\bar{u} \nabla w_1) - \chi_1 \int_{\mathbb{T}^2} \bar{u} \nabla \cdot (u_2 \nabla \bar{w}) \\ & + r_1 \int_{\mathbb{T}^2} \bar{u}^2 - r_1 \int_{\mathbb{T}^2} \bar{u}^2 (u_1 + u_2) - a_1 r_1 \int_{\mathbb{T}^2} \bar{u} (\bar{u} v_1 + u_2 \bar{v}) \\ = & - \int_{\mathbb{T}^2} |\Lambda^{\frac{s}{2}} \bar{u}|^2 + \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4 + \mathrm{II}_5. \end{aligned} \tag{3.28}$$

Integrating twice by parts for the term II_1, we have

$$\mathrm{II}_1 = \chi_1 \int_{\mathbb{T}^2} \bar{u} \nabla w_1 \cdot \nabla \bar{u} = \frac{\chi_1}{2} \int_{\mathbb{T}^2} \nabla w_1 \cdot \nabla (\bar{u}^2) = -\frac{\chi_1}{2} \int_{\mathbb{T}^2} \bar{u}^2 \Delta w_1, \tag{3.29}$$

and Hölder's inequality yields

$$\mathrm{II}_1 \le \frac{\chi_1}{2} \|\Delta w_1\|_{L^\infty} \|\bar{u}\|_{L^2}^2 \le C \|\bar{u}\|_{L^2}^2. \tag{3.30}$$

By Hölder's inequality and the third equation of (3.27), we immediately obtain the estimates of the remaining parts

$$\begin{aligned} \mathrm{II}_2 &= -\chi_1 \int_{\mathbb{T}^2} \bar{u} \nabla u_2 \cdot \nabla \bar{w} - \chi_1 \int_{\mathbb{T}^2} \bar{u} u_2 \Delta \bar{w} \\ &\le \chi_1 \|\nabla u_2\|_{L^\infty} \|\bar{u}\|_{L^2} \|\nabla \bar{w}\|_{L^2} + \chi_1 \|u_2\|_{L^\infty} \|\bar{u}\|_{L^2} \|\Delta \bar{w}\|_{L^2} \\ &\le C \|\bar{u}\|_{L^2} \|\bar{w}\|_{H^2} \\ &\le C \|\bar{u}\|_{L^2} (\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}), \end{aligned} \tag{3.31}$$

$$\mathrm{II}_3 = r_1 \|\bar{u}\|_{L^2}^2, \tag{3.32}$$

$$\mathrm{II}_4 \le r_1 (\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}) \|\bar{u}\|_{L^2}^2 \le C \|\bar{u}\|_{L^2}^2 \tag{3.33}$$

and

$$\begin{aligned} \mathrm{II}_5 &= -a_1 r_1 \int_{\mathbb{T}^2} v_1 \bar{u}^2 - a_1 r_1 \int_{\mathbb{T}^2} u_2 \bar{u} \bar{v} \\ &\le a_1 r_1 \|v_1\|_{L^\infty} \|\bar{u}\|_{L^2}^2 + a_1 r_1 \|u_2\|_{L^\infty} \|\bar{u}\|_{L^2} \|\bar{v}\|_{L^2} \\ &\le C (\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}) \|\bar{u}\|_{L^2}. \end{aligned} \tag{3.34}$$

Plugging (3.30)–(3.34) into (3.28), we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_{L^2}^2 \le C(\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2})\|\bar{u}\|_{L^2}, \tag{3.35}$$

i.e.,

$$\frac{d}{dt}\|\bar{u}\|_{L^2} \le C(\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}). \tag{3.36}$$

As for the second equation of (3.27), in the same way, we have

$$\frac{d}{dt}\|\bar{v}\|_{L^2} \le C(\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}). \tag{3.37}$$

Combining (3.36) with (3.37), we obtain

$$\frac{d}{dt}(\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}) \le C(\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2}). \tag{3.38}$$

Hence the Gronwall's inequality immediately yields

$$\|\bar{u}\|_{L^2} + \|\bar{v}\|_{L^2} \le (\|\bar{u}(x,0)\|_{L^2} + \|\bar{v}(x,0)\|_{L^2})e^{Ct}. \tag{3.39}$$

From the last inequality we can obtain the uniqueness of solutions.

Part 4. (Positivity) We will show that we can obtain a nonnegative solution from nonnegative initial data to complete the proof of Lemma 3.1.

Let (u, v, w) be a smooth solution with nonnegative initial data and denote

$$\underline{u}(t) = \min_{x\in\mathbb{T}^2} u(x,t) = u(x_t,t),\ \underline{v} = \min_{x\in\mathbb{T}^2} v(x,t) = v(x_{t_*},t_*) \text{ and}$$
$$\underline{w} = \min_{x\in\mathbb{T}^2} w(x,t) = w(x_{t^*},t^*).$$

Evaluating the first equation of (1.1) at the minimum point of u and using the kernel expression for Λ^s as well the equation of w, we have

$$\begin{aligned}\frac{d}{dt}\underline{u}(t) &= -\Lambda^s u(x_t,t) - \chi_1\underline{u}\Delta w(x_t,t) + r_1\underline{u}(1-\underline{u}-a_1 v(x_t,t))\\ &\ge \underline{u}(t)\big[r_1 + (\alpha\chi_1 - r_1)\underline{u} + (\beta\chi_1 - a_1 r_1)v(x_t,t) - \gamma\chi_1 w(x_t,t)\big].\end{aligned} \tag{3.40}$$

By a comparison theorem, we obtain

$$\begin{aligned}u(x_t,t) \ge u_0(x)\exp\Big(\int_0^t &\big[r_1 + (\alpha\chi_1 - r_1)\underline{u}(x_s,s)\\ &+ (\beta\chi_1 - a_1 r_1)v(x_s,s) - \gamma\chi_1 w(x_s,s)\big]\mathrm{d}s\Big).\end{aligned} \tag{3.41}$$

Thus,

$$\underline{u} \ge 0$$

since $u_0(x) \ge 0$. As for the positivity of v, making through the same part of the procedure for the second equation of (1.1), we can get

$$\underline{v} \ge 0$$

because of $v_0(x) \ge 0$. Using the third equation of (1.1), we have

$$\gamma\underline{w} = \alpha u(x_{t^*},t^*) + \beta v(x_{t^*},t^*) \ge \alpha\underline{u} + \beta\underline{v} \ge 0.$$

Since then, we have completed the proof of Lemma 3.1. □

Lemma 3.2 (Weak estimates). *Let $0 < T < \infty$ be arbitrary and (u, v, w) be a nonnegative solution to (1.1). Take any $m > 0$ such that*

$$(m+1)(\alpha\chi_1 - r_1) \le \alpha\chi_1,\ (m+1)(\beta\chi_1 - a_1 r_1) \le \beta\chi_1,$$
$$(m+1)(\beta\chi_2 - r_2) \le \beta\chi_2 \text{ and } (m+1)(\alpha\chi_2 - a_2 r_2) \le \alpha\chi_2. \tag{3.42}$$

Then for any $t \in [0, T]$, it holds

$$\sup_{t\in[0,T]} \|u(t)\|_{L^{m+1}} \le e^{r_1 T}\|u_0\|_{L^{m+1}} \text{ and}$$
$$\sup_{t\in[0,T]} \|v(t)\|_{L^{m+1}} \le e^{r_2 T}\|v_0\|_{L^{m+1}}. \tag{3.43}$$

Proof. Testing the first equation of (1.1) with u^m, we can get

$$\begin{aligned} \int_{\mathbb{T}^2} u^m u_t = & -\int_{\mathbb{T}^2} u^m \Lambda^s u - \chi_1 \int_{\mathbb{T}^2} u^m \nabla\cdot(u\nabla w) \\ & + r_1 \int_{\mathbb{T}^2} u^{m+1}(1 - u - a_1 v). \end{aligned} \tag{3.44}$$

Integrating by parts yields

$$\begin{aligned} & \frac{1}{m+1}\frac{d}{dt}\int_{\mathbb{T}^2} u^{m+1} + \int_{\mathbb{T}^2} u^m \Lambda^s u \\ = & \int_{\mathbb{T}^2} \chi_1 m u^m \nabla w\cdot\nabla u + r_1 u^{m+1} - r_1 u^{m+2} - a_1 r_1 u^{m+1} v \\ = & \int_{\mathbb{T}^2} -\frac{\chi_1 m}{m+1} u^{m+1}\Delta w + r_1 u^{m+1} - r_1 u^{m+2} - a_1 r_1 u^{m+1} v \\ = & \int_{\mathbb{T}^2} r_1 u^{m+1} + (\frac{\alpha\chi_1 m}{m+1} - r_1) u^{m+2} \\ & + (\frac{\beta\chi_1 m}{m+1} - a_1 r_1) u^{m+1} v - \frac{\gamma\chi_1 m}{m+1} u^{m+1} w. \end{aligned} \tag{3.45}$$

Via the Stroock-Varopoulos inequality in Lemma 2.1 and (3.45), we have

$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{T}^2} u^{m+1} + \frac{4m}{m+1}\int_{\mathbb{T}^2} |\Lambda^{\frac{s}{2}}(u^{\frac{m+1}{2}})|^2 \\ \le & \int_{\mathbb{T}^2} r_1(m+1)u^{m+1} + \Big[\alpha\chi_1 m - r_1(m+1)\Big] u^{m+2} \\ & + \Big[\beta\chi_1 m - a_1 r_1(m+1)\Big] u^{m+1} v - \gamma\chi_1 m u^{m+1} w \\ \le & \int_{\mathbb{T}^2} r_1(m+1) u^{m+1}. \end{aligned} \tag{3.46}$$

Therefore we conclude

$$\frac{d}{dt}\int_{\mathbb{T}^2} u^{m+1} \le r_1(m+1)\int_{\mathbb{T}^2} u^{m+1}. \tag{3.47}$$

On integration, we can obtain

$$\sup_{t\in[0,T]} \|u(t)\|_{L^{m+1}} \le e^{r_1 T}\|u_0\|_{L^{m+1}}. \tag{3.48}$$

Testing the second equation of (1.1) with v^m and integrating by parts, in the same way, we can get

$$\sup_{t\in[0,T]} \|v(t)\|_{L^{m+1}} \le e^{r_2 T}\|v_0\|_{L^{m+1}}. \tag{3.49}$$

□

Lemma 3.3 (Strong estimates)**.** *Assume* $0 < T < \infty$ *be arbitrary. Let* (u, v, w) *be a nonnegative solution to* (1.1)*. If* $2 > s > 2p_0$ *with*

$$p_0 = \max\Big\{\frac{(\alpha\chi_1 - r_1)_+}{\alpha\chi_1}, \frac{(\beta\chi_1 - a_1 r_1)_+}{\beta\chi_1}, \frac{(\beta\chi_2 - r_2)_+}{\beta\chi_2}, \frac{(\alpha\chi_2 - a_2 r_2)_+}{\alpha\chi_2}\Big\} \in (0,1).$$

Then for any finite $p \ge 1$*, there exist finite constants* C_1 *and* C_2 *such that*

$$\begin{aligned} &\|u\|_{L^\infty(0,T;L^p)} + \|v\|_{L^\infty(0,T;L^p)} \\ &\le C\Big[(e^{C_1 T}+1)\|u_0\|_{L^p}^{C_1} + (e^{C_2 T}+1)\|v_0\|_{L^p}^{C_2}\Big]. \end{aligned} \tag{3.50}$$

Proof. Let us recall (3.46) and use the Sobolev embedding $H^{\frac{s}{2}}(\mathbb{T}^2) \subset L^{\frac{2}{1-\frac{s}{2}}}(\mathbb{T}^2)$ for $u^{\frac{m+1}{2}}$, we can get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^2} u^{m+1} + C_{m,s}\Big(\int_{\mathbb{T}^2} |u|^{\frac{m+1}{1-\frac{s}{2}}}\Big)^{1-\frac{s}{2}} \\ \le &\int_{\mathbb{T}^2} r_1(m+1)u^{m+1} + \Big[\alpha\chi_1 m - r_1(m+1)\Big]u^{m+2} \\ &+ \Big[\beta\chi_1 m - a_1 r_1(m+1)\Big]u^{m+1}v. \end{aligned} \tag{3.51}$$

Using Young's inequality for the last term on the right hand side and inequality $u^{m+1} - u^{m+2} \le 1$, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^2} u^{m+1} + C_{m,s}\Big(\int_{\mathbb{T}^2} |u|^{\frac{m+1}{1-\frac{s}{2}}}\Big)^{1-\frac{s}{2}} \\ \le &\int_{\mathbb{T}^2} r_1(m+1)u^{m+1} - r_1(m+1)u^{m+2} + \Big[\alpha\chi_1 m \\ &+ \frac{m+1}{m+2}\big|\beta\chi_1 m - a_1 r_1(m+1)\big|\Big]u^{m+2} \\ &+ \frac{1}{m+2}\big|\beta\chi_1 m - a_1 r_1(m+1)\big|v^{m+2} \\ \le &\, 4\pi^2 r_1(m+1) + \int_{\mathbb{T}^2}\Big[\alpha\chi_1 m + \frac{m+1}{m+2}\big|\beta\chi_1 m - a_1 r_1(m+1)\big|\Big]u^{m+2} \\ &+ \frac{1}{m+2}\big|\beta\chi_1 m - a_1 r_1(m+1)\big|v^{m+2}. \end{aligned} \tag{3.52}$$

After going through a similar process for v, we can arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^2} v^{m+1}+C_{m,s}\Big(\int_{\mathbb{T}^2}|v|^{\frac{m+1}{1-\frac{s}{2}}}\Big)^{1-\frac{s}{2}}\\ &\le 4\pi^2 r_2(m+1)+\int_{\mathbb{T}^2}\Big[\beta\chi_2 m+\frac{m+1}{m+2}\big|\alpha\chi_2 m-a_2r_2(m+1)\big|\Big]v^{m+2}\\ &\quad+\frac{1}{m+2}\big|\alpha\chi_2 m-a_2r_2(m+1)\big|u^{m+2}.\end{aligned}\tag{3.53}$$

Adding (3.52) and (3.53) together, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{T}^2}u^{m+1}+\int_{\mathbb{T}^2}v^{m+1}\Big)+C_{m,s}\|u\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}+C_{m,s}\|v\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\\ &\le 4\pi^2(m+1)(r_1+r_2)+A_1\int_{\mathbb{T}^2}u^{m+2}+A_2\int_{\mathbb{T}^2}v^{m+2},\end{aligned}\tag{3.54}$$

where

$$A_1:=\alpha\chi_1 m+\frac{m+1}{m+2}\big|\beta\chi_1 m-a_1r_1(m+1)\big|+\frac{1}{m+2}\big|\alpha\chi_2 m-a_2r_2(m+1)\big|$$

and

$$A_2:=\beta\chi_2 m+\frac{m+1}{m+2}\big|\alpha\chi_2 m-a_2r_2(m+1)\big|+\frac{1}{m+2}\big|\beta\chi_1 m-a_1r_1(m+1)\big|.$$

Let $q>1$ be a finite number to be fixed further, and using the interpolation inequality

$$\|u\|_{L^{m+2}}\le C\|u\|^{\theta}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\|u\|^{1-\theta}_{L^q}\ \text{and}\ \|v\|_{L^{m+2}}\le C\|v\|^{\theta}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\|v\|^{1-\theta}_{L^q},\tag{3.55}$$

with

$$\theta=\frac{2(m+2-q)(m+1)}{(m+2)\big[2(m+1)-q(2-s)\big]}.\tag{3.56}$$

As long as the interpolation inequality holds, we need the condition $q<m+2<\frac{m+1}{1-\frac{s}{2}}$, i.e.,

$$s(m+2)>2\ \text{and}\ q<m+2.\tag{3.57}$$

Under conditions (3.57), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{T}^2}u^{m+1}+\int_{\mathbb{T}^2}v^{m+1}\Big)+C_{m,s}\|u\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}+C_{m,s}\|v\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\\ &\le 4\pi^2(m+1)(r_1+r_2)+C\|u\|^{\theta(m+2)}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\|u\|^{(1-\theta)(m+2)}_{L^q}\\ &\quad+C\|v\|^{\theta(m+2)}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\|v\|^{(1-\theta)(m+2)}_{L^q}.\end{aligned}\tag{3.58}$$

Furthermore, if we have $\frac{m+1}{\theta(m+2)}>1$, by (3.56), i.e.,

$$qs>2$$

then Young's inequality yields

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{T}^2} u^{m+1}+\int_{\mathbb{T}^2} v^{m+1}\Big)+\frac{C_{m,s}}{2}\|u\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}+\frac{C_{m,s}}{2}\|v\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\\ &\le 4\pi^2(m+1)(r_1+r_2)+C\|u\|_{L^q}^{\frac{(1-\theta)(m+2)(m+1)}{m+1-\theta(m+2)}}+C\|v\|_{L^q}^{\frac{(1-\theta)(m+2)(m+1)}{m+1-\theta(m+2)}}.\end{aligned}\tag{3.59}$$

Recalling the expression of θ and assuming that

$$s(m+2)>2,\ q<m+2 \text{ and } qs>2,\tag{3.60}$$

we can arrive at

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{T}^2} u^{m+1}+\int_{\mathbb{T}^2} v^{m+1}\Big)+\frac{C_{m,s}}{2}\|u\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}+\frac{C_{m,s}}{2}\|v\|^{m+1}_{L^{\frac{m+1}{1-\frac{s}{2}}}}\\ &\le C\big(\|u\|_{L^q}^{\frac{q[s(m+2)-2]}{qs-2}}+\|v\|_{L^q}^{\frac{q[s(m+2)-2]}{qs-2}}+1\big).\end{aligned}\tag{3.61}$$

Denoting $\lambda := \frac{q[s(m+2)-2]}{qs-2} < \infty$, integrating above inequality in time and neglecting the second and the third terms on its left hand side, we have

$$\begin{aligned}&\|u\|_{L^\infty(0,T;L^{m+1})}+\|v\|_{L^\infty(0,T;L^{m+1})}\\ &\le C\big(\|u\|^{\frac{\lambda}{m+1}}_{L^\lambda(0,T;L^q)}+\|v\|^{\frac{\lambda}{m+1}}_{L^\lambda(0,T;L^q)}+T^{\frac{1}{m+1}}\big)+\|u_0\|_{L^{m+1}}+\|v_0\|_{L^{m+1}}\\ &\le CT^{\frac{1}{m+1}}\big(\|u\|^{\frac{\lambda}{m+1}}_{L^\infty(0,T;L^q)}+\|v\|^{\frac{\lambda}{m+1}}_{L^\infty(0,T;L^q)}+1\big)+\|u_0\|_{L^{m+1}}+\|v_0\|_{L^{m+1}},\end{aligned}\tag{3.62}$$

where the conditions (3.60) are required. Let

$$\begin{aligned}q=p_0^{-1}=\min\Big\{&\frac{\alpha\chi_1}{(\alpha\chi_1-r_1)_+},\frac{\beta\chi_1}{(\beta\chi_1-a_1r_1)_+},\frac{\beta\chi_2}{(\beta\chi_2-r_2)_+},\frac{\alpha\chi_2}{(\alpha\chi_2-a_2r_2)_+}\Big\}\\ &\in(1,+\infty),\end{aligned}\tag{3.63}$$

by Lemma 3.2, we have

$$\|u\|_{L^\infty(0,T;L^q)}\le e^{r_1T}\|u_0\|_{L^q} \text{ and } \|v\|_{L^\infty(0,T;L^q)}\le e^{r_2T}\|v_0\|_{L^q}.\tag{3.64}$$

Thus, we obtain

$$\begin{aligned}&\|u\|_{L^\infty(0,T;L^{m+1})}+\|v\|_{L^\infty(0,T;L^{m+1})}\\ &\le CT^{\frac{1}{m+1}}\Big[\big(e^{r_1T}\|u_0\|_{L^q}\big)^{\frac{\lambda}{m+1}}+\big(e^{r_2T}\|v_0\|_{L^q}\big)^{\frac{\lambda}{m+1}}+1\Big]\\ &\quad+\|u_0\|_{L^{m+1}}+\|v_0\|_{L^{m+1}}.\end{aligned}\tag{3.65}$$

Noticed that we already have $sq=sp_0^{-1}>2$, if $1<m+1\le q=p_0^{-1}$, Lemma 3.2 immediately yields

$$\|u\|_{L^\infty(0,T;L^{m+1})}+\|v\|_{L^\infty(0,T;L^{m+1})}\le e^{r_1T}\|u_0\|_{L^{m+1}}+e^{r_2T}\|v_0\|_{L^{m+1}}.\tag{3.66}$$

If $m+1>q$, the required conditions (3.60) are meet, interpolation in (3.65) of $\|u_0\|_{L^q}$ between $\|u_0\|_{L^1}$ and $\|u_0\|_{L^{m+1}}$ as well $\|v_0\|_{L^q}$ between $\|v_0\|_{L^1}$ and

$\|v_0\|_{L^{m+1}}$, respectively, we can obtain

$$\begin{aligned}&\|u\|_{L^\infty(0,T;L^{m+1})}+\|v\|_{L^\infty(0,T;L^{m+1})}\\ &\le C\Big[\big(e^{C_1T}+1\big)\|u_0\|_{L^{m+1}}^{C_1}+\big(e^{C_2T}+1\big)\|v_0\|_{L^{m+1}}^{C_2}\Big].\end{aligned}\tag{3.67}$$

□

Lemma 3.4. *Under the condition of Lemma 3.1, if*

$$\|u\|_{L^\infty(0,T_{\max};L^\infty(\mathbb{T}^2))}+\|v\|_{L^\infty(0,T_{\max};L^\infty(\mathbb{T}^2))}\le F(T_{\max})<\infty,$$

then we have

$$\limsup_{t\to T_{\max}}\|u(t)\|_{H^4(\mathbb{T}^2)}+\|v(t)\|_{H^4(\mathbb{T}^2)}\le G(T_{\max})<\infty.$$

Proof. In view of the definition of Sobolev space H^4, we have

$$\|u\|_{H^4}+\|v\|_{H^4}=\|u\|_{L^2}+\|v\|_{L^2}+\sum_{k=1}^{4}(\|\partial_x^k u\|_{L^2}+\|\partial_x^k v\|_{L^2}).\tag{3.68}$$

By Hölder's inequality, we immediately obtain

$$\|u\|_{L^2}+\|v\|_{L^2}\le C\big[\|u\|_{L^\infty(0,T_{\max};L^\infty(\mathbb{T}^2))}+\|v\|_{L^\infty(0,T_{\max};L^\infty(\mathbb{T}^2))}\big]<\infty.\tag{3.69}$$

Let us take $k=1$ for an example. Taking the derivative of $\frac{1}{2}\|\partial_x u\|_{L^2}^2$ in time and using the first equation of (1.1) yield

$$\begin{aligned}\int_{\mathbb{T}^2}\partial_x u\partial_x u_t=&-\int_{\mathbb{T}^2}\partial_x u\partial_x\Lambda^s u-\chi_1\int_{\mathbb{T}^2}\partial_x u\partial_x\nabla\cdot(u\nabla w)\\ &+r_1\int_{\mathbb{T}^2}\partial_x u\partial_x[u(1-u-a_1v)]\\ =&-\int_{\mathbb{T}^2}|\Lambda^{\frac{s}{2}}\partial_x u|^2+\mathrm{III}_1+\mathrm{III}_2.\end{aligned}\tag{3.70}$$

Integrating by parts and using Hölder's inequality for the term III_1, we obtain

$$\begin{aligned}\mathrm{III}_1=&-\chi_1\int_{\mathbb{T}^2}\partial_x u\partial_x(\nabla u\cdot\nabla w)-\chi_1\int_{\mathbb{T}^2}\partial_x\partial_x(u\Delta w)\\ =&-\chi_1\int_{\mathbb{T}^2}\partial_x u\partial_x\nabla u\cdot\nabla w-\chi_1\int_{\mathbb{T}^2}\partial_x u\nabla u\cdot\partial_x\nabla w\\ &-\chi_1\int_{\mathbb{T}^2}\partial_x u\partial_x u\Delta w-\chi_1\int_{\mathbb{T}^2}\partial_x uu\partial_x\Delta w\\ \le&-\frac{\chi_1}{2}\int_{\mathbb{T}^2}\nabla(\partial_x u)^2\cdot\nabla w+\chi_1\|\partial_x\nabla w\|_{L^\infty}\|\partial_x u\|_{L^2}\|\nabla u\|_{L^2}\\ &+\chi_1\|\Delta w\|_{L^\infty}\|\partial_x u\|_{L^2}^2+\chi_1\|u\|_{L^\infty}\|\partial_x u\|_{L^2}\|\partial_x\Delta w\|_{L^2}\\ \le&\frac{\chi_1}{2}\int_{\mathbb{T}^2}(\partial_x u)^2\Delta w+C\|\Delta w\|_{L^\infty}\|\partial_x u\|_{L^2}^2\\ &+C\|\partial_x u\|_{L^2}(\|\partial_x u\|_{L^2}+\|\partial_x v\|_{L^2}+\|\partial_x w\|_{L^2})\\ \le&\, C\|\Delta w\|_{L^\infty}\|\partial_x u\|_{L^2}^2+C\|\partial_x u\|_{L^2}(\|\partial_x u\|_{L^2}+\|\partial_x v\|_{L^2}+\|\partial_x w\|_{L^2})\end{aligned}\tag{3.71}$$

$$\leq C\|\partial_x u\|_{L^2}(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1),$$

where we have used

$$\|\partial_x w\|_{L^2} \leq \|w\|_{H^2} \leq \|u\|_{L^2} + \|v\|_{L^2} < \infty \tag{3.72}$$

and

$$\begin{aligned} \|\Delta w\|_{L^\infty} &\leq C(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{L^\infty}) \\ &\leq C(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{H^2}) < \infty. \end{aligned} \tag{3.73}$$

As for the term III_2, Hölder's inequality immediately yields

$$\begin{aligned} \text{III}_2 &= r_1 \int_{\mathbb{T}^2} \partial_x u \partial_x u - r_1 \int_{\mathbb{T}^2} \partial_x u \partial_x (u^2) - a_1 r_1 \int_{\mathbb{T}^2} \partial_x u \partial_x (uv) \\ &= r_1 \|\partial_x u\|_{L^2}^2 - 2r_1 \int_{\mathbb{T}^2} \partial_x u u \partial_x u - a_1 r_1 \int_{\mathbb{T}^2} \partial_x u \partial_x u v \\ &\quad - a_1 r_1 \int_{\mathbb{T}^2} \partial_x u u \partial_x v \\ &\leq r_1 \|\partial_x u\|_{L^2}^2 + 2r_1 \|u\|_{L^\infty} \|\partial_x u\|_{L^2}^2 + a_1 r_1 \|v\|_{L^\infty} \|\partial_x u\|_{L^2}^2 \\ &\quad + a_1 r_1 \|u\|_{L^\infty} \|\partial_x u\|_{L^2} \|\partial_x v\|_{L^2} \\ &\leq C\|\partial_x u\|_{L^2}(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2}). \end{aligned} \tag{3.74}$$

Plugging (3.71) and (3.74) into (3.70), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_x u\|_{L^2}^2 \leq C\|\partial_x u\|_{L^2}(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1), \tag{3.75}$$

i.e.,

$$\frac{d}{dt}\|\partial_x u\|_{L^2} \leq C(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1). \tag{3.76}$$

As for v, in the same way, we have

$$\frac{d}{dt}\|\partial_x v\|_{L^2} \leq C(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1). \tag{3.77}$$

Combining (3.76) with (3.77) yields

$$\frac{d}{dt}(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2}) \leq C(\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1). \tag{3.78}$$

By a comparison theorem, we obtain

$$\|\partial_x u\|_{L^2} + \|\partial_x v\|_{L^2} + 1 \leq C(\|\partial_x u_0\|_{L^2} + \|\partial_x v_0\|_{L^2} + 1)e^{CT} < \infty. \tag{3.79}$$

Similarly, we have

$$\|\partial_x^k u\|_{L^2} + \|\partial_x^k v\|_{L^2} < \infty \quad \text{for } k = 2, 3, 4.$$

Hence, we can arrive at the conclusion. □

4. Global existence

Taking any u_0, $v_0 \in H^4$ and taking advantage of Lemma 3.1, it is proved that there admits a local classical solution on the time interval $[0, T_{\max})$ for problem (1.1), and the standard continuation argument for an autonomous ODE in Banach spaces implies that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and

$$\limsup_{t\to T_{\max}} \|u(t)\|_{H^4} + \|v(t)\|_{H^4} = \infty. \tag{4.1}$$

Due to Lemma 3.4, in order to prove the global existence of solutions, we only need to show the boundedness of the sum of $\|u(t)\|_{L^\infty(0,T_{\max};L^\infty(T^2))}$ and $\|v(t)\|_{L^\infty(0,T_{\max};L^\infty(T^2))}$.

The proof of Theorem 1.1. Let us denote by x_{t^1} the point such that

$$u(x_{t^1}, t^1) = \max_{x\in\mathbb{T}^2} u(x,t)$$

and x_{t^2} the point such that

$$v(x_{t^2}, t^2) = \max_{x\in\mathbb{T}^2} v(x,t),$$

meanwhile, we write

$$\bar{u}(x,t) = u(x_{t^1}, t^1) \text{ and } \bar{v}(x,t) = v(x_{t^2}, t^2).$$

Since u, $v \in C^{2,1}(T^2 \times (0, T_{\max}))$, the function $\bar{u}$ and $\bar{v}$ are Lipschitz and therefore they possess a derivative, respectively almost everywhere. As the derivative vanishes at the maximum point, evaluating the first equation of (1.1) at the maximum point of u and using the kernel expression for Λ^s, we obtain

$$\bar{u}_t = -\Lambda^s u(x_{t^1}, t^1) - \chi_1 \bar{u} \Delta w(x_{t^1}, t^1) + r_1 \bar{u}\big[1 - \bar{u} - a_1 v(x_{t^1}, t^1)\big]. \tag{4.2}$$

Taking advantage of the third equation of (1.1) and the nonnegativity of solutions, we can get

$$\begin{aligned} &\frac{d}{dt}\bar{u}(t) + \Lambda^s u(x_{t^1}, t^1) \\ &= -\chi_1 \bar{u}\big[\gamma w(x_{t^1}, t^1) - \alpha \bar{u} - \beta v(x_{t^1}, t^1)\big] + r_1 \bar{u}\big[1 - \bar{u} - a_1 v(x_{t^1}, t^1)\big] \\ &= r_1 \bar{u} + (\alpha\chi_1 - r_1)\bar{u}^2 + (\beta\chi_1 - a_1 r_1)\bar{u} v(x_{t^1}, t^1) - \gamma\chi_1 \bar{u} w(x_{t^1}, t^1) \\ &\le r_1 \bar{u} + (\alpha\chi_1 - r_1)_+ \bar{u}^2 + (\beta\chi_1 - a_1 r_1)_+ \bar{u} v(x_{t^1}, t^1). \end{aligned} \tag{4.3}$$

As for the second equation of (1.1), in the same way, we have

$$\frac{d}{dt}\bar{v}(t) + \Lambda^s v(x_{t^2}, t^2) \le r_2 \bar{v} + (\beta\chi_2 - r_2)_+ \bar{v}^2 + (\alpha\chi_2 - a_2 r_2)_+ \bar{v} u(x_{t^2}, t^2). \tag{4.4}$$

If

$$\alpha\chi_1 - r_1 \le 0,\ \beta\chi_1 - a_1 r_1 \le 0,\ \beta\chi_2 - r_2 \le 0 \text{ and } \alpha\chi_2 - a_2 r_2 \le 0.$$

The inequalities (4.3) and (4.4) turn into

$$\frac{d}{dt}\bar{u} \le r_1 \bar{u} - (r_1 - \alpha\chi_1)\bar{u}^2 \text{ and } \frac{d}{dt}\bar{v} \le r_2 \bar{v} - (r_2 - \beta\chi_2)\bar{v}^2.$$

By a comparison theorem we conclude

$$u(t) \le \max\left\{\|u_0\|_{L^\infty}; \frac{r_1}{r_1-\alpha\chi_1}\right\} \text{ and } v(t) \le \max\left\{\|v_0\|_{L^\infty}; \frac{r_2}{r_2-\beta\chi_2}\right\}.$$

Thus

$$\|u\|_{L^\infty(0,T_{\max};L^\infty(T^2))} + \|v\|_{L^\infty(0,T_{\max};L^\infty(T^2))} < \infty.$$

Else,

$$p_0 = \max\left\{\frac{(\alpha\chi_1 - r_1)_+}{\alpha\chi_1}, \frac{(\beta\chi_1 - a_1 r_1)_+}{\beta\chi_1}, \frac{(\beta\chi_2 - r_2)_+}{\beta\chi_2}, \frac{(\alpha\chi_2 - a_2 r_2)_+}{\alpha\chi_2}\right\} \in (0,1),$$

using Lemma 2.2 for u with $p = p_0^{-1} \ge 1$, we have either

$$\bar{u}(t) \le M_1(s)\|u\|_{L^p},$$

or

$$\Lambda^s u(x_{t^1}, t^1) \ge M_2(s)\frac{\bar{u}(t)^{1+\frac{sp}{2}}}{\|u\|_{L^p}^{\frac{sp}{2}}},$$

and Lemma 3.3 yields

$$\begin{aligned} &\|u\|_{L^\infty(0,T_{\max};L^p)} + \|v\|_{L^\infty(0,T_{\max};L^p)} \\ &\le C\left[\left(e^{C_1 T_{\max}}+1\right)\|u_0\|_{L^p}^{C_1} + \left(e^{C_2 T_{\max}}+1\right)\|v_0\|_{L^p}^{C_2}\right] \\ &:= C_0(T_{\max}). \end{aligned} \tag{4.5}$$

Then we have the following alternative: either

$$\begin{aligned} \|u\|_{L^\infty(0,T_{\max};L^\infty)} &\le M_1(s)\|u\|_{L^\infty(0,T_{\max};L^p)} \\ &\le M_1(s)C_0(T_{\max}) \\ &\le \tilde{C}_0(T_{\max}), \end{aligned} \tag{4.6}$$

or

$$\begin{aligned} \Lambda^s u(x_{t^1}, t^1) &\ge M_2(s)\frac{\bar{u}(t)^{1+\frac{sp}{2}}}{\|u\|_{L^\infty(0,T_{\max};L^p)}^{\frac{sp}{2}}} \\ &\ge M_2(s)\frac{\bar{u}(t)^{1+\frac{sp}{2}}}{C_0^{\frac{sp}{2}}(T_{\max})} \\ &\ge \bar{C}_0(T_{\max})\bar{u}(t)^{1+\frac{sp}{2}}, \end{aligned} \tag{4.7}$$

i.e., we have either

$$\|u\|_{L^\infty(0,T_{\max};L^\infty)} \le \tilde{C}_0(T_{\max}), \tag{4.8}$$

or

$$\Lambda^s u(x_{t^1}, t^1) \ge \bar{C}_0(T_{\max})\bar{u}(t)^{1+\frac{sp}{2}}. \tag{4.9}$$

As for v, after going through the same process, we have a similar conclusion: either

$$\|v\|_{L^\infty(0,T_{\max};L^\infty)} \le \tilde{C}_0(T_{\max}), \tag{4.10}$$

or

$$\Lambda^s v(x_{t^2}, t^2) \geq \bar{C}_0(T_{\max})\bar{v}(t)^{1+\frac{sp}{2}}. \tag{4.11}$$

If (4.8) and (4.10) are true, we immediately arrive at

$$\|u\|_{L^\infty(0,T_{\max};L^\infty)} + \|v\|_{L^\infty(0,T_{\max};L^\infty)} < \infty. \tag{4.12}$$

In the case (4.8) and (4.11) are satisfied, we have

$$\|u\|_{L^\infty(0,T_{\max};L^\infty)} < \infty, \tag{4.13}$$

and

$$\begin{aligned} &\frac{d}{dt}\bar{v}(t) + \bar{C}_0(T_{\max})\bar{v}^{1+\frac{sp}{2}} \\ &\leq r_2\bar{v} + (\beta\chi_2 - r_2)_+\bar{v}^2 + (\alpha\chi_2 - a_2r_2)_+\bar{v}u(x_{t^2}, t^2) \\ &\leq (\beta\chi_2 - r_2)_+\bar{v}^2 + \big[r_2 + (\alpha\chi_2 - a_2r_2)_+\|u\|_{L^\infty(0,T_{\max};L^\infty)}\big]\bar{v}. \end{aligned} \tag{4.14}$$

In fact, we can get $1 + \frac{sp}{2} > 2$ from $s > 2p_0$. Using Young's inequality on the right hand side, we obtain

$$\frac{d}{dt}\bar{v} \leq C_3(T_{\max}). \tag{4.15}$$

Integration (4.15) in time gives us

$$\|v\|_{L^\infty(0,T_{\max};L^\infty)} \leq \|v_0\|_{L^\infty(0,T_{\max};L^\infty)}e^{C_3(T_{\max})T_{\max}}. \tag{4.16}$$

Combining (4.13) with (4.16) also leads to (4.12).

The case (4.9) and (4.10) hold can be treated in a similar fashion.

Finally, when (4.9) and (4.11) are correct, Young's inequality yields

$$\begin{aligned} &\frac{d}{dt}\bar{u}(t) + \bar{C}_0(T_{\max})\bar{u}^{1+\frac{sp}{2}} \\ &\leq r_1\bar{u} + (\alpha\chi_1 - r_1)_+\bar{u}^2 + (\beta\chi_1 - a_1r_1)_+\bar{u}\bar{v} \\ &\leq r_1\bar{u} + \big[(\alpha\chi_1 - r_1)_+ + \frac{1}{2}(\beta\chi_1 - a_1r_1)_+\big]\bar{u}^2 + \frac{1}{2}(\beta\chi_1 - a_1r_1)_+\bar{v}^2 \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} &\frac{d}{dt}\bar{v}(t) + \bar{C}_0(T_{\max})\bar{v}^{1+\frac{sp}{2}} \\ &\leq r_2\bar{v} + (\beta\chi_2 - r_2)_+\bar{v}^2 + (\alpha\chi_2 - a_2r_2)_+\bar{u}\bar{v} \\ &\leq r_2\bar{v} + \big[(\beta\chi_2 - r_2)_+ + \frac{1}{2}(\alpha\chi_2 - a_2r_2)_+\big]\bar{v}^2 + \frac{1}{2}(\alpha\chi_2 - a_2r_2)_+\bar{u}^2. \end{aligned} \tag{4.18}$$

Adding (4.18) into (4.17), we obtain

$$\begin{aligned} &\frac{d}{dt}\big(\bar{u}(t) + \bar{v}(t)\big) + \bar{C}_0(T_{\max})\bar{u}^{1+\frac{sp}{2}} + \bar{C}_0(T_{\max})\bar{v}^{1+\frac{sp}{2}} \\ &\leq r_1\bar{u} + r_2\bar{v} + \big[(\alpha\chi_1 - r_1)_+ + \frac{1}{2}(\beta\chi_1 - a_1r_1)_+ + \frac{1}{2}(\alpha\chi_2 - a_2r_2)_+\big]\bar{u}^2 \\ &\quad + \big[(\beta\chi_2 - r_2)_+ + \frac{1}{2}(\alpha\chi_2 - a_2r_2)_+ + \frac{1}{2}(\beta\chi_1 - a_1r_1)_+\big]\bar{v}^2. \end{aligned} \tag{4.19}$$

Making use of Young's inequality and neglecting the second and the third terms on the left hand side of (4.19), we have

$$\frac{d}{dt}\big(\bar{u}(t)+\bar{v}(t)\big) \le C_4(T_{\max}). \tag{4.20}$$

Integration (4.20) in time yields

$$\begin{aligned} &\|u\|_{L^\infty(0,T_{\max};L^\infty)} + \|v\|_{L^\infty(0,T_{\max};L^\infty)} \\ \le\ & (\|u_0\|_{L^\infty(0,T_{\max};L^\infty)} + \|v_0\|_{L^\infty(0,T_{\max};L^\infty)})e^{C_4(T_{\max})T_{\max}} \\ <\ & \infty. \end{aligned} \tag{4.21}$$

In all above four cases, taking advantage of Lemma 3.4, we arrive at a contradiction with (4.1). Hence we obtain the global existence of solutions. That is, the proof of Theorem 1.1 is complete. □

5. Global asymptotic stability

In this section, we will take advantage of the proof method of [8] to discuss the asymptotic behavior of solutions, for the sake of completeness, we present the main process here.

From the proof of Theorem 1.1, we know that u and v exist globally and they are both bounded and nonnegative, so we can define some finite nonnegative real numbers L_1, l_1, L_2, l_2 by

$$L_1 := \limsup_{t\to\infty}\big(\max_{x\in\mathbb{T}^2} u(x,t)\big),\ l_1 := \liminf_{t\to\infty}\big(\min_{x\in\mathbb{T}^2} u(x,t)\big),$$

$$L_2 := \limsup_{t\to\infty}\big(\max_{x\in\mathbb{T}^2} v(x,t)\big),\ l_2 := \liminf_{t\to\infty}\big(\min_{x\in\mathbb{T}^2} v(x,t)\big).$$

By the definitions, we obtain that for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\begin{aligned} &l_1 - \varepsilon < u(x,t) < L_1 + \varepsilon \quad \text{and} \\ &l_2 - \varepsilon < v(x,t) < L_2 + \varepsilon \quad \text{for all} \quad t > T_\varepsilon \quad \text{and all} \quad x \in \mathbb{T}^2. \end{aligned} \tag{5.1}$$

Via the maximum principle applied to the third equation of (1.1), we can get

$$\begin{aligned} &\min_{\xi\in\mathbb{T}^2}(\alpha u(\xi,t) + \beta v(\xi,t)) \\ \le\ & \gamma w(x,t) \\ \le\ & \max_{\xi\in\mathbb{T}^2}(\alpha u(\xi,t) + \beta v(\xi,t)) \text{ for all } t > 0 \text{ and all } x \in \mathbb{T}^2. \end{aligned} \tag{5.2}$$

Therefore, we have for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\alpha l_1 + \beta l_2 - 2\varepsilon < \gamma w(x,t) < \alpha L_1 + \beta L_2 + 2\varepsilon \text{ for all } t > T_\varepsilon \text{ and all } x \in \mathbb{T}^2. \tag{5.3}$$

Having the boundedness of u, v and w at hand, it is sufficient for us to obtain the conclusion.

Proof of Theorem 1.2. At first, by the first and the third equations of (1.1), we have

$$u_t + \Lambda^s u + \chi_1 \nabla u \cdot \nabla w = r_1 u + (\alpha\chi_1 - r_1)u^2 + (\beta\chi_1 - a_1 r_1)uv - \gamma\chi_1 uw. \tag{5.4}$$

Assuming $\varepsilon > 0$ and $t > T_\varepsilon$, taking advantage of (5.1), (5.3) and (1.8) as well the nonnegativity of u, we obtain

$$\begin{aligned} &u_t + \Lambda^s u + \chi_1 \nabla u \cdot \nabla w \\ \leq\ & r_1 u + (\alpha\chi_1 - r_1)u^2 + (\beta\chi_1 - a_1 r_1)u(l_2 - \varepsilon) - \chi_1 u(\alpha l_1 + \beta l_2 - 2\varepsilon) \\ =\ & -(r_1 - \alpha\chi_1)u^2 + \big[r_1 + l_2(\beta\chi_1 - a_1 r_1) - \chi_1(\alpha l_1 + \beta l_2) \\ & + (-(\beta\chi_1 - a_1 r_1) + 2\chi_1)\varepsilon\big]u. \end{aligned} \tag{5.5}$$

Choosing $\bar{u}_\varepsilon \in (0, \infty)$ such that $u(\cdot, T_\varepsilon) \leq \bar{u}_\varepsilon$ in $\mathbb{T}^2$ and denoting by $\bar{y} : [T_\varepsilon, +\infty) \to \mathbb{R}$ the function solving

$$\begin{cases} \bar{y}' = -(r_1 - \alpha\chi_1)\bar{y}^2 + \big[r_1 + l_2(\beta\chi_1 - a_1 r_1) \\ \qquad\quad - \chi_1(\alpha l_1 + \beta l_2) + (-(\beta\chi_1 - a_1 r_1) + 2\chi_1)\varepsilon\big]\bar{y}, \\ \bar{y}(T_\varepsilon) = \bar{u}_\varepsilon. \end{cases} \tag{5.6}$$

Thus we have

$$\bar{y}(t) \to \frac{\big[r_1 - a_1 r_1 l_2 - \alpha\chi_1 l_1 + (a_1 r_1 - \beta\chi_1 + 2\chi_1)\varepsilon\big]_+}{r_1 - \alpha\chi_1} \text{ as } t \to \infty. \tag{5.7}$$

By comparison as well the arbitrarity of ε, we can get

$$L_1 \leq \limsup_{t\to\infty} \bar{y}(t) = \frac{(r_1 - a_1 r_1 l_2 - \alpha\chi_1 l_1)_+}{r_1 - \alpha\chi_1}. \tag{5.8}$$

Making use of (1.8), we have

$$(r_1 - \alpha\chi_1)L_1 \leq (r_1 - a_1 r_1 l_2 - \alpha\chi_1 l_1)_+. \tag{5.9}$$

On the other hand, making use of (5.1), (5.3) and (1.8) as well the nonnegativity of u again, for all $\varepsilon > 0$ and $t > T_\varepsilon$, we also have

$$\begin{aligned} &u_t + \Lambda^s u + \chi_1 \nabla u \cdot \nabla w \\ \geq\ & r_1 u + (\alpha\chi_1 - r_1)u^2 + (\beta\chi_1 - a_1 r_1)u(L_2 + \varepsilon) \\ & - \chi_1 u(\alpha L_1 + \beta L_2 + 2\varepsilon) \\ =\ & -(r_1 - \alpha\chi_1)u^2 + \big[r_1 + L_2(\beta\chi_1 - a_1 r_1) - \chi_1(\alpha L_1 + \beta L_2) \\ & + (\beta\chi_1 - a_1 r_1 - 2\chi_1)\varepsilon\big]u. \end{aligned} \tag{5.10}$$

Let us choose $\underline{u}_\varepsilon > 0$ such that $u(\cdot, T_\varepsilon) \geq \underline{u}_\varepsilon$ in $\mathbb{T}^2$ and denote by $\underline{y} : [T_\varepsilon, +\infty) \to \mathbb{R}$ the function solving

$$\begin{cases} \underline{y}' = -(r_1 - \alpha\chi_1)\underline{y}^2 + \big[r_1 + L_2(\beta\chi_1 - a_1 r_1) \\ \qquad\quad - \chi_1(\alpha L_1 + \beta L_2) + (\beta\chi_1 - a_1 r_1 - 2\chi_1)\varepsilon\big]\underline{y}, \\ \underline{y}(T_\varepsilon) = \underline{u}_\varepsilon. \end{cases} \tag{5.11}$$

Then we arrive at

$$\underline{y}(t) \to \frac{\left[r_1 - a_1 r_1 L_2 - \alpha\chi_1 L_1 + (\beta\chi_1 - a_1 r_1 - 2\chi_1)\varepsilon\right]_+}{r_1 - \alpha\chi_1} \text{ as } t \to \infty. \tag{5.12}$$

From the comparison theorem and the arbitrarity of ε, we can obtain

$$l_1 \geq \liminf_{t\to\infty} \underline{y}(t) = \frac{r_1 - a_1 r_1 L_2 - \alpha\chi_1 L_1}{r_1 - \alpha\chi_1}. \tag{5.13}$$

By (1.8), we have

$$(r_1 - \alpha\chi_1) l_1 \geq r_1 - a_1 r_1 L_2 - \alpha\chi_1 L_1. \tag{5.14}$$

Making the same process to the second and the third equations of (1.1), we get

$$L_2 \leq \frac{(r_2 - a_2 r_2 l_1 - \beta\chi_2 l_2)_+}{r_2 - \beta\chi_2} \tag{5.15}$$

and

$$l_2 \geq \frac{r_2 - a_2 r_2 L_1 - \beta\chi_2 L_2}{r_2 - \beta\chi_2}. \tag{5.16}$$

Using (1.8), we have

$$(r_2 - \beta\chi_2) L_2 \leq (r_2 - a_2 r_2 l_1 - \beta\chi_2 l_2)_+ \tag{5.17}$$

and

$$(r_2 - \beta\chi_2) l_2 \geq r_2 - a_2 r_2 L_1 - \beta\chi_2 L_2. \tag{5.18}$$

Next, before going on, we will verify that the quantities of the upper bounds for L_1 and L_2 are in fact nonnegative, so we can neglect the positive part operator $(\cdot)_+$. By Lemma 3.3 of [8], we obtain

$$r_1 - a_1 r_1 l_2 - \alpha\chi_1 l_1 \geq 0 \text{ and } r_2 - a_2 r_2 l_1 - \beta\chi_2 l_2 \geq 0. \tag{5.19}$$

Thus (5.9) and (5.17) turn into

$$(r_1 - \alpha\chi_1) L_1 \leq r_1 - a_1 r_1 l_2 - \alpha\chi_1 l_1 \tag{5.20}$$

and

$$(r_2 - \beta\chi_2) L_2 \leq r_2 - a_2 r_2 l_1 - \beta\chi_2 l_2. \tag{5.21}$$

Since then, we have get explicit bounds for u and v, next we will calculate the exact limit values of u and v as $t \to \infty$. Let us start with the convergence of u and v, i.e., $L_1 = l_1$ and $L_2 = l_2$ hold. Collecting (5.14) and (5.20) as well Making use of (1.8), we can obtain

$$L_1 - l_1 \leq \frac{a_1 r_1}{r_1 - 2\alpha\chi_1}(L_2 - l_2). \tag{5.22}$$

By (5.18), (5.21) and (1.8), in the same way, we have

$$L_2 - l_2 \leq \frac{a_2 r_2}{r_2 - 2\beta\chi_2}(L_1 - l_1). \tag{5.23}$$

Combining (5.22) and (5.23) yields

$$L_2 - l_2 \leq \frac{a_2 r_2}{r_2 - 2\beta\chi_2} \frac{a_1 r_1}{r_1 - 2\alpha\chi_1}(L_2 - l_2). \tag{5.24}$$

Hence we can obtain $L_2 = l_2$ from the small coefficient in (1.8). By (5.22), we can also get $L_1 = l_1$.

Finally, we will show

$$L_1 = l_1 = u^* \text{ and } L_2 = l_2 = v^*. \tag{5.25}$$

From (5.14), (5.18), (5.20), (5.21) and (5.25), we have

$$(r_1 - \alpha\chi_1)L_1 = r_1 - a_1 r_1 L_2 - \alpha\chi_1 L_1 \text{ and } (r_2 - \beta\chi_2)L_2 = r_2 - a_2 r_2 L_1 - \beta\chi_2 L_2.$$

Therefore, we can obtain

$$L_1 = \frac{1 - a_1}{1 - a_1 a_2} = u^* = l_1 \text{ and } L_2 = \frac{1 - a_2}{1 - a_1 a_2} = v^* = l_2.$$

These imply that

$$u(t) \to u^* \text{ and } v(t) \to v^* \text{ as } t \to \infty$$

uniformly in $\mathbb{T}^2$. According to (5.3), for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\alpha u^* + \beta v^* - 2\varepsilon < \gamma w(x,t) < \alpha u^* + \beta v^* + 2\varepsilon \text{ for all } t > T_\varepsilon \text{ and all } x \in \mathbb{T}^2.$$

Therefore $w(x,t) \to \frac{\alpha u^* + \beta v^*}{\gamma}$ as $t \to \infty$ uniformly in $\mathbb{T}^2$. With these results, we have accomplished the proof of Theorem 1.2. □

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