

ON A CHANGE OF RINGS FOR MIXED MULTIPLICITIES

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ABSTRACT. This paper establishes a formula changing the ring from a Noetherian local ring A of dimension $d > 0$ containing the residue field k to the polynomial ring in d variables $k[X_1, X_2, \dots, X_d]$ for mixed multiplicities. And as consequences, we get a formula for the multiplicity of Rees rings and formulas for mixed multiplicities and the multiplicity of Rees rings of quotient rings of A by highest dimensional associated prime ideals of A .

1. Introduction

Let (A, \mathfrak{m}, k) be a Noetherian local ring with maximal ideal \mathfrak{m} and the residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A -module. Let J, I_1, \dots, I_s be ideals of A such that J is an \mathfrak{m} -primary ideal and $I = I_1 \cdots I_s$ is not contained in $\sqrt{\text{Ann}M}$. Set $\dim M/0_M : I^\infty = q$. Then $\ell_A \left[\frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s} M}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s} M} \right]$ is a polynomial of degree $q - 1$ for all large n_0, n_1, \dots, n_s [18, Proposition 3.1(i)]. The terms of total degree $q - 1$ in this polynomial have the form

$$\sum_{k_0 + k_1 + \cdots + k_s = q-1} e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{k_0! k_1! \cdots k_s!}.$$

Then $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}, M)$ is called the *mixed multiplicity of M with respect to ideals J, I_1, \dots, I_s of the type $(k_0 + 1, k_1, \dots, k_s)$* .

It has been known that mixed multiplicities are an important object of Algebraic Geometry and Commutative Algebra. In past years, one obtained interesting results for this theory. Apart from the results for the positivity and characterizations mixed multiplicities in terms of the Hilbert-Samuel multiplicity (see e.g. [4–6, 10–16, 18–22, 24, 25, 27–29]) and the Euler-Poincaré characteristic (see e.g. [15, 32]), the representation of the multiplicity of Rees modules as the sum of mixed multiplicities has been established (see e.g. [7, 9, 16, 17, 23, 26]). Moreover, recent papers showed that many important properties of the Hilbert-Samuel multiplicity can be expanded to mixed multiplicities (see e.g. [30, 31]).

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In this paper, we study mixed multiplicities and the multiplicity of Rees rings over a Noetherian local ring A containing the residue field k . Let a_1, a_2, \dots, a_d be a system of parameters for A . Denote by $S = k[X_1, X_2, \dots, X_d]$ the polynomial ring in d variables X_1, X_2, \dots, X_d over k , and by $R = S_{(X_1, X_2, \dots, X_d)}$ the localization of S at the maximal ideal (X_1, X_2, \dots, X_d) . For any ideal \mathfrak{J} of S , put $\tilde{\mathfrak{J}}_R = \mathfrak{J}R$ and \mathfrak{J}_A the ideal of A generated by $\{f(a_1, a_2, \dots, a_d) \mid f \in \mathfrak{J}\}$.

Boda and Schenzel in [3] investigated the relationship between the Hilbert-Samuel multiplicity of A and the Hilbert-Samuel multiplicity of R .

Being inspired by the main result of [3], we want to build formulas for the relationship between mixed multiplicities of A and R . In fact, we obtain the following.

Theorem 1.1 (Theorem 2.2). *Let (A, \mathfrak{m}, k) be a Noetherian local ring of dimension $d > 0$ containing the residue field k . Let $\mathfrak{q} = a_1, a_2, \dots, a_d$ be a system of parameters for A . Assume that $J, I_1, \dots, I_s \subset (X_1, X_2, \dots, X_d)S$ are ideals of S with J_R being $(X_1, X_2, \dots, X_d)R$ -primary and $I_1 \cdots I_s \neq 0$. Then we have*

$$e_A(J_A^{[k_0+1]}, I_{1A}^{[k_1]}, \dots, I_{sA}^{[k_s]}; A) = e_R(J_R^{[k_0+1]}, I_{1R}^{[k_1]}, \dots, I_{sR}^{[k_s]}; R)e_A(\mathfrak{q}; A).$$

Using this theorem one can transfer the computation of a class of mixed multiplicities satisfying the assumptions of Theorem 1.1 from Noetherian local rings containing the residue field k to polynomial rings over k . As corollaries of Theorem 1.1, we obtain a formula on the relationship between the multiplicity of Rees rings of A and R (see Corollary 2.3), and formulas for mixed multiplicities and the multiplicity of Rees rings in the case of A/\mathfrak{p} for \mathfrak{p} -highest dimensional associated prime ideal of A (see Corollary 2.4).

2. On some formulas transferring multiplicities

This section states and proves the main theorem together with corollaries for the multiplicity of Rees rings and mixed multiplicities in the case of A/\mathfrak{p} for \mathfrak{p} -highest dimensional associated prime ideal of A . And to prove the main theorem we show in Note 2.1 that mixed multiplicities of a module are the same as that of its completion.

Let (A, \mathfrak{m}) be a Noetherian local ring with maximal ideal \mathfrak{m} and the residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A -module. Let I_1, \dots, I_s be ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\text{Ann}M}$. Let J be an \mathfrak{m} -primary ideal. We put $\mathbf{0} = (0, \dots, 0)$; $\mathbf{k} = (k_1, \dots, k_s)$; $\mathbf{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$ and $\mathbf{k}! = k_1! \cdots k_s!$; $|\mathbf{k}| = k_1 + \dots + k_s$; $\mathbf{n}^{\mathbf{k}} = n_1^{k_1} \cdots n_s^{k_s}$. Moreover, set

$$\mathbf{I} = I_1, \dots, I_s; \mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \dots, I_s^{[k_s]}; \mathbb{I}^{\mathbf{n}} = I_1^{n_1} \cdots I_s^{n_s}.$$

Suppose that $\dim M/0_M : I^\infty = q$. Recall that the author of [18, Proposition 3.1(i)] in 2000 (see [11, Proposition 3.1(i)]) proved that $\ell_A \left[\frac{J^{n_0} \mathbb{I}^{\mathbf{n}} M}{J^{n_0+1} \mathbb{I}^{\mathbf{n}} M} \right]$ is a polynomial of degree $q - 1$ for all large n_0, \mathbf{n} . The terms of total degree $q - 1$

in this polynomial have the form

$$\sum_{k_0 + |\mathbf{k}| = q-1} e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; M) \frac{n_0^{k_0} \mathbf{n}^{\mathbf{k}}}{k_0! \mathbf{k}!}.$$

Then $e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; M)$ is called the *mixed multiplicity of M with respect to ideals J, \mathbf{I} of the type $(k_0 + 1, \mathbf{k})$* .

We need the following note which will be used in the proof of the main result.

Note 2.1. For any A -module N , denote by \widehat{N} the \mathfrak{m} -adic completion of N . Now assume that $\ell_A(N) = t < \infty$, i.e., N has a composition series of length t :

$$N = N_0 \supset N_1 \supset \dots \supset N_t = \{0\},$$

where $N_{i-1}/N_i \cong k$ for all $1 \leq i \leq t$. Then we get a decreasing sequence of \widehat{A} -submodules of \widehat{N} :

$$(1) \quad \widehat{N} = \widehat{N}_0 \supset \widehat{N}_1 \supset \dots \supset \widehat{N}_t = \{0\}.$$

Note that $\widehat{N}_{i-1}/\widehat{N}_i \cong \widehat{N_{i-1}/N_i} \cong \widehat{k}$ and $\widehat{k} \cong k$ for all $1 \leq i \leq t$. Therefore (1) is also a composition series of \widehat{N} . Hence \widehat{N} is an \widehat{A} -module of finite length and $\ell_A(N) = \ell_{\widehat{A}}(\widehat{N})$. From this it follows that

$$\ell_A \left[\frac{J^{n_0} \mathbb{I}^{\mathbf{n}} M}{J^{n_0+1} \mathbb{I}^{\mathbf{n}} M} \right] = \ell_{\widehat{A}} \left[\frac{\widehat{J^{n_0} \mathbb{I}^{\mathbf{n}} M}}{\widehat{J^{n_0+1} \mathbb{I}^{\mathbf{n}} M}} \right] = \ell_{\widehat{A}} \left[\frac{\widehat{J}^{n_0} \widehat{I}_1^{n_1} \dots \widehat{I}_s^{n_s} \widehat{M}}{\widehat{J}^{n_0+1} \widehat{I}_1^{n_1} \dots \widehat{I}_s^{n_s} \widehat{M}} \right].$$

Consequently $e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; M) = e_{\widehat{A}}(\widehat{J}^{[k_0+1]}, \widehat{I}_1^{[k_1]}, \dots, \widehat{I}_s^{[k_s]}; \widehat{M})$.

Next suppose further that A contains the residue field k . Let a_1, a_2, \dots, a_d be a system of parameters for A . Denote by $S = k[X_1, X_2, \dots, X_d]$ the polynomial ring in d variables X_1, X_2, \dots, X_d over k , and by $R = S_{(X_1, X_2, \dots, X_d)}$ the localization of S at the maximal ideal (X_1, X_2, \dots, X_d) .

For any ideal \mathfrak{J} of S , put $\mathfrak{J}_R = \mathfrak{J}R$ and \mathfrak{J}_A the ideal of A generated by

$$\{f(a_1, a_2, \dots, a_d) \mid f \in \mathfrak{J}\}.$$

Now assume that $J, I_1, \dots, I_s \subset (X_1, X_2, \dots, X_d)S$ are ideals of S with J_R being $(X_1, X_2, \dots, X_d)R$ -primary. We need to determine the relationship between mixed multiplicities of A with respect to ideals $J_A, I_{1A}, \dots, I_{sA}$ and mixed multiplicities of R with respect to ideals $J_R, I_{1R}, \dots, I_{sR}$ of the same type.

And as one might expect, the our aim is achieved by the following theorem.

Theorem 2.2. *Let (A, \mathfrak{m}, k) be a Noetherian local ring of dimension $d > 0$ containing the residue field k . Let $\mathfrak{q} = a_1, a_2, \dots, a_d$ be a system of parameters for A . Assume that $J, I_1, \dots, I_s \subset (X_1, X_2, \dots, X_d)S$ are ideals of S with $I_1 \cdots I_s \neq 0$ and J_R being $(X_1, X_2, \dots, X_d)R$ -primary. Set $\mathbf{I}_A = I_{1A}, \dots, I_{sA}$; $\mathbf{I}_R = I_{1R}, \dots, I_{sR}$. Then we have*

$$e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[\mathbf{k}]}; A) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R)e_A(\mathfrak{q}; A).$$

Proof. Set $A' = k[a_1, a_2, \dots, a_d]$ and $\mathfrak{n} = (a_1, a_2, \dots, a_d)A'$. Define

$$F : S \rightarrow A' \text{ by } F(f(X_1, X_2, \dots, X_d)) = f(a_1, a_2, \dots, a_d).$$

Remember that the system of parameters a_1, a_2, \dots, a_d for A is algebraically independent over k (see e.g. [1, Corollary 11.21]), so F is an isomorphism. Moreover

$$F((X_1, X_2, \dots, X_d)) = \mathfrak{n}.$$

Hence $\dim A' = d$ and \mathfrak{n} is a maximal ideal of A' . Set $C = A'_\mathfrak{n}$. It is clear that $\mathfrak{n} = \mathfrak{m} \cap A'$. From this it follows that C is a subring of $A_\mathfrak{m}$. Note that $A_\mathfrak{m} = A$ because A is a local ring with maximal ideal \mathfrak{m} . Hence C is a subring of A .

It can be verified that the isomorphism F yields an isomorphism $F^* : R \rightarrow C$ given by

$$F^*\left(\frac{f}{g}\right) = \frac{f(a_1, a_2, \dots, a_d)}{g(a_1, a_2, \dots, a_d)}.$$

Consequently, C is a d -dimensional Noetherian local domain with maximal ideal $\mathfrak{n}C$ and the residue field k . Now for any ideal I of S , we put $I_C = F(I)C$ and $\mathbf{I}_C = I_{1C}, \dots, I_{sC}$. Then $F^*(I_R) = I_C$ for any ideal I of S . So J_C is $\mathfrak{n}C$ -primary since J_R is $(X_1, X_2, \dots, X_d)R$ -primary. Moreover, we get

$$\begin{aligned} e_C(a_1, a_2, \dots, a_d; C) &= e_R(X_1, X_2, \dots, X_d; R); \\ e_C(J_C^{[k_0+1]}, \mathbf{I}_C^{[\mathbf{k}]}; C) &= e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R). \end{aligned}$$

It can easily be seen that $I_C A = I_A$ for any ideal I of S . Since a_1, a_2, \dots, a_d is a system of parameters for A , we get $\mathfrak{m}^n \subset (a_1, a_2, \dots, a_d)A$ for a certain integer n . On the other hand $[(a_1, a_2, \dots, a_d)C]^u \subset J_C$ for a certain integer u because J_C is $\mathfrak{n}C$ -primary. So we have

$$[(a_1, a_2, \dots, a_d)A]^u = [(a_1, a_2, \dots, a_d)C]^u A \subset J_C A = J_A.$$

Hence $\mathfrak{m}^{nu} \subset J_A$. Consequently J_A is \mathfrak{m} -primary.

Set $\mathbb{I}_A^\mathfrak{n} = I_{1A}^{n_1} \cdots I_{sA}^{n_s}$ and $\mathbb{I}_C^\mathfrak{n} = I_{1C}^{n_1} \cdots I_{sC}^{n_s}$. Note that any A -module is also a C -module. Moreover, for any composition series of an A -module U is also a composition series of U as a C -module because the residue fields of A and C are the same. So

$$\ell_A \left[\frac{J_A^{n_0} \mathbb{I}_A^\mathfrak{n} A}{J_A^{n_0+1} \mathbb{I}_A^\mathfrak{n} A} \right] = \ell_C \left[\frac{J_A^{n_0} \mathbb{I}_A^\mathfrak{n} A}{J_A^{n_0+1} \mathbb{I}_A^\mathfrak{n} A} \right].$$

Now, consider A as a C -module, then since $I_C A = I_A$ for any ideal I of S , it follows that

$$\ell_C \left[\frac{J_A^{n_0} \mathbb{I}_A^\mathfrak{n} A}{J_A^{n_0+1} \mathbb{I}_A^\mathfrak{n} A} \right] = \ell_C \left[\frac{J_C^{n_0} \mathbb{I}_C^\mathfrak{n} A}{J_C^{n_0+1} \mathbb{I}_C^\mathfrak{n} A} \right].$$

Therefore

$$(2) \quad \ell_A \left[\frac{J_A^{n_0} \mathbb{I}_A^\mathfrak{n} A}{J_A^{n_0+1} \mathbb{I}_A^\mathfrak{n} A} \right] = \ell_C \left[\frac{J_C^{n_0} \mathbb{I}_C^\mathfrak{n} A}{J_C^{n_0+1} \mathbb{I}_C^\mathfrak{n} A} \right].$$

By Note 2.1, without loss of generality, in this proof, we can consider $A = \widehat{A}$; $C = \widehat{C}$ and $R = \widehat{R}$. Then we have

$$k[[a_1, a_2, \dots, a_d]] = C \cong R = \widehat{R} = k[[X_1, X_2, \dots, X_d]].$$

So in this case, A is a Noetherian complete local ring containing the residue field k and $C = k[[a_1, a_2, \dots, a_d]]$. Hence A is a finite generated C -module (see e.g. [2, Theorem A.22]). Consequently, by (2) and the definition of the mixed multiplicity, it shows that

$$e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[k]}; A) = e_C(J_C^{[k_0+1]}, \mathbf{I}_C^{[k]}; A).$$

As well as this formula, we have the following formula for the Hilbert-Samuel multiplicity

$$e_A(a_1, a_2, \dots, a_d; A) = e_C(a_1, a_2, \dots, a_d; A).$$

Denote by K the field of fractions of C . Then by [31, Corollary 3.6] we have

$$e_C(J_C^{[k_0+1]}, \mathbf{I}_C^{[k]}; A) = e_C(J_C^{[k_0+1]}, \mathbf{I}_C^{[k]}; C) \dim_K(K \otimes A).$$

Consequently, since $e_C(J_C^{[k_0+1]}, \mathbf{I}_C^{[k]}; C) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[k]}; R)$,

$$e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[k]}; A) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[k]}; R) \dim_K(K \otimes A).$$

Recall that

$$e_C(a_1, a_2, \dots, a_d; A) = e_C(a_1, a_2, \dots, a_d; C) \dim_K(K \otimes A)$$

(see e.g. [2, Corollary 4.7.9] or [8, Corollary 11.2.6]). On the other hand,

$$e_C(a_1, a_2, \dots, a_d; C) = e_R(X_1, X_2, \dots, X_d; R) = 1$$

because R is a regular local ring. Hence

$$e_A(a_1, a_2, \dots, a_d; A) = \dim_K(K \otimes A).$$

Thus, $e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[k]}; A) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[k]}; R)e_A(a_1, a_2, \dots, a_d; A)$. □

Denote by $\mathfrak{R}(\mathbf{I}; A) = \bigoplus_{\mathbf{n} \geq \mathbf{0}} \mathbb{I}^{\mathbf{n}}$ the Rees algebra of ideals \mathbf{I} and by

$$\mathfrak{R}(\mathbf{I}; M) = \bigoplus_{\mathbf{n} \geq \mathbf{0}} \mathbb{I}^{\mathbf{n}} M$$

the Rees module of ideals \mathbf{I} with respect to M . Set $\mathfrak{R}(\mathbf{I}; A)_+ = \bigoplus_{|\mathbf{n}| > 0} \mathbb{I}^{\mathbf{n}}$.

Then as a corollary of Theorem 2.2, we get the following formula for Rees rings.

Corollary 2.3. *Set $\mathfrak{J}_A = (J_A, \mathfrak{R}(\mathbf{I}_A; A)_+)$; $\mathfrak{J}_R = (J_R, \mathfrak{R}(\mathbf{I}_R; R)_+)$. Then with the previous notions and the assumptions as in Theorem 2.2 we have*

$$e(\mathfrak{J}_A; \mathfrak{R}(\mathbf{I}_A; A)) = e(\mathfrak{J}_R; \mathfrak{R}(\mathbf{I}_R; R))e_A(\mathfrak{q}; A).$$

Proof. By Theorem 2.2, we have

$$e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[\mathbf{k}]}; A) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R)e_A(\mathfrak{q}; A).$$

Now, since $C \cong R$ are domains and $I_1 \cdots I_s \neq 0$ in S , $\text{ht}_R(I_{1R} \cdots I_{sR}) > 0$ and $\text{ht}_C(I_{1C} \cdots I_{sC}) > 0$, here C as in the proof of Theorem 2.2. Note that A is integral over C since $\mathfrak{q} = a_1, a_2, \dots, a_d$ is a system of parameters for A . Then it is easily seen that $\text{ht}_A(I_{1A} \cdots I_{sA}) > 0$ since $\text{ht}_C(I_{1C} \cdots I_{sC}) > 0$ and A is integral over C . Hence by [17, Theorem 1.4], we get

$$\begin{aligned} e(\mathfrak{J}_A; \mathfrak{R}(\mathbf{I}_A; A)) &= \sum_{k_0 + |\mathbf{k}| = d-1} e_A(J_A^{[k_0+1]}, \mathbf{I}_A^{[\mathbf{k}]}; A) \\ &= \sum_{k_0 + |\mathbf{k}| = d-1} e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R)e_A(\mathfrak{q}; A) \\ &= \left[\sum_{k_0 + |\mathbf{k}| = d-1} e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R) \right] e_A(\mathfrak{q}; A) \\ &= e(\mathfrak{J}_R; \mathfrak{R}(\mathbf{I}_R; R))e_A(\mathfrak{q}; A). \end{aligned}$$

Consequently, $e(\mathfrak{J}_A; \mathfrak{R}(\mathbf{I}_A; A)) = e(\mathfrak{J}_R; \mathfrak{R}(\mathbf{I}_R; R))e_A(\mathfrak{q}; A)$. □

Finally, denote by Π the set of all prime ideals \mathfrak{p} of A such that $\mathfrak{p} \in \text{Min}A$ and $\dim A/\mathfrak{p} = \dim A$. For any $\mathfrak{p} \in \Pi$, denote by $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_d; J_{A/\mathfrak{p}}; \mathbf{I}_{A/\mathfrak{p}}$ the images of $a_1, a_2, \dots, a_d; J_A; \mathbf{I}_A$ in A/\mathfrak{p} , respectively. Set $\mathfrak{J}_{A/\mathfrak{p}} = (J_{A/\mathfrak{p}}, \mathfrak{R}(\mathbf{I}_{A/\mathfrak{p}}; A/\mathfrak{p}))_+$. Then since $\dim A/\mathfrak{p} = \dim A$, it follows that $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_d$ is a system of parameters for A/\mathfrak{p} . Since C is a domain and A is integral over C , we get $\mathfrak{p} \cap C = 0$ for any $\mathfrak{p} \in \Pi$, here C as in the proof of Theorem 2.2. Consequently, $\frac{C+\mathfrak{p}}{\mathfrak{p}} \cong C$. Hence we can consider C as a subring of A/\mathfrak{p} .

So one can replace A by A/\mathfrak{p} in Theorem 2.2 and Corollary 2.3, that means $e_{A/\mathfrak{p}}(J_{A/\mathfrak{p}}^{[k_0+1]}, \mathbf{I}_{A/\mathfrak{p}}^{[\mathbf{k}]}; A/\mathfrak{p}) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R)e_{A/\mathfrak{p}}((a_1, a_2, \dots, a_d)A/\mathfrak{p}; A/\mathfrak{p})$ and $e(\mathfrak{J}_{A/\mathfrak{p}}; \mathfrak{R}(\mathbf{I}_{A/\mathfrak{p}}; A/\mathfrak{p})) = e(\mathfrak{J}_R; \mathfrak{R}(\mathbf{I}_R; R))e_{A/\mathfrak{p}}((a_1, a_2, \dots, a_d)A/\mathfrak{p}; A/\mathfrak{p})$.

Hence we get the following result.

Corollary 2.4. *Denote by Π the set of all prime ideals $\mathfrak{p} \in \text{Min}A$ such that $\dim A/\mathfrak{p} = \dim A$. Then with the previous notions and the assumptions as in Theorem 2.2, for any $\mathfrak{p} \in \Pi$ we have*

- (i) $e_{A/\mathfrak{p}}(J_{A/\mathfrak{p}}^{[k_0+1]}, \mathbf{I}_{A/\mathfrak{p}}^{[\mathbf{k}]}; A/\mathfrak{p}) = e_R(J_R^{[k_0+1]}, \mathbf{I}_R^{[\mathbf{k}]}; R)e_{A/\mathfrak{p}}((\mathfrak{q})A/\mathfrak{p}; A/\mathfrak{p})$.
- (ii) $e(\mathfrak{J}_{A/\mathfrak{p}}; \mathfrak{R}(\mathbf{I}_{A/\mathfrak{p}}; A/\mathfrak{p})) = e(\mathfrak{J}_R; \mathfrak{R}(\mathbf{I}_R; R))e_{A/\mathfrak{p}}((\mathfrak{q})A/\mathfrak{p}; A/\mathfrak{p})$.

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