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A NOTE ON THE BOUNDARY BEHAVIOUR OF THE SQUEEZING FUNCTION AND FRIDMAN INVARIANT

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Abstract. Let Ω be a domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is smooth pseudoconvex of D'Angelo finite type near a boundary point $\xi_0 \in \partial\Omega$ and the Levi form has corank at most 1 at ξ_0. Our goal is to show that if the squeezing function $s_\Omega(\eta_j)$ tends to 1 or the Fridman invariant $h_\Omega(\eta_j)$ tends to 0 for some sequence $\{\eta_j\} \subset \Omega$ converging to ξ_0, then this point must be strongly pseudoconvex.

1. Introduction and the main result

The study of biholomorphic invariants has been attracted much attention in the complex differential geometry to enhance the comprehension and application of biholomorphic classification of complex domains. The squeezing function, the Fridman invariant, and the quotient invariant by using the Carathéodory and Kobayashi-Eisenman volume elements, have received increasing interest as biholomorphic invariants in recent years (see [2], [12], [14], [15] and the references therein). We particularly consider both the squeezing function and the Fridman invariant associated to a certain class of pseudoconvex domains in $\mathbb{C}^n$ in this paper.

Let Ω be a domain in $\mathbb{C}^n$ and $p \in \Omega$. For a holomorphic embedding $f\colon \Omega \to \mathbb{B}^n$ with $f(p) = 0$, let us define

$$s_{\Omega,f}(p) := \sup\{r > 0\colon \mathbb{B}(0;r) \subset f(\Omega)\},$$

where $\mathbb{B}(z_0;r) \subset \mathbb{C}^n$ denotes the complex ball of radius r with center at z_0 and $\mathbb{B}^n$ denotes the complex unit ball $\mathbb{B}(0;1)$. Then the *squeezing function* $s_\Omega : \Omega \to \mathbb{R}$ is defined in [4] as

$$s_\Omega(p) := \sup_f \{s_{\Omega,f}(p)\}.$$

Note that $0 < s_\Omega(z) \leq 1$ for any point $z \in \Omega$.

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Next, let us recall the Fridman invariant. Let Ω be a bounded domain in $\mathbb{C}^n$ and let $B_\Omega(p,r)$ be the Kobayashi ball around p of radius $r>0$. Let $\mathcal{R}$ be the set of all $r>0$ such that there is a holomorphic embedding $f\colon \mathbb{B}^n\to\Omega$ with $B_\Omega(p,r)\subset f(\mathbb{B}^n)$. Note that $\mathcal{R}$ is non-empty (cf. [12]). Then the *Fridman invariant* is defined by

$$h_\Omega(p)=\inf_{r\in\mathcal{R}}\frac{1}{r}.$$

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$ and $\xi_0\in\partial\Omega$. Suppose that $\partial\Omega$ is pseudoconvex of D'Angelo finite type near ξ_0. Then it is proved in [5], [6] and [10] that ξ_0 is strongly pseudoconvex if $\lim\limits_{\Omega\ni z\to\xi_0} s_\Omega(z)=1$.

Now we consider a sequence $\{\eta_j\}\subset\Omega$ converging to ξ_0. Suppose that $\partial\Omega$ is pseudoconvex of D'Angelo finite type near ξ_0 and $\lim_{j\to\infty} s_\Omega(\eta_j)=1$ or $\lim_{j\to\infty} h_\Omega(\eta_j)=0$. In [9] and [12], they proved that if the sequence $\{\eta_j\}\subset\Omega$ converges to ξ_0 along the inner normal line to $\partial\Omega$ at ξ_0, then ξ_0 must be strongly pseudoconvex (for details, see [9] for $n=2$ and [12] for general case). Moreover, this result was obtained in [13] for the case that $\{\eta_j\}\subset\Omega$ converges nontangentially to ξ_0 and in [15] for the case that $\{\eta_j\}\subset\Omega$ converges $\left(\frac{1}{m_1},\ldots,\frac{1}{m_{n-1}}\right)$-nontangentially to an h-extendible boundary point ξ_0 (for definition, see [15]). Here $(1,m_1,\ldots,m_{n-1})$ is the *multitype of* $\partial\Omega$ *at* ξ_0 and the *h-extendiblility at* ξ_0 means that the Catlin multitype and D'Angelo multitype of $\partial\Omega$ at ξ_0 coincide (see [16]).

Throughout this paper, we consider a smooth bounded domain Ω in $\mathbb{C}^n$ and a point $\xi_0\in\partial\Omega$ such that $\partial\Omega$ is pseudoconvex of D'Angelo finite type near ξ_0 and the Levi form has corank at most 1 at ξ_0. In this paper, we prove the following theorem.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary. If ξ_0 is a boundary point of Ω of D'Angelo finite type such that the Levi form has corank at most 1 at ξ_0 and if there exists a sequence $\{\eta_j\}\subset\Omega$ such that $\lim_{j\to\infty}\eta_j=\xi_0$ and $\lim_{j\to\infty} s_\Omega(\eta_j)=1$ or $\lim_{j\to\infty} h_\Omega(\eta_j)=0$, then $\partial\Omega$ is strongly pseudoconvex at ξ_0.*

As a consequence, we obtain the following well-known result (see [2, 9, 12]).

Corollary 1.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary. If ξ_0 is a boundary point of Ω of D'Angelo finite type such that the Levi form has corank at most 1 at ξ_0 and if $\lim\limits_{\Omega\ni z\to\xi_0} s_\Omega(z)=1$ or $\lim\limits_{\Omega\ni z\to\xi_0} h_\Omega(z)=0$, then $\partial\Omega$ is strongly pseudoconvex at ξ_0.*

Remark 1.3. It is known that the boundary point ξ_0 in our situation is h-extendible. Therefore, if $\{\eta_j\}$ converges $\left(\frac{1}{m_1},\ldots,\frac{1}{m_{n-1}}\right)$-nontangentially to ξ_0, then ξ_0 is strongly pseudoconvex as mentioned above. However, we emphasize here that $\{\eta_j\}\subset\Omega$ is an arbitrary sequence converging to ξ_0. For the proof of Theorem 1.1, as in [9] we also utilize the scaling method by Pinchuk to show that the complex unit ball $\mathbb{B}^n$ is biholomorphically equivalent to a model

$$M_P = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \mathrm{Re}(z_n) + P(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 < 0 \right\},$$

where P is a non-zero real-valued subharmonic polynomial of degree $2m$, where $2m$ is the D'Angelo type of $\partial\Omega$ at ξ_0. Then, this yields $2m = 2$ and hence our theorem follows.

The organization of the paper is described as follows: For the convenience of the reader, we exploit a constructive procedure of the scaling sequence in higher dimension in Section 2, based on the results in [3] and [7]. Then we investigate the normality of our scaling sequence which is crucial in determining the fact that $\mathbb{B}^n$ and M_P are biholomorphically equivalent. We finalize the proof of Theorem 1.1 in Section 3, after applying a technical lemma [1, Lemma 3.2] related to the biholomorphic equivalence among models.

2. The scaling sequence in higher dimension

This section is devoted to a proof of the normality of our scaling sequence. Then, by using this normality result the biholomorphic equivalence between M_P and the complex unit ball $\mathbb{B}^n$ will be shown.

First of all, we recall the following definition which will be used for the proof in this section (see [7] or [8]).

Definition. Let $\{\Omega_j\}_{j=1}^{\infty}$ be a sequence of open sets in $\mathbb{C}^n$ and Ω_0 be an open set of $\mathbb{C}^n$. The sequence $\{\Omega_j\}_{j=1}^{\infty}$ is said to converge to Ω_0 (written $\lim \Omega_j = \Omega_0$) if and only if

(i) For any compact set $K \subset \Omega_0$, there is a $j_0 = j_0(K)$ such that $j \geq j_0$ implies that $K \subset \Omega_i$, and

(ii) If K is a compact set which is contained in Ω_i for all sufficiently large j, then $K \subset \Omega_0$.

Throughout this section, the domain Ω and the boundary point $\xi_0 \in \partial\Omega$ are assumed to satisfy the hypothesis of Theorem 1.1. Let $2m$ be the D'Angelo type of $\partial\Omega$ at ξ_0. Without loss of generality, we may assume that $\xi_0 = 0 \in \mathbb{C}^n$ and the rank of Levi form at ξ_0 is exactly $n-2$. Let ρ be a smooth defining function for Ω. After a linear change of coordinates, we can find the coordinate functions $z_1, \dots, z_n$ defined on a neighborhood U_0 of ξ_0 such that

$$\begin{aligned}\rho(z) = \mathrm{Re}(z_n) &+ \sum_{\substack{j+k \leq 2m \\ j,k>0}} a_{j,k} z_1^j \bar{z}_1^k \\ &+ \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k\leq m \\ j,k>0}} \mathrm{Re}((b_{j,k}^{\alpha} z_1^j \bar{z}_1^k) z_\alpha) \\ &+ O(|z_n||z| + |z^*|^2|z| + |z^*|^2|z_1|^{m+1} + |z_1|^{2m+1}),\end{aligned}$$

where $z=(z_1,\ldots,z_n)$, $z^*=(0,z_2,\ldots,z_{n-1},0)$, and $a_{j,k}, b^\alpha_{j,k}$ $(2\le\alpha\le n-1)$ are $\mathcal{C}^\infty$-smooth functions in a small neighborhood of the origin in $\mathbb{C}^n$.

By [3, Proposition 2.2] (see also [7, Proposition 3.1]), for each point η in a small neighborhood of the origin, there exists a unique biholomorphism Φ_η of $\mathbb{C}^n$, $z=\Phi_\eta^{-1}(w)$, such that

$$\begin{aligned}\rho(\Phi_\eta^{-1}(w))-\rho(\eta) &= \mathrm{Re}(w_n)+\sum_{\substack{j+k\le 2m\\ j,k>0}} a_{j,k}(\eta) w_1^j\bar w_1^k\\ &\quad+\sum_{\alpha=2}^{n-1}|w_\alpha|^2+\sum_{\alpha=2}^{n-1}\sum_{\substack{j+k\le m\\ j,k>0}}\mathrm{Re}[(b^\alpha_{j,k}(\eta)w_1^j\bar w_1^k)w_\alpha]\\ &\quad+O(|w_n||w|+|w^*|^2|w|+|w^*|^2|w_1|^{m+1}+|w_1|^{2m+1}),\end{aligned}\tag{2.1}$$

where $w^*=(0,w_2,\ldots,w_{n-1},0)$.

Now let us denote by

$$\begin{aligned}A_l(\eta)&=\max\{|a_{j,k}(\eta)|:\ j+k=l\}\ (2\le l\le 2m),\\ B_{l'}(\eta)&=\max\{|b^\alpha_{j,k}(\eta)|:\ j+k=l',\ 2\le\alpha\le n-1\}\ (2\le l'\le m).\end{aligned}\tag{2.2}$$

For each $\delta>0$, we define $\tau(\eta,\delta)$ as follows.

$$\tau(\eta,\delta)=\min\left\{\big(\delta/A_l(\eta)\big)^{1/l},\ \big(\delta^{\frac12}/B_{l'}(\eta)\big)^{1/l'}\ :\ 2\le l\le 2m,\ 2\le l'\le m\right\}.$$

We note that the D'Angelo type of $\partial\Omega$ at ξ_0 equals $2m$ and the Levi form has rank at least $n-2$ at ξ_0. Therefore, $A_{2m}(\xi_0)\neq 0$ and hence there exists a sufficiently small neighborhood U of ξ_0 such that $|A_{2m}(\eta)|\ge c>0$ for all $\eta\in U$. This yields the relation

$$\delta^{1/2}\lesssim\tau(\eta,\delta)\lesssim\delta^{1/(2m)}\ (\eta\in U).\tag{2.3}$$

Let us define an anisotropic dilation Δ^ϵ_η by

$$\Delta^\epsilon_\eta(w_1,\ldots,w_n)=\left(\frac{w_1}{\tau_1(\eta,\epsilon)},\ldots,\frac{w_n}{\tau_n(\eta,\epsilon)}\right),$$

where $\tau_1(\eta,\epsilon)=\tau(\eta,\epsilon),\ \tau_k(\eta,\epsilon)=\sqrt{\epsilon}\ (2\le k\le n-1),\ \tau_n(\eta,\epsilon)=\epsilon$. For each $\eta\in\partial\Omega$, if we set $\rho^\epsilon_\eta(w)=\epsilon^{-1}\rho\circ\Phi_\eta^{-1}\circ(\Delta^\epsilon_\eta)^{-1}(w)$, then (2.1) and (2.3) imply that

$$\begin{aligned}\rho^\epsilon_\eta(w)&=\mathrm{Re}(w_n)+\sum_{\substack{j+k\le 2m\\ j,k>0}}a_{j,k}(\eta)\epsilon^{-1}\tau(\eta,\epsilon)^{j+k}w_1^j\bar w_1^k+\sum_{\alpha=2}^{n-1}|w_\alpha|^2\\ &\quad+\sum_{\alpha=2}^{n-1}\sum_{\substack{j+k\le m\\ j,k>0}}\mathrm{Re}(b^\alpha_{j,k}(\eta)\epsilon^{-1/2}\tau(\eta,\epsilon)^{j+k}w_1^j\bar w_1^k w_\alpha)+O(\tau(\eta,\epsilon)).\end{aligned}\tag{2.4}$$

In what follows, let us fix a sufficiently small neighborhood U_0 of ξ_0 and let $\{\eta_j\}\subset\Omega$ be a sequence converging to ξ_0. Further, we may also assume that

$\eta_j \in U_0^- := U_0 \cap \{\rho < 0\}$ for all j. For this sequence $\{\eta_j\}$, one associates with a sequence of points $\eta_j' = (\eta_{1j}, \ldots, \eta_{(n-1)j}, \eta_{nj} + \epsilon_j)$, $\epsilon_j > 0$, η_j' in the hypersurface $\{\rho = 0\}$. Let us consider the sequence of dilations $\Delta_{\eta_j'}^{\epsilon_j}$. Then $\Delta_{\eta_j'}^{\epsilon_j} \circ \Phi_{\eta_j'}(\eta_j) = (0, \ldots, 0, -1)$ and moreover it follows from (2.4) that $\Delta_{\eta_j'}^{\epsilon_j} \circ \Phi_{\eta_j'}(\{\rho = 0\})$ is defined by

$$\mathrm{Re}(w_n) + P_{\eta_j'}(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 + \sum_{\alpha=2}^{n-1} \mathrm{Re}(Q^\alpha_{\eta_j'}(w_1, \bar{w}_1) w_\alpha) + O(\tau(\eta_j', \epsilon_j)) = 0,$$

where

$$P_{\eta_j'}(w_1, \bar{w}_1) := \sum_{\substack{j+k \le 2m \\ j,k>0}} a_{j,k}(\eta_j') \epsilon_j^{-1} \tau(\eta_j', \epsilon_j)^{j+k} w_1^j \bar{w}_1^k,$$

$$Q^\alpha_{\eta_j'}(w_1, \bar{w}_1) := \sum_{\substack{j+k \le m \\ j,k>0}} b^\alpha_{j,k}(\eta_j') \epsilon_j^{-1/2} \tau(\eta_j', \epsilon_j)^{j+k} w_1^j \bar{w}_1^k.$$

Then one can deduce from (2.2) that the coefficients of $P_{\eta_j'}$ and $Q^\alpha_{\eta_j'}$ are bounded by one. Therefore, after taking a subsequence, we may assume that $\{P_{\eta_j'}\}$ converges uniformly on every compact subset of $\mathbb{C}$ to a polynomial $P(z_1, \bar{z}_1)$. Moreover, $\{Q^\alpha_{\eta_j'}\}$ $(2 \le \alpha \le n-1)$ converge uniformly on every compact subset of $\mathbb{C}$ to 0 by the following lemma.

Lemma 2.1 (see Lemma 2.4 in [3])**.** $|Q^\alpha_{\eta_j'}(w_1, \bar{w}_1)| \le \tau(\eta_j', \epsilon_j)^{\frac{1}{10}}$ *for all* $\alpha = 2, \ldots, n-1$ *and* $|w_1| \le 1$, *provided that* τ *is sufficiently small.*

Then, by Lemma 2.1, after taking a subsequence, one can deduce that $\Delta_{\eta_j'}^{\epsilon_j} \circ \Phi_{\eta_j'}(U_0^-)$ converges to the following model

$$M_P := \left\{ \hat{\rho} := \mathrm{Re}(w_n) + P(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\}, \tag{2.5}$$

where $P(w_1, \bar{w}_1)$ is a polynomial of degree $\le 2m$ without harmonic terms (cf. [7, p. 153]).

Remark 2.2. It is well-known that M_P is a smooth limit of the pseudoconvex domains $\Delta_{\eta_j'}^{\epsilon_j} \circ \Phi_{\eta_j'}(U_0^-)$. Then, M_P becomes to be a pseudoconvex domain. Therefore, the function $\hat{\rho}$ in (2.5) is plurisubharmonic, and thus P is a subharmonic polynomial whose Laplacian does not vanish identically.

Now let us recall the following theorem, which ensures the normality of the scaling sequence that will be given in the proof of Proposition 2.4.

Proposition 2.3 (see Theorem 3.11 in [7])**.** *Let* Ω *be a domain in* $\mathbb{C}^n$. *Suppose that* $\partial\Omega$ *is pseudoconvex, of D'Angelo finite type and is* $\mathcal{C}^\infty$*-smooth near a boundary point* $(0, \ldots, 0) \in \partial\Omega$. *Suppose that the Levi form has corank at most*

1 at $(0,\dots,0)$. Let D be a domain in $\mathbb{C}^k$ and $\varphi_j : D \to \Omega$ be a sequence of holomorphic mappings such that $\eta_j := \varphi_j(a)$ converges to $(0,\dots,0)$ for some point $a \in D$. Let $\{T_j\}$ be a sequence of automorphisms of $\mathbb{C}^n$ which associates with the sequence $\{\eta_j\}$ by the method of the dilation of coordinates (i.e., $T_j = \Delta^{\epsilon_j}_{\eta'_j} \circ \Phi_{\eta'_j}$). Then $\{T_j \circ \varphi_j\}$ is normal and its limits are holomorphic mappings from D to the domain of the form

$$M_P = \left\{ (w_1,\dots,w_n) \in \mathbb{C}^n : \mathrm{Re}(w_n) + P(w_1,\bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0 \right\},$$

where $P \in \mathcal{P}_{2m}$. Here $\mathcal{P}_{2m}$ denotes the space of real-valued polynomials on $\mathbb{C}$ of degree $\le 2m$ without harmonic terms.

Proposition 2.4. *M_P is biholomorphically equivalent to the complex unit ball $\mathbb{B}^n$.*

Proof. Let $\{\eta_j\} \subset \Omega$ be a sequence as in Theorem 1.1, that is, $\eta_j \to \xi_0 = 0$ as $j \to \infty$. We now split the proof into two following cases:

Case 1: $\lim_{j\to\infty} s_\Omega(\eta_j) = 1$. Let us set $\delta_j = 2(1 - s_\Omega(\eta_j))$ for all j. Then by our assumption, for each j, there exists an injective holomorphic map $f_j : \Omega \to \mathbb{B}^n$ such that $f_j(\eta_j) = (0',0)$ and $\mathbb{B}(0; 1-\delta_j) \subset f_j(\Omega)$. Then by [7, Proposition 2.2] and the hypothesis of Theorem 1.1, after choosing a suitable sequence of injective holomorphic mappings $f_j : \Omega \to \mathbb{B}^n$ whose existence is assured by the assumption on the squeezing function s_Ω, for each compact subset $K \Subset \mathbb{B}^n$ and each neighborhood U_0 of ξ_0, there exists an integer j_0 such that $f_j^{-1}(K) \subset \Omega \cap U_0$ for all $j \ge j_0$, i.e., $f_j(\Omega \cap U_0)$ converges to $\mathbb{B}^n$. Then it follows from Proposition 2.3 that the sequence $T_j \circ f_j^{-1} \colon f_j(\Omega\cap U_0) \to T_j(\Omega \cap U_0)$ is normal and its limits are holomorphic mappings from $\mathbb{B}^n$ to M_P. Moreover, by Montel's theorem the sequence $f_j \circ T_j^{-1} \colon T_j(\Omega \cap U_0) \to f_j(\Omega \cap U_0) \subset \mathbb{B}^n$ is also normal. We further note that the sequence $T_j \circ f_j^{-1}$ is not compactly divergent since $T_j \circ f_j^{-1}(0',0) = (0',-1)$. Then by [7, Proposition 2.1], after taking some subsequence of $\{T_j \circ f_j^{-1}\}$, we may assume that such a subsequence converges uniformly on every compact subset of $\mathbb{B}^n$ to a biholomorphism F from $\mathbb{B}^n$ onto M_P, as desired.

Case 2: $\lim_{j\to\infty} h_\Omega(\eta_j) = 0$.

Since the point ξ_0 is a local peak point (cf. [16]), by [11, Proposition 3.4], one has $\lim_{j\to\infty} h_{U_0\cap\Omega}(\eta_j) = 0$. Moreover, by our assumption, there exist a sequence of positive real numbers $R_j \to +\infty$ and a sequence of biholomorphic embeddings $g_j : \mathbb{B}^n \to U_0 \cap \Omega$ such that $g_j(0) = \eta_j$ and $B_{U_0\cap\Omega}(\eta_j, R_j) \subset g_j(\mathbb{B}^n)$. Then it follows from Proposition 2.3 that the sequence $T_j \circ g_j : \mathbb{B}^n \to T_j(\Omega \cap U_0)$ is normal and its limits are holomorphic mappings from $\mathbb{B}^n$ to M_P. Moreover, by Montel's theorem the sequence $g_j^{-1} \circ T_j^{-1} \colon T_j(\Omega \cap U_0) \to g_j^{-1}(\Omega \cap U_0) \subset \mathbb{B}^n$ is also normal. We also note that the sequence $T_j \circ g_j$ is not compactly divergent since $T_j \circ g_j(0',0) = (0',-1)$. Then by [7, Proposition 2.1], after taking some subsequence of $\{T_j \circ g_j\}$, we may assume that such a subsequence converges uniformly on every compact subset of $\mathbb{B}^n$ to a biholomorphism G from $\mathbb{B}^n$ onto M_P, as desired.

Altogether, the proof is now complete. □

Remark 2.5. As in [9], the sequence $\{\eta_j\}$ can be chosen so that η_j converges to ξ_0 along the direction normal to the boundary. Therefore, $P(z_1, \bar{z}_1)$ must be homogeneous subharmonic polynomial of degree $2m$. However, by using the argument as in [1, Sections 3 and 4] (see also [7, Section 4]), in our situation, P is also a homogeneous subharmonic polynomial of degree $2m$ without harmonic terms. Moreover, one sees from Remark 2.2 in particular that $\Delta P \not\equiv 0$.

3. Proof of the main theorem

We shall complete the proof of Theorem 1.1 as our main result in this section. Recall from Remark 2.5 that

$$M_P = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon \mathrm{Re}(z_n) + P(z_1, \bar{z}_1) + |z_2|^2 + \cdots + |z_{n-1}|^2 < 0\right\},$$

where P is a non-zero real-valued subharmonic polynomial of degree $2m$. We define a space $\mathcal{H}_{2m}$ by setting

$$\mathcal{H}_{2m} := \{H \in \mathcal{P}_{2m} : \deg H = 2m,\ H \text{ is homogeneous and subharmonic}\},$$

where the space $\mathcal{P}_{2m}$ is given as in Proposition 2.3.

With these notations, we prepare one more lemma in order to prove Theorem 1.1.

Lemma 3.1 (see Lemma 3.2 in [1])**.** *Let $Q \in \mathcal{P}_{2m}$ and $H \in \mathcal{H}_{2m}$. If M_Q and M_H are biholomorphically equivalent, then the homogeneous part of higher degree in Q is equal to $\lambda H(e^{i\nu} z)$ for some $\lambda > 0$ and $\nu \in [0, 2\pi]$.*

We note first that the complex unit ball $\mathbb{B}^n$ is biholomorphic to the Siegel half-space $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon \mathrm{Re}(z_n) + |z_1|^2 + |z_2|^2 + \cdots + |z_{n-1}|^2 < 0\}$. In addition, Proposition 2.4 and Lemma 3.1 imply that $P(z_1, \bar{z}_1) = c|z_1|^2$ for some $c > 0$, that is, $m = 1$. Combining these two facts, we conclude that $\partial\Omega$ is strongly pseudoconvex at ξ_0 (ξ_0 is of the D'Angelo type 2), which ends the proof of Theorem 1.1.

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