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GRADED BETTI NUMBERS OF GOOD FILTRATIONS

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Abstract. The asymptotic behavior of graded Betti numbers of powers of homogeneous ideals in a polynomial ring over a field has recently been reviewed. We extend quasi-polynomial behavior of graded Betti numbers of powers of homogenous ideals to $\mathbb{Z}$-graded algebra over Noetherian local ring. Furthermore our main result treats the Betti table of filtrations which is finite or integral over the Rees algebra.

1. Introduction

A significant result on the Castelnuovo-Mumford regularity of powers of homogeneous ideal I in a polynomial ring S shows that the maximal degree of the i-th syzygy of I^t is a linear function of t for t large enough. This result have various generalizations including a case I is a homogeneous ideal in standard graded algebras over a Noetherian ring. Bagheri, Chardin and Hà promote the content in [1] through investigating of the eventual behavior of all the minimal generators of the i-th syzygy module of MI^t where M is a finitely generated Z-graded S-module. Their approach formed from the fact that the module $\oplus_t \operatorname{Tor}_i^S(MI^t, k)$ for a homogeneous ideal I in graded ring S, has a structure of a finitely generated graded module over a non-standard graded polynomial ring over k, from which one can conclude the behavior of $\operatorname{Tor}_i^S(I^t, k)$ when t varies.

In the case that I is generated by the forms of the same degree, an interesting result in [1, Theorem 3.1] shows that the Betti tables of modules MI^t could be encoded by a variant of Hilbert-Serre polynomials. This is a refinement of asymptotic stability of total Betti numbers proved by Kodiyalam [9]. The question that track the asymptotic behavior of graded Betti numbers in the case that ideal I is generated by forms of degrees (not necessarily equal) $d_1, \ldots, d_r$ was the core of our previous work [2]. The fact that $\oplus_t \operatorname{Tor}_i^S(I^t, k)$ is a finitely generated over a multigraded polynomial ring $B = K[T_1, \ldots, T_r]$ endowed $(d_i, 1)$ to each variables T_i for $1 \leq i \leq r$, guarantees a bigraded minimal free B-resolution with free module $\oplus B(-a, -b)^{\beta(i,(a,b))}$ at the homological

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degree i. As a consequence, we proved in [2] that $\mathbb{Z}^2$ can be splitted into a finite number of regions (see Figure 1) such that each region corresponds to quasi-polynomial behavior of the Betti numbers of homogeneous ideals in a polynomial ring over a field. Accordingly, the graded Betti tables of power I^n could be encoded by the a set of polynomials for n large enough. More generally on the stabilization of graded Betti numbers for a collection of graded ideals we use the fact that the module

$$B_i := \oplus_{t_1,\dots,t_s} \operatorname{Tor}_i^R(MI_1^{t_1}\cdots I_s^{t_s}, k)$$

is a finitely generated $(\mathbb{Z}^p \times \mathbb{Z}^s)$-graded ring over $k[T_{i,j}]$ setting $\deg(T_{i,j}) = (\deg(f_{i,j}), e_i)$ with e_i the i-th canonical generator of $\mathbb{Z}^s$ and for fixed i the $f_{i,j}$ form a set of minimal generators of I_i. We proved in [2] that the chamber associated to a positive $\mathbb{Z}^d$-grading of $R := k[T_{i,j}]$ splitted $\mathbb{Z}^d$ into a finite maximal cells where the eventual behaviour of graded Betti numbers is encoded by the a set of polynomials. Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a commutative Noetherian local ring A with residue field k and $I \subseteq S$ be a homogeneous ideal. There is a ring homomorphism $A \to k$ bringing the spectral sequence $E^2_{p,q} = \operatorname{Tor}_p(\operatorname{Tor}_q^S(I^t, A)_\nu, k) \Rightarrow \operatorname{Tor}_{p+q}^S(I^t, k)_\nu$ in which one can write the Hilbert function of $\operatorname{Tor}_i^S(I^t, k)_\nu$ in terms of Hilbert function of $\operatorname{Tor}_p(\operatorname{Tor}_q^S(I^t, A)_\nu, k)$ for $i = p + q$. Bagheri, Chardin and Hà investigating the support of the module $\operatorname{Tor}_i^S(I^t, A)$ [1, Theorem 4.6] as the t get large. In Theorem 3.1 we generalize the quasi-polynomial behavior of the Betti numbers [2, Theorem 4.6] to the case of homogeneous ideals in G-graded algebra over Noetherian local ring. By using the concept of vector partition function we present an effective method to find such a set of polynomials. Roughly speaking, a vector partition function is the number of ways where a vector decompose as a linear combination with nonnegative integral coefficients of a fixed set of vectors. On the other hand, let e_i be the i-th standard basis of the space $\mathbb{R}^r$ for $1 \leqslant i \leqslant r$ and suppose that a linear map $f : \mathbb{R}^n \to \mathbb{R}^d$ is defined by $f(e_i) = v_i$. The Rational convex polytope can be written as

$$P(b) := f^{-1}(b) \cap \mathbb{R}^r_{\geqslant 0} = \{x \in \mathbb{R}^r \mid Ax = a; x \geqslant 0\},$$

where A is the matrix of f. If b is in the interior of $\operatorname{Pos}(A) := \{\sum_{i=1}^n \lambda_i a_i \in \mathbb{R}^d \mid \lambda_i \geq 0, 1 \leq i \leq n\}$, the polytope $P(b)$ has dimension $n - d$. One can see that evaluating the vector partition is equivalent to computing the number of integral points in a rational convex polytope. Accordingly in order to compute the Hilbert function $HF(S; b)$ of a polynomial rings S over an infinite field K of characteristic 0 weighted by a set of vectors in $\mathbb{Z}^r$ we use an algorithm of enumeration the integral points in the polytope $P(b)$.

A classical result of G. Pick (1899) for a two-dimensional polygone, states that if $P \subset \mathbb{R}^2$ is an integer polygon, then the number of integer points inside P is $|P \cap \mathbb{Z}^2| = area(P) + \frac{|\partial P \cap \mathbb{Z}^2|}{2} + 1$. One of important generalizations of the Pick's formula is the theorem of Ehrhart. In the general, for any rational

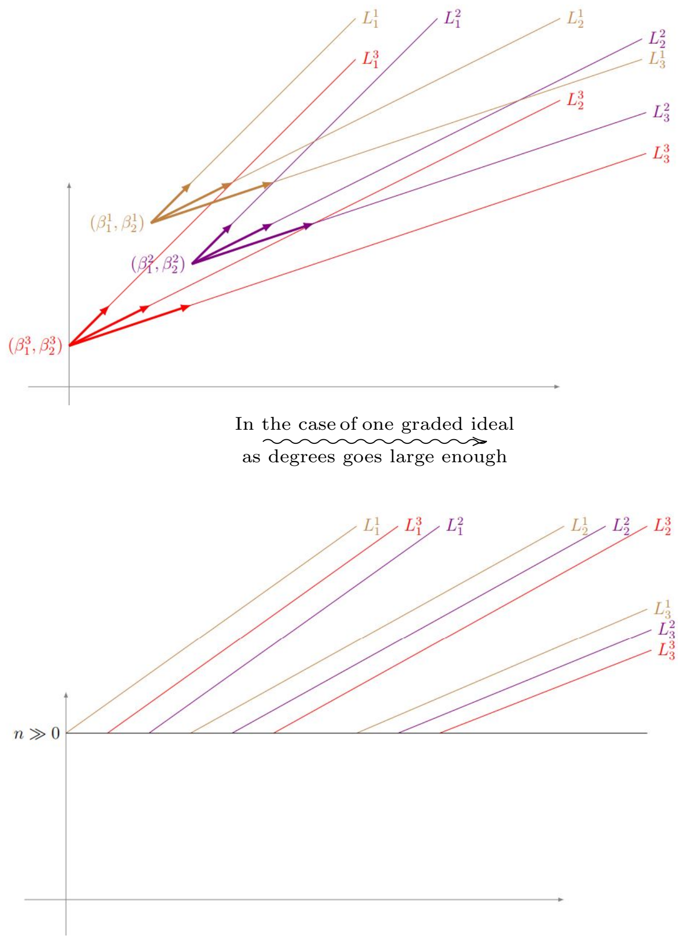

FIGURE 1.

polyhedron $P \subset \mathbb{R}^n$ we consider following generating function:

$$f(P, \mathbf{x}) = \sum_{m \in P \cap \mathbb{Z}^n} \mathbf{x}^m,$$

where $m = (m_1, \ldots, m_n)$ and $\mathbf{x}^m = x_1^{m_1} \cdots x_n^{m_n}$. By Brion's theorem [5], the generating function of the polytope P is equal to the sum of the generating functions of its vertex cones. More precisely,

$$f(P; \mathbf{x}) = \sum_{m \in P \cap \mathbb{Z}^n} \mathbf{x}^m = \sum_{v \in \Omega(P)} f(K; \mathbf{v}),$$

where $\Omega(P)$ is the set of vertices of P. In order to find the generation function of arbitrary pointed cones, Stanley [14] gives a triangulation of a rational cone into simplicial cones. Barvinok proved [3] that every rational polyhedral cone can be triangulated into unimodular cones. The method of Barvinok enabling us to calculate the generating function of the above polytope depending on b, before that we should mention that the polytope $P(b)$ associated to the matrix A is not full dimensional so to use the Barvinok method we need to transform $P(b)$ to polytope Q which is full dimensional and the integer points of Q are in one-to-one correspondence to the integer points of $P(b)$. In Example 3.2 we show the implementation of above method in order to find out an effective method for our objective.

Our interest here is to investigate the behavior of graded Betti numbers of integral closures and Rattliff-Rush closure of powers of ideals by using the notion of I-good filtrations, which it generalize our previous results in [2]. This is a series $\mathcal{J} = \{\mathcal{J}_n\}_{n\geq 0}$ of ideals such that $\oplus_{n\geqslant 0}\mathcal{J}_n$ is a finite module over the Rees ring. Our result on graded Betti numbers of I-good filtrations takes the following form:

Theorem. *Let $S = A\,[x_1, \ldots, x_n]$ be a graded algebra over a Noetherian local ring $(A, m) \subset S_0$. Let $\mathcal{J} = \{\mathcal{J}_n\}_{n\geq 0}$ be an I-good filtration of ideals $\mathcal{J}_n$ of S and $\mathcal{J}_1 = (f_1, f_2, \ldots, f_r)$ with $\deg f_i = d_i$ be a $\mathbb{Z}$-homogenous ideal in S, and let $R = S[T_1, \ldots, T_r]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg_{\mathbb{Z}}(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$. Then,*

- *For all i, $\operatorname{Tor}_i^R(\mathcal{R}_{\mathcal{J}}, k)$ is a finitely generated $k[T_1, \ldots, T_r]$-module.*
- *There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \ldots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \ldots, m$ and $j \in 1, \ldots, D$, such that, for $t \geq t_0$,*
  - (i) *$L_i(t) < L_j(t) \;\Leftrightarrow\; i < j$,*
  - (ii) *if $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\operatorname{Tor}_i^S(\mathcal{J}_t, k)_\mu = 0$, and*
  - (iii) *if $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \mod (D)$, then*

$$\dim_k \operatorname{Tor}_i^S(\mathcal{J}_t, k)_\mu = Q_{i,j}(\mu, t).$$

It's worth mentioning that even in a simple situation, namely when I is a complete intersection ideal quite many polynomials are involved to give the asymptotic behavior of the graded Betti numbers of powers of I.

2. Preliminaries

In this section, we give a brief recall on necessary notations and terminology used in the article.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Let $\mathbb{G}$ be an abelian group. A $\mathbb{G}$-grading of S is a morphism $\deg : \mathbb{Z}^n \longrightarrow \mathbb{G}$ and if G is torsion-free and $S_0 = k$, the grading is positive. Criterions for positivity are given in

[10, Theorem 8.6]. When $\mathbb{G} = \mathbb{Z}^d$ and the grading is positive, (generalized) Laurent series are associated to finitely generated graded modules:

Definition 2.1. The Hilbert function of a finitely generated module M over a positively graded polynomial ring is the map:

$$\begin{array}{rccc} HF(M;-): & \mathbb{Z}^d & \longrightarrow & \mathbb{N} \\ & \mu & \longmapsto & \dim_k(M_\mu). \end{array}$$

The Hilbert series of M is the Laurent series

$$H(M;t) = \sum_{\mu \in \mathbb{Z}^d} \dim_k(M_\mu) t^\mu.$$

Let M be a finitely generated $\mathbb{Z}^d$-graded S-module. It admits a finite minimal graded free S-resolution:

$$\mathbb{F}_\bullet : 0 \to F_u \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Writing

$$F_i = \oplus_\mu S(-\mu)^{\beta_{i,\mu}(M)},$$

the minimality shows that $\beta_{i,\mu}(M) = \dim_k \left(\operatorname{Tor}_i^S(M,k)\right)_\mu$, as the maps of $\mathbb{F}_\bullet \otimes_S k$ are zero. We also recall that the support of a $\mathbb{Z}^d$-graded module N is

$$\operatorname{Supp}_{\mathbb{Z}^d}(N) := \{\mu \in \mathbb{Z}^d \mid N_\mu \neq 0\}.$$

2.1. Partition function

Let e_i be the standard basis of the space $\mathbb{R}^r$ for $1 \leqslant i \leqslant r$. Let f be a linear map $f : \mathbb{R}^r \to \mathbb{R}^d$ defined by $f(e_i) = v_i$ and denote by V the linear span of $[v_1, \ldots, v_r]$. For $a \in V$ consider the following convex polytope:

$$P(a) := f^{-1}(a) \cap \mathbb{R}^r_{\geqslant 0} = \left\{ x = (x_1, \ldots, x_r) \in \mathbb{R}^r \mid \sum_{i=1}^r x_i v_i = a; x \geqslant 0 \right\}.$$

Definition 2.2. The function $\varphi : \mathbb{N}^d \to \mathbb{N}$ defined by $\varphi_A(a) = \sharp(f^{-1}(a) \cap \mathbb{Z}^r_{\geqslant 0})$ is called vector partition function corresponding to the matrix $A = (v_1, \ldots, v_r)$.

For more details about the vector partition functions in particular the definition of chambers, chamber complex and quasipolynomials we use the terminology of [6, 15].

Now, we recall the vector partition function theorem:

Theorem 2.3 (See [15, Theorem 1])**.** *For each chamber C of maximal dimension in the chamber complex of A, there exist a polynomial P of degree $n-d$, a collection of polynomials Q_σ and functions $\Omega_\sigma : G_\sigma \setminus \{0\} \to \mathbb{Q}$ indexed by non-trivial $\sigma \in \Delta(C)$ such that, if $u \in \mathbb{N}A \cap \overline{C}$,*

$$\varphi_A(u) = P(u) + \sum \left\{ \Omega_\sigma([u]_\sigma).Q_\sigma(u) : \sigma \in \Delta(C), [u]_\sigma \neq 0 \right\},$$

where $[u]_\sigma$ denotes the image of u in G_σ. Furthermore, $\deg(Q_\sigma) = \#\sigma - d$.

Corollary 2.4 ([2]). *For each chamber C of maximal dimension in the chamber complex of A, there exists a collection of polynomials Q_τ for $\tau \in \mathbb{Z}^d/\Lambda$ such that*

$$\varphi_A(u) = Q_\tau(u), \text{ if } u \in \mathbb{N}A \cap \overline{C} \text{ and } u \in \tau + \Lambda_C,$$

where $\Lambda_C = \cap_{\sigma\in\Delta(C)}\Lambda_\sigma$

Notice that setting Λ for the intersection of the lattices Λ_σ with σ maximal, the class of u mod Λ determines the class of u mod Λ_C, hence the corollary holds with Λ in place of Λ_C.

3. Structure of Tor module of Rees algebra

Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a commutative Noetherian local ring $S_0 = (A, m)$ with residue field k and set $R = S[T_1, \dots, T_r]$ and $B = k[T_1, \dots, T_r]$. We set $\deg(T_i) = (d_i, 1)$ and extended the grading from S to R by setting $\deg(x_i) = (\deg_{\mathbb{Z}}(x_i), 0)$. Let M be a finitely generated graded S-module and I be a graded S-ideal generated in degrees $d_1, \dots, d_r$. In this section we use the important fact which it provides a B-structure on $\oplus_t \operatorname{Tor}_i^S(MI^t, k)$ that were already at the center of the work [1] .

Theorem 3.1. *Let $S = A[x_1, \dots, x_n]$ be a $\mathbb{Z}$-graded algebra over Noetherian local ring (A, m, k). Let $I = (f_1, f_2, \dots, f_r)$ be a homogeneous ideal in S with $\deg f_i = d_i$, and let $R = S[T_1, \dots, T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg_{\mathbb{Z}}(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$. Then there exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \dots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \dots, m$ and $j \in 1, \dots, D$, such that, for $t \geq t_0$,*

(i) $L_i(t) < L_j(t) \;\Leftrightarrow\; i < j$,
(ii) *if $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\operatorname{Tor}_i^S(I^t, k)_\mu = 0$, and*
(iii) *if $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \mod (D)$, then*

$$\dim_k \operatorname{Tor}_i^S(I^t, k)_\mu = Q_{i,j}(\mu, t).$$

Proof. The natural anto map $R \to \mathcal{R}_I := \bigoplus_{t\geqslant 0} I^t$ sending T_i to f_i makes $\mathcal{R}_I = \bigoplus_{t\geqslant 0} I^t$ a finitely generated graded R-module. It was shown in [1] that $\operatorname{Tor}_i^S(I^t, A)$ is a finitely generated $k[T_1, \dots, T_r]$-module. Let $f : S \to A$ be the canonical map then there is a graded spectral sequence with second term $E^2_{p,q} = \operatorname{Tor}_p^A(\operatorname{Tor}_q^S(I^t, A)_\nu, k) \Rightarrow \operatorname{Tor}_{p+q}^S(I^t, k)_\nu$ therefore $\operatorname{Tor}_i^S(I^t, k)$ is a finitely generated $k[T_1, \dots, T_r]$-module. Then the result follows from [2, Proposition 4.5]. $\square$

Example 3.2. Let $S = A[x_1, \dots, x_n]$ be a graded algebra over a commutative Noetherian local ring $S_0 = (A, m)$. Let $I \subseteq S$ be a complete intersection ideal of three forms f_1, f_2, f_3, f_4 of degrees $3, 5, 7, 9$ respectively. Let $R = S[T_1, T_2, T_3, T_4]$ be a $\mathbb{Z}\times\mathbb{Z}$-graded polynomial extension of S with $\deg x_i = (1, 0)$

and $\deg(T_1) = (3,1)$, $\deg(T_2) = (5,1)$, $\deg(T_3) = (8,1)$, $\deg(T_1) = (9,1)$. By Theorem 3.1 $\mathrm{Tor}_i^R(\mathcal{R}_I, k)$ is a finitely generated $B = k[T_1, \dots, T_4]$-module with assigned weight. Let $A = \left(\begin{smallmatrix} 3 & 5 & 8 & 9 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)$ be the matrix of degrees of B then by [2, Proposition 3.1] the Hilbert function of B at degree (ν, n) equals to enumeration lattice points of the following convex polytope

$$P(\nu, n) = \{x \in \mathbb{R}^r \mid Ax = (\nu, n); x \geqslant 0\}.$$

To take advantage of the Barvinok method of enumeration lattice points we need a transformation of polytope with one-to-one correspondence to the lattice points by the following procedure explained in [7].

(1) Let $P = \{x \in \mathbb{R}^n \mid Ax = a, Bx \leqslant b\}$ be a polytope related to full row-rank $d \times n$ matrix A.
(2) Find the generators $\{g_1, \dots, g_{n-d}\}$ of the integer null-space of A.
(3) Find integer solution x_0 to $Ax = a$.
(4) Substituting the general integer solution $x = x_0 + \sum_{i=1}^{n-d} \beta_i g_i$ into the inequalities $Bx \leqslant b$.
(5) By substitution of (4) we arrive at a new system $C\beta \leqslant c$ which defines the new polytope $Q = \{\beta \in \mathbb{R}^{n-d} \mid C\beta \leqslant c\}$.

In order to find out the integer null-space of A we first calculate the Hermite Normal form (HNF) of A which is

$$H := HFN(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 1 & 4 & 3 & 2 \\ -2 & -4 & -5 & -3 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here U is a unimodular matrix such that $H = AU$. Then the columns of $U_1 = \left(\begin{smallmatrix} 1 & 4 \\ -2 & -4 \\ 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ gives the generators of the integer null-space of A. Hence by the above procedure the polytope $Q(\nu, n)$ defined by the solutions of following system of linear inequalities

$$\begin{cases} -4\lambda_2 - 4n - \nu \leq \lambda_1, \\ \lambda_1 + 2\lambda_2 \leq -\nu - 2n, \\ \lambda_1 + \lambda_2 \geq -\nu - n, \end{cases}$$

4. Structure of Tor module of Hilbert filtrations

To study blowup algebras, Northcott and Rees defined the notion of reduction of an ideal I in a commutative ring R. An ideal $J \subseteq I$ is a reduction of I if there exists r such that $JI^r = I^{r+1}$ (equivalently this hold for $r \gg 0$). An important fact about reduction of ideals is that this property is equivalent to the fact that

$$\mathcal{R}_J = \oplus_n J^n \to \mathcal{R}_I = \oplus_n I^n$$

is a finite morphism. Okon and Ratliff in [11] extended the above notion of reduction to the case of filtrations by setting the following definition:

Definition 4.1. Let R be a ring, I an R-ideal and $\mathcal{J} = \{\mathcal{J}_n\}_{n\geq 0}$ and $\mathcal{I} = \{\mathcal{I}_n\}_{n\geq 0}$ two filtrations on R:

(1) $\mathcal{J} \leq \mathcal{I}$ if $\mathcal{J}_n \subseteq \mathcal{I}_n$ for all $n \geq 0$.
(2) $\mathcal{J}$ is a reduction of $\mathcal{I}$ if $\mathcal{J} \leq \mathcal{I}$ and there exists a positive integer d such that $\mathcal{I}_n = \sum_{i=0}^{d} \mathcal{J}_{n-i}\mathcal{I}_i$ for all $n \geq 1$.
(3) $\mathcal{J}$ is an I-good filtration if $I\mathcal{J}_i \subseteq \mathcal{J}_{i+1}$ for all $i \geq 0$ and $\mathcal{J}_{n+1} = I\mathcal{J}_n$ for all $n \gg 0$, $\mathcal{J}$ is called a good filtration if it is an I-good filtration for some ideal I of R.

Remark 4.2. $\mathcal{J}$ is a good filtration if and only if $\mathcal{J}$ is a $\mathcal{J}_1$-good filtration.

Opposite to the ideal case, minimal reductions of a filtration does not exist in general. But Hoa and Zarzuela showed in [8] the existence of a minimal reduction for I-good filtrations.

If $\mathcal{J} = \{\mathcal{J}_n\}_{n\geq 0}$ is an I-good filtration on R, then $\mathcal{R}_{\mathcal{J}} := \oplus_{n\geqslant 0}\mathcal{J}_n$ is a finitely generated $\mathcal{R}_I$-module [4, Theorem III.3.1.1]. This is why we are interested about I-good filtration to generalize the previous results. The following theorem explain the structure of Tor module of I-good filtrations:

Theorem 4.3. *Let $S = A\,[x_1, \ldots, x_n]$ be a graded algebra over a Noetherian local ring $(A, m, k) \subset S_0$. Let $\mathcal{J} = \{\mathcal{J}_n\}_{n\geq 0}$ be an I-good filtration of $\mathbb{Z}$-homogeneous ideals in S, and $\mathcal{J}_1 = (f_1, f_2, \ldots, f_r)$ with $\deg f_i = d_i$. Let $R = S[T_1, \ldots, T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg_{\mathbb{Z}}(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$.*

(1) *Then for all i: $\mathrm{Tor}_i^R(\mathcal{R}_{\mathcal{J}}, k)$ is a finitely generated $k[T_1, \ldots, T_r]$-module.*
(2) *There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0, \ldots, m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1, \ldots, m$ and $j \in 1, \ldots, D$, such that, for $t \geq t_0$,*
   (i) *$L_i(t) < L_j(t) \;\Leftrightarrow\; i < j$,*
   (ii) *if $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\mathrm{Tor}_i^S(\varphi(t), k)_\mu = 0$, and*
   (iii) *if $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \mod (D)$, then*

$$\dim_k \mathrm{Tor}_i^S(\mathcal{J}_t, k)_\mu = Q_{i,j}(\mu, t).$$

Proof. Recall that $\mathcal{R}_{\mathcal{J}}$ is a finite $\mathcal{R}_I$-module, hence a finitely generated $\mathbb{Z}^2$-graded R-module. Let $F_\bullet$ be a $\mathbb{Z} \times \mathbb{Z}$-graded minimal free resolution of $\mathcal{R}_{\mathcal{J}}$ over R. Each $F_i = \oplus_{\mu,t} R(-\mu, -n)^{\beta^i_{\mu,n}}$ is of finite rank due to the Noetherianity of A. The graded stand $F^t_\bullet := (F_\bullet)_{*,t}$ is a $\mathbb{Z}$-graded free resolution of $\mathcal{J}_t$ over

$S = R_{(*,0)}$. Meanwhile there is a S-graded isomorphism

$$R(-\mu,-n)_{(*,t)} \simeq R(-\mu)_{(*,t-n)}.$$

Thus $\operatorname{Tor}_i^S(\mathcal{J}_t, A) = H_i(F_\bullet^t \otimes_S A)$ and by the above isomorphism $\operatorname{Tor}_i^S(\mathcal{J}_t, A)$ is subquotient of $A(-\mu)^{\beta^i_{\mu,n}} \otimes_A (A[T_1,\dots,T_r])_{t-n}$ Since the A is Noetherian it follows that $\operatorname{Tor}_i^S(\mathcal{J}_t, A)$ is a finitely generated $A[T_1,\dots,T_r]$-module. With a similar approach one has $\operatorname{Tor}_i^S(\mathcal{J}_t, k) = H_i(F_\bullet^t \otimes_S k)$. Moreover, taking homology respects the graded structure and therefore,

$$H_i(F_\bullet^t \otimes_S k) = H_i(F_\bullet \otimes_R R/\mathfrak{m} + \mathfrak{n}R)_{(*,t)},$$

where $\mathfrak{n} = (x_1,\dots,x_n)$ is the homogeneous irrelevant ideal of S. It follows that $\operatorname{Tor}_j^R(\mathcal{R}_\mathcal{J}, k)$ is a finitely generated graded $k[T_1,\dots,T_r]$-module. To proof the second fact let $E := \{d_1,\dots,d_r\}$ with $d_1 < \cdots < d_r$ be a set of positive integers. For ℓ from 1 up to $r-1$, let

$$\Omega_\ell := \left\{ a\binom{d_\ell}{1} + b\binom{d_{\ell+1}}{1},\ (a,b) \in \mathbb{R}^2_{\geq 0} \right\}$$

be the closed cone spanned by $\binom{d_\ell}{1}$ and $\binom{d_{\ell+1}}{1}$. For integers $i \neq j$, let $\Lambda_{i,j}$ be the lattice spanned by $\binom{d_i}{1}$ and $\binom{d_j}{1}$ and $\Lambda_\ell := \bigcap_{i\leq \ell<j} \Lambda_{i,j}$. Also we set $\Lambda := \bigcap_{i<j} \Lambda_{i,j}$ with $\Delta = \det(\Lambda)$. It follows from Theorem 2.3 that $\dim_k B_{\mu,t} = 0$ if $(\mu,t) \notin \Omega := \bigcup_\ell \Omega_\ell$. Part (ii) and (iii) then follow from [2, Lemma 4.4 and Proposition 4.5]. □

This in particular applies to the following situations:

- I is a graded ideal of S and S is an analytically unramified ring without nilpotent elements. Then the integral closure filtration $\mathcal{J} = \{\overline{I^n}\}$ is I-good filtration [13].
- I is a graded ideal of S, then the Rattliff-Rush closure filtration $\mathcal{J} = \{\widetilde{I^n}\}$ is an I-good filtration [12].

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