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ON STRONGLY 1-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with $1 \neq 0$. In this paper, we introduce a subclass of the class of 1-absorbing primary ideals called the class of strongly 1-absorbing primary ideals. A proper ideal I of R is called strongly 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{0}$. Firstly, we investigate basic properties of strongly 1-absorbing primary ideals. Hence, we use strongly 1-absorbing primary ideals to characterize rings with exactly one prime ideal (the UN-rings) and local rings with exactly one non maximal prime ideal. Many other results are given to disclose the relations between this new concept and others that already exist. Namely, the prime ideals, the primary ideals and the 1-absorbing primary ideals. In the end of this paper, we give an idea about some strongly 1-absorbing primary ideals of the quotient rings, the polynomial rings, and the power series rings.

1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. An ideal I of a ring R is said to be proper if $I \neq R$. Let R be a ring and I be an ideal of R. Then, $\sqrt{0}$ denotes the nilradical of R and $\sqrt{I}$ denotes the radical of I.

Recall that an ideal $\mathfrak{q}$ of a ring R is said to be primary if, whenever $a, b \in R$ with $ab \in \mathfrak{q}$, then $a \in \mathfrak{q}$ or $b \in \sqrt{\mathfrak{q}}$. In this case $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is a prime ideal of R, and $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary. In [2], Badawi, Tekir and Yetkin introduced a generalization of primary ideals called 2-absorbing primary ideals. A proper ideal I of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. The concept of 2-absorbing primary ideals is generalized in many ways (see for example [1, 3]). Recently, Badawi and Yetkin [4] consider an new class of ideals called the class of 1-absorbing primary ideals. A proper ideal I of R is called a 1-absorbing primary ideal of R if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. They showed that

$$\{\text{primary ideals}\} \subseteq \{\text{1-absorbing primary ideals}\} \subseteq \{\text{2-absorbing primary ideals}\},$$

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and these inclusion may be strict.

In this paper, we study a subclass of the class of 1-absorbing primary ideals that does not necessarily contain all primary ideals. Let R be a ring. A proper ideal I of R is called strongly 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{0}$. Firstly, we investigate basic properties of strongly 1-absorbing primary ideals. Hence, we use strongly 1-absorbing primary ideals to characterize rings with exactly one prime ideal (the UN-rings) and local rings with exactly one non maximal prime ideal. Many other results are given to disclose the relations between this new concept and others that already exist. Namely, the prime ideals, the primary ideals and the 1-absorbing primary ideals. In the end of this paper, we give an idea about some strongly 1-absorbing primary ideals of the quotient rings, the polynomial rings, and the power series rings.

2. Strongly 1-absorbing primary ideals of commutative rings

Definition. A proper ideal I of R is called strongly 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{0}$.

Clearly, every strongly 1-absorbing primary ideal is a 1-absorbing primary ideal. However, the next example shows that these are different concepts. The same example shows that a prime ideal (in particular a primary ideal) is not necessarily a strongly 1-absorbing primary ideal.

Example 2.1. Let $R = \mathbb{Z}$ and $I = 3\mathbb{Z}$. It is clear that I is prime, and then 1-absorbing primary. However, I is not a strongly 1-absorbing primary. Indeed, $2 \cdot 2 \cdot 3 \in I$ but neither $2 \cdot 2 \in I$ nor $3 \in \sqrt{0} = \{0\}$.

Our first theorem gives a characterization of strongly 1-absorbing primary ideals.

Theorem 2.2. *Let I be a proper ideal of R. Then, I is strongly 1-absorbing primary if and only if*

(1) *I is 1-absorbing primary and $\sqrt{I} = \sqrt{0}$, or*
(2) *R is local with maximal ideal $\mathfrak{m} = \sqrt{I}$ and $\mathfrak{m}^2 \subseteq I$.*

Proof. ($\Rightarrow$) It is obvious that every strongly 1-absorbing primary ideal is 1-absorbing primary.

Suppose that $\sqrt{I} \neq \sqrt{0}$. Assume that there exist nonunit elements $x, y \in R$ such that $xy \notin I$. For each $a \in I$, we have $xya \in I$, and then $a \in \sqrt{0}$ since I is a strongly 1-absorbing primary. Hence, $I \subseteq \sqrt{0}$. Then, $\sqrt{I} = \sqrt{0}$, a contradiction. Hence, $xy \in I$ for each nonunit elements $x, y \in R$. Let $\mathfrak{m}$ be a maximal ideal of R. We have $\mathfrak{m}^2 \subseteq I$, and then $\mathfrak{m} = \sqrt{\mathfrak{m}^2} \subseteq \sqrt{I}$. Thus, $\mathfrak{m} = \sqrt{I}$ for each maximal ideal $\mathfrak{m}$ of R. Hence, we conclude that R is local with maximal ideal $\mathfrak{m} = \sqrt{I}$ and $\mathfrak{m}^2 \subseteq I$.

($\Leftarrow$) If (1) holds, then I is clearly a strongly 1-absorbing primary ideal of R. Now, suppose that (2) holds. Then, for each nonunit elements $x, y \in R$, $xy \in \mathfrak{m}^2 \subseteq I$. Then, I is trivially a strongly 1-absorbing primary ideal of R. □

Remark 2.3. (1) Note that if R is non local, then 1-absorbing ideals coincide with primary ideals [4, Theorem 3]. Hence, if R is non local with $\sqrt{0}$ prime, then the class of strongly 1-absorbing primary ideals coincide with the class of $\sqrt{0}$-primary ideals of R.

(2) Let R be a local ring. If I is a strongly 1-absorbing primary ideal of R, then $\sqrt{I}$ needs not to be maximal. To see that, it suffices to consider any local domain which is not a field. Then, $(0) = \sqrt{0}$ is a strongly 1-absorbing primary which is not maximal.

We deduce a characterization of strongly 1-absorbing primary prime ideals.

Corollary 2.4. *Let $\mathfrak{p}$ be a prime ideal of R. Then, $\mathfrak{p}$ is a strongly 1-absorbing primary if and only if*

(1) $\mathfrak{p} = \sqrt{0}$*, or*
(2) *R is local with maximal ideal $\mathfrak{p}$.*

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ be a prime ideal of R. Then, $\mathfrak{p}\mathfrak{m}$ is a strongly 1-absorbing primary ideal of R if and only if $\mathfrak{p} = \sqrt{0}$ or $\mathfrak{p} = \mathfrak{m}$.*

Proof. Note first that $\mathfrak{p}\mathfrak{m}$ is a 1-absorbing primary ideal of R ([4, Theorem 7]). Hence, using Theorem 2.2, $\mathfrak{p}\mathfrak{m}$ is a strongly 1-absorbing primary ideal of R if and only if $\mathfrak{p} = \sqrt{\mathfrak{p}\mathfrak{m}} = \sqrt{0}$ or $\mathfrak{p} = \sqrt{\mathfrak{p}\mathfrak{m}} = \mathfrak{m}$ and $\mathfrak{m}^2 \subseteq \mathfrak{p}\mathfrak{m}$. Hence, we have the desired result. □

The next result characterizes rings admitting strongly 1-absorbing primary ideals.

Theorem 2.6. *Let R be a ring. Then, there exists a strongly 1-absorbing primary ideal of R if and only if $\sqrt{0}$ is prime or R is local*

Proof. ($\Rightarrow$) Suppose that R contains a strongly 1-absorbing primary I. If R is not local, then by Theorem 2.2, $\sqrt{I} = \sqrt{0}$. Now, by [4, Theorem 2], $\sqrt{I}$ is prime since I is 1-absorbing primary. Thus, $\sqrt{0}$ is prime.

($\Leftarrow$) Following Corollary 2.4, if $(R, \mathfrak{m})$ is local, then $\mathfrak{m}$ is a strongly 1-absorbing primary ideal, and if $\mathfrak{p} = \sqrt{0}$ is prime, then it is a strongly 1-absorbing primary ideal. □

Corollary 2.7. *Let $n \geq 2$ be an integer. Then, $\mathbb{Z}/n\mathbb{Z}$ has a strongly 1-absorbing primary ideal if and only if $n = p^k$ where p is a prime number and k is a positive integer.*

Proof. Following Theorem 2.6, $\mathbb{Z}/n\mathbb{Z}$ has a strongly 1-absorbing primary ideal if and only if and only if it is local or $\sqrt{0}$ is prime. On the other hand, $\mathbb{Z}/n\mathbb{Z}$ is artinian, and then $\mathbb{Z}/n\mathbb{Z}$ local if and only it is a field; that is n is prime. Now, if $m = p_1^{n_1} \cdots p_r^{n_r}$ is the primary decomposition of n, then $\sqrt{0} = p_1 \cdots p_r \mathbb{Z}/n\mathbb{Z}$. Hence, $\sqrt{0}$ is prime if and only if $r = 1$. □

Corollary 2.8. *Let R_1 and R_2 be two rings. Then, $R_1 \times R_2$ has no strongly 1-absorbing primary ideal.*

Proof. This is because $R_1 \times R_2$ is not local and $\sqrt{0_{R_1 \times R_2}} = \sqrt{0_{R_1}} \times \sqrt{0_{R_2}}$ is never prime in $R_1 \times R_2$. □

Proposition 2.9. *Let I be a proper ideal of R. Then, I is a strongly 1-absorbing primary ideal of R if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some proper ideals I_1, I_2, and I_3 of R, then $I_1 I_2 \subseteq I$ or $I_3 \subseteq \sqrt{0}$.*

Proof. ($\Rightarrow$) Suppose that $I_1 I_2 \not\subseteq I$. Then, there exist $x \in I_1$ and $y \in I_2$ such that $xy \notin I$. For each $z \in I_3$, we have $xyz \in I$, and then $z \in \sqrt{0}$. Hence, $I_3 \subseteq \sqrt{0}$.

($\Leftarrow$) Let nonunit elements $a, b, c \in R$ with $abc \in I$ and set $I_1 = (a)$, $I_2 = (b)$, and $I_3 = (c)$. Then, by hypothesis, $ab \in I$ or $c \in \sqrt{0}$. Hence, I is a strongly 1-absorbing primary ideal. □

Proposition 2.10. *Let R be a ring and I and J be two proper ideals of R. If I and J are strongly 1-absorbing primary, then so is $I \cap J$.*

Proof. It is clear. □

Recall from [6], that a ring R is said to be an UN-ring if every nonunit element a of R is a product a unit and a nilpotent elements, or equivalently every element of R is either nilpotent or unit ([7, Proposition 2.25]). A simple example of UN-rings is $\mathbb{Z}/9\mathbb{Z}$.

Proposition 2.11. *For any ring R, the followings are equivalent:*
(1) *R is a UN-ring.*
(2) *Every proper principal ideal is strongly 1-absorbing primary.*
(3) *Every proper ideal is strongly 1-absorbing primary.*
(4) *$\sqrt{0}$ is a maximal ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let x be a nonunit element of R, and consider nonunit elements $a, b, c \in R$ with $abc \in (x)$. Then, $c \in \sqrt{0}$ (since it is a nonunit element and R is UN). Hence, (x) is a strongly 1-absorbing primary ideal of R.

(2) $\Rightarrow$ (3) Let I be a proper ideal of R. Consider nonunit elements $a, b, c \in R$ with $abc \in I$ and $c \notin \sqrt{0}$. We have $abc \in (abc)$ and (abc) is strongly 1-absorbing primary. Then, $ab \in (abc) \subseteq I$. Hence, I is a strongly 1-absorbing primary ideal of R.

(3) $\Rightarrow$ (4) Suppose that R is non local and let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two different maximal ideals of R. Using Theorem 2.2 we get $\mathfrak{m}_1 = \sqrt{0} = \mathfrak{m}_2$, a contradiction. Hence, R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be a prime ideal. By Corollary 2.4, $\mathfrak{p} = \sqrt{0}$ or $\mathfrak{p} = \mathfrak{m}$. Hence, $\sqrt{0}$ and $\mathfrak{m}$ are the only prime ideals of R. We claim that $\sqrt{0} = \mathfrak{m}$. Suppose that $\sqrt{0} \neq \mathfrak{m}$. Hence, $\mathfrak{m}^2 \not\subseteq \sqrt{0}$. Let $x \in \mathfrak{m}^2 \setminus \sqrt{0}$.

Since xR is a strongly 1-absorbing primary ideal and $\sqrt{xR} \neq \sqrt{0}$, we obtain that $\mathfrak{m}^2 \subseteq xR$ (by Theorem 2.2). Then, $\mathfrak{m}^2 = xR$. On the other hand, $\mathfrak{m}^3$ is a strongly 1-absorbing primary ideal with $\sqrt{\mathfrak{m}^3} = \mathfrak{m} \neq \sqrt{0}$. Hence, $\mathfrak{m}^2 \subseteq \mathfrak{m}^3$. Thus, $\mathfrak{m}^2 = \mathfrak{m}^3 = \mathfrak{m}.\mathfrak{m}^2 = \mathfrak{m}.\mathfrak{m}^3 = \mathfrak{m}^4$. Thus, $x = x^2r$ for some $r \in R$. Thus, $x(1 - xr) = 0$. But $1 - xr \notin \mathfrak{m}$. Thus, $1 - xr$ is a unit, and then $x = 0$, a contradiction since $x \notin \sqrt{0}$. Consequently, $\sqrt{0} = \mathfrak{m}$, as desired.

(4) ⇒ (1) It is clear. □

Proposition 2.12. *For any ring R, the followings are equivalent*:

1. *Every prime ideal of R is strongly 1-absorbing primary.*
2. *R is local and has at most one non maximal prime ideal.*

Proof. (⇒) Suppose that R is not local and let $\mathfrak{p}$ be a prime ideal of R. Then, since $\mathfrak{p}$ is strongly 1-absorbing primary, we get $\sqrt{0} = \mathfrak{p}$ (by Corollary 2.4). Hence, $\sqrt{0}$ is the only prime ideal of R. Thus, R is local, a contradiction. Hence, R is local with maximal ideal $\mathfrak{m}$. Now let $\mathfrak{p}$ be a prime ideal which is not maximal (if there exist). Then, $\sqrt{\mathfrak{p}} = \sqrt{0}$. Hence, R has at most two prime ideals $\sqrt{0}$ and $\mathfrak{m}$.

(⇐) If R is a UN-ring the result is trivial. Otherwise, R has exactly two prime ideals $\mathfrak{p} \subsetneq \mathfrak{m}$. Since $\sqrt{0}$ is the intersection of prime ideals, we get that $\sqrt{0} = \mathfrak{p}$. It is clear that both of $\sqrt{0}$ and $\mathfrak{m}$ are strongly 1-absorbing primary ideals. Hence, we have the desired result. □

Remark 2.13. Examples of local rings having one non maximal prime ideal are discrete valuation rings. An easy example of such rings is

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b \right\}$$

for any prime integer p.

Proposition 2.14. *Let R be a ring. Then, every primary ideal of R is strongly 1-absorbing primary if and only if*

1. *R is a UN-ring or*
2. *R is local with maximal ideal m, one non maximal prime ideal (which is $\sqrt{0}$), and every $\mathfrak{m}$-primary ideal contains $\mathfrak{m}^2$.*

Proof. (⇒) If R is not a UN-ring, then, by Proposition 2.12, R is local with exactly tow prime ideals which are $\sqrt{0}$ and $\mathfrak{m}$ (the maximal ideal). Now, let $\mathfrak{p}$ be an $\mathfrak{m}$-primary ideal of R. Then, since $\sqrt{0} \neq \sqrt{\mathfrak{p}} = \mathfrak{m}$, we have necessarily $\mathfrak{m}^2 \subseteq \mathfrak{p}$ (by Theorem 2.2).

(⇐) If R is a UN-ring the result follows trivially. Hence, suppose that (2) holds. Let $\mathfrak{p}$ be a primary ideal. Then, $\sqrt{\mathfrak{p}} = \sqrt{0}$ or $\sqrt{\mathfrak{p}} = \mathfrak{m}$. If $\sqrt{\mathfrak{p}} = \sqrt{0}$ then, by Theorem 2.2, $\mathfrak{p}$ is strongly 1-absorbing primary since every primary is 1-absorbing primary. Now, if $\sqrt{\mathfrak{p}} = \mathfrak{m}$, then also $\mathfrak{p}$ is strongly 1-absorbing primary since $\mathfrak{m}^2 \subseteq \mathfrak{p}$. □

Corollary 2.15. *For any Noetherian ring R, the following are equivalent*:

(1) *Every primary ideal of R is strongly 1-absorbing primary.*
(2) *R is a UN-ring.*

Proof. (1) $\Rightarrow$ (2) If R is not a UN-ring then, following Proposition 2.14, R is local with maximal ideal $\mathfrak{m}$ and every $\mathfrak{m}$-primary contains $\mathfrak{m}^2$. Hence, since $\mathfrak{m}^4$ is $\mathfrak{m}$-primary, we have $\mathfrak{m}^2 \subseteq \mathfrak{m}^4$. Thus, $\mathfrak{m}^2 = \mathfrak{m}^4$. Hence, $\mathfrak{m}^2$ is an idempotent ideal. Now, since R is Noetherian, $\mathfrak{m}^2$ is generated by an idempotent element of R. But R is local and so the only idempotent elements are 0 and 1. Hence, $\mathfrak{m}^2 = (0)$. Consequently, $\mathfrak{m}^2 \subseteq \sqrt{0}$. Then, $\mathfrak{m} = \sqrt{0}$, and so R is a UN-ring, a contradiction. Hence, R is a UN-ring. □

Proposition 2.16. *Let R be a Noetherian ring. Then, R is a UN-ring if and only if $\mathfrak{m}^k$ is a strongly 1-absorbing primary ideal for some maximal ideal $\mathfrak{m}$ and some integer $k \geq 3$.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Suppose that there exist a maximal ideal $\mathfrak{m}$ and an integer $k \geq 3$ such that $\mathfrak{m}^k$ is a strongly 1-absorbing primary ideal. If $\mathfrak{m} = \sqrt{\mathfrak{m}^k} = \sqrt{0}$, then R is UN, as desired. Now, suppose that $\mathfrak{m} \neq \sqrt{0}$. Then, $(R, \mathfrak{m})$ is local and $\mathfrak{m}^2 \subseteq \mathfrak{m}^k$. Thus, $\mathfrak{m}^2 = \mathfrak{m}^k$. If $k = 3$, then $\mathfrak{m}^2 = \mathfrak{m}^3 = \mathfrak{m}.\mathfrak{m}^2 = \mathfrak{m}^4$. Hence, $\mathfrak{m}^2$ is an idempotent ideal. We obtain the same thing if $k = 4$. Now, if $k > 4$, then $\mathfrak{m}^{k-2} = \mathfrak{m}^{k-4}\mathfrak{m}^2 = \mathfrak{m}^{2k-4}$. Thus, $\mathfrak{m}^{k-2}$ is an idempotent ideal. Then, there is always some integer n such that $\mathfrak{m}^n$ is an idempotent ideal. Since R is Noetherian, $\mathfrak{m}^n$ is generated by an idempotent element of R. But R is local and so the only idempotent elements are 0 and 1. Thus, $\mathfrak{m}^n = (0)$. Hence, $\mathfrak{m} \subseteq \sqrt{0}$, a contradiction. □

Proposition 2.17. *Let R be a ring. Then, (0) is the only strongly 1-absorbing primary ideal of R if and only if R is a field or a non local domain.*

Proof. ($\Rightarrow$) If R is local with maximal ideal $\mathfrak{m}$, then $\mathfrak{m} = (0)$ since $\mathfrak{m}$ is a strongly 1-absorbing primary ideal of R. Hence, R is a field. Now, if R is non local, then $\sqrt{0} = (0)$ is prime since $\sqrt{0}$ is a strongly 1-absorbing primary ideal. Thus, R is a domain.

($\Leftarrow$) Clear. □

Lemma 2.18. *Let I be a 1-absorbing primary ideal of R and $J \nsubseteq I$ a proper ideal of R. Then, $(I : J)$ is primary.*

Proof. Note $(I : J)$ is a proper ideal of R since $J \nsubseteq I$. Let $a, b \in R$ with $ab \in (I : J)$ and $a \notin (I : J)$. Clearly, b is a nonunit element. If a is unit, then $b \in (I : J) \subseteq \sqrt{(I : J)}$. Hence, we may assume that a and b are nonunit elements of R. Since $a \notin (I : J)$, there is $c \in J$ such that $ac \notin I$. But $abc \in I$ and I is 1-absorbing primary. Then, $b \in \sqrt{I} \subseteq \sqrt{(I : J)}$. Thus, $(I : J)$ is primary. □

Proposition 2.19. *Let I be a strongly 1-absorbing primary ideal of R and $J \nsubseteq \sqrt{I}$ a proper ideal of R. Then, $(I : J)$ is a strongly 1-absorbing primary ideal of R.*

Proof. If $\sqrt{I} \neq \sqrt{0}$, then R is local with maximal ideal $\sqrt{I}$. In this situation, our assumption $J \nsubseteq \sqrt{I}$ is not satisfied. Then, we must have that $\sqrt{I} = \sqrt{0}$ is prime. Let $x \in (I : J)$. Then, $xJ \subseteq I \subseteq \sqrt{I}$. Since $J \nsubseteq \sqrt{I}$, we get $x \in \sqrt{I}$. Hence, $I \subseteq (I : J) \subseteq \sqrt{I}$. Thus, $\sqrt{(I : J)} = \sqrt{I} = \sqrt{0}$. Now, the result follows from Theorem 2.2 and Lemma 2.18. □

3. Strongly 1-absorbing primary ideals and change of ring

Proposition 3.1. *Let $f : R \to S$ be a ring homomorphism. Then, the followings hold:*

(1) *If f is an epimorphism and I is a strongly 1-absorbing primary ideal of R containing $\ker(f)$, then $f(I)$ is a strongly 1-absorbing primary ideal of S.*
(2) *If f is a monomorphism and J is a strongly 1-absorbing primary ideal of S, then $f^{-1}(J)$ is a strongly 1-absorbing primary ideal of R.*

Proof. (1) Let $x, y, z \in S$ be nonunit elements such that $xyz \in f(I)$. Since f is an epimorphism, there exist nonunit elements $a, b, c \in R$ such that $x = f(a)$, $y = f(b)$, and $z = f(c)$. Suppose that $xy \notin f(I)$. Then, $ab \notin I$. Hence, since I is a strongly 1-absorbing primary ideal of R and $abc \in I$ (because $\ker(f) \subseteq I$), we get that $c \in \sqrt{0_R}$. Thus, $z = f(c) \in \sqrt{0_S}$. Consequently, $f(I)$ is a strongly 1-absorbing primary ideal of S.

(2) Let $x, y, z \in R$ be nonunit elements such that $xyz \in f^{-1}(J)$. Suppose that $xy \notin f^{-1}(J)$. Then, $f(x)f(y) \notin J$. Hence, since J is a strongly 1-absorbing primary ideal of S and $f(x)f(y)f(z) \in J$, we conclude that $f(z) \in \sqrt{0_S}$. Thus, since f is a monomorphism, $z \in \sqrt{0_R}$. Consequently, $f^{-1}(J)$ is a strongly 1-absorbing primary ideal of R. □

Corollary 3.2. (1) *Let $J \subseteq I$ be two proper ideals of a ring R. If I is a strongly 1-absorbing primary ideal of R, then I/J is a strongly 1-absorbing primary ideal of R/J.*
(2) *Let S be a sub-ring of a ring R. If J is a strongly 1-absorbing primary ideal of R, then $J \cap S$ is a strongly 1-absorbing primary ideal of S.*

Proof. Follows by applying Proposition 3.1 to the canonical surjection $\pi : R \to R/J$ and to the natural injection $\iota : S \hookrightarrow R$, respectively. □

Proposition 3.3. *Let R be a ring and S be a multiplicatively closed subset of R such that $I \cap S = \emptyset$. Then, if I is a strongly 1-absorbing primary ideal of R, then $S^{-1}I$ is a strongly 1-absorbing primary ideal of $S^{-1}R$.*

In particular, if I is a strongly 1-absorbing primary ideal of R and $\mathfrak{m}$ is a maximal ideal of R, then $I_{\mathfrak{m}}$ is a strongly 1-absorbing primary ideal of $R_{\mathfrak{m}}$.

Proof. Since $I \cap S = \emptyset$, $S^{-1}I$ is a proper ideal of $S^{-1}R$. Let $\frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$ for some nonunit elements $a, b, c \in R$ and $s_1, s_2, s_3 \in S$. Then, there is $u \in S$ such that $uabc \in I$. Suppose that $\frac{a}{s_1}\frac{b}{s_2} \notin S^{-1}I$. Then, $uab \notin I$. Hence, $c \in \sqrt{0_R}$ since I is a strongly 1-absorbing primary ideal of R. Thus, $\frac{c}{s_3} \in S^{-1}\sqrt{0_R} = \sqrt{0_{S^{-1}R}}$.

To end the proof, if suffices to prove that in the particular case $S = R \setminus \mathfrak{m}$ we have $S \cap I = \emptyset$. Seen Theorem 2.2, it is easy to see that $I \subseteq \mathrm{rad}(R)$. Hence, for $S \cap I \subseteq (R \setminus \mathfrak{m}) \cap \mathfrak{m} = \emptyset$. □

Proposition 3.4. *Let R be a ring and I be a proper ideal of R.*

1. *$R[X]$ has a strongly 1-absorbing primary ideal if and only if $\sqrt{0_R}$ is a prime ideal of R.*
2. *If $I[X]$ is a strongly 1-absorbing primary ideal of $R[X]$, then I is a strongly 1-absorbing primary ideal of R.*
3. *The ideal $I + XR[X]$ is never a strongly 1-absorbing primary ideal of $R[X]$.*
4. *$I[X]$ is a strongly 1-absorbing primary ideal of $R[X]$ if and only if $I[X]$ is primary and $\sqrt{I} = \sqrt{0}$.*

Proof. (1) From Theorem 2.6, $R[X]$ has a strongly 1-absorbing primary ideal if and only if $\sqrt{0_{R[X]}}$ is prime (since $R[X]$ is never local). Moreover, $\sqrt{0_{R[X]}} = (\sqrt{0_R})[X]$, and $(\sqrt{0_R})[X]$ is a prime ideal of $R[X]$ if and only $\sqrt{0_R}$ is a prime ideal of R.

(2) If $I[X]$ is a strongly 1-absorbing primary ideal of $R[X]$, then, by Corollary 3.2, $I = I[X] \cap R$ is a strongly 1-absorbing primary ideal of R.

(3) Since $R[X]$ is not local and $I + XR[X] \nsubseteq \sqrt{0_{R[X]}}$ (because $X \notin \sqrt{0_{R[X]}}$), $I + XR[X]$ is never a strongly 1-absorbing primary ideal of $R[X]$.

(4) Since $R[X]$ is not local, $I[X]$ is a strongly 1-absorbing primary ideal of $R[X]$ if and only if $I[X]$ is $\sqrt{0_{R[X]}}$-primary if and only if $I[X]$ is primary and $\sqrt{0_R}[X] = \sqrt{0_{R[X]}} = \sqrt{I[X]} = \sqrt{I}[X]$ if and only if $I[X]$ is primary and $\sqrt{0} = \sqrt{I}$ □

Proposition 3.5. *Let I be a proper ideal of a ring R. Then, $I + XR[[X]]$ is a strongly 1-absorbing primary ideal of $R[[X]]$ if and only if $(R, \sqrt{I})$ is local with $(\sqrt{I})^2 \subseteq I$.*

In particular, $XR[[X]]$ is a strongly 1-absorbing primary ideal of $R[[X]]$ if and only if R is a UN-ring with $(\sqrt{0})^2 = \{0\}$.

Proof. ($\Rightarrow$) By Corollary 3.2, $I = (I + XR[[X]]) \cap R$ is a strongly 1-absorbing primary ideal of R. We have that $X \notin \sqrt{0_{R[[X]]}}$, and then $\sqrt{I + XR[[X]]} \neq \sqrt{0_{R[[X]]}}$. Then, $R[[X]]$ must be local with maximal ideal $\sqrt{I + XR[[X]]} = \sqrt{I} + XR[[X]]$. Now, by [5, Theorem 2], R is local with maximal ideal $\sqrt{I}$. Moreover, $(\sqrt{I} + XR[[X]])^2 \subseteq (\sqrt{I})^2 + XR[[X]] \subseteq I + XR[[X]]$. Then, $\sqrt{I}^2 \subseteq I$.

($\Leftarrow$) Since $(R, \sqrt{I})$ is local, then $R[[X]]$ is local with maximal ideal $\sqrt{I} + XR[[X]] = \sqrt{I + XR[[X]]}$ (by [5, Theorem 2]). Moreover, $(\sqrt{I} + XR[[X]])^2 \subseteq (\sqrt{I})^2 + XR[[X]] \subseteq I + XR[[X]]$. Then, $I + XR[[X]]$ is a strongly 1-absorbing primary ideal of $R[[X]]$. □

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