Bull. Korean Math. Soc. **57** (2020), No. 5, pp. 1195–1204
https://doi.org/10.4134/BKMS.b190865
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE DEFECTS OF HOLOMORPHIC CURVES

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Abstract. In this paper we consider the holomorphic curves (or derived holomorphic curves introduced by Toda in [15]) with maximal defect sum in the complex plane. Some well-known theorems on meromorphic functions of finite order with maximal sum of defects are extended to holomorphic curves in projective space.

1. Introduction

In this paper, we use the usual notation of Nevanlinna theory of holomorphic curve. For detail, see [5,11,20]. Let f be a holomorphic curve from the complex plane $\mathbf{C}$ into the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$ and let

$$\tilde{f} = (f_1, \ldots, f_{n+1}) : \mathbf{C} \to \mathbf{C}^{n+1} \backslash \{\mathbf{0}\}$$

be a reduced representation of f, where n is a positive integer. We use the following notation:

$$\|\tilde{f}(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{\frac{1}{2}}.$$

The Nevanlinna-Cartan characteristic function of f is defined as following:

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\tilde{f}(re^{i\theta})\| d\theta - \log \|\tilde{f}(0)\|.$$

Remark 1.1. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

We suppose throughout the paper that f is transcendental, that is to say,

$$\lim_{r \to \infty} \frac{T_f(r)}{\log r} = \infty$$

and that f is linearly non-degenerate over $\mathbf{C}$, namely, $f_1, \ldots, f_{n+1}$ are linearly independent over $\mathbf{C}$.

Received September 26, 2019; Revised December 12, 2019; Accepted January 17, 2020.
2010 *Mathematics Subject Classification.* Primary 32H30, 30D35.
Key words and phrases. Holomorphic curve, maximum defect sum, characteristic function.
This work was financially supported by NNSF of China (11701006), and also by NSF of Anhui Province (1808085QA02), China.

We denote the order of f by ρ_f and the lower order of f by μ_f respectively:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}, \quad \mu_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

It is said that f is of regular growth if $\rho_f = \mu_f$.

For a vector $\boldsymbol{a} = (a_1, \ldots, a_{n+1}) \in \mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$, we write

$$\langle \boldsymbol{a}, \tilde{f}\rangle = a_1 f_1 + \cdots + a_{n+1} f_{n+1}.$$

The proximity function $m_f(r, \boldsymbol{a})$ is defined as

$$m_f(r, \boldsymbol{a}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\|\boldsymbol{a}\| \cdot \|\tilde{f}(re^{i\theta})\|}{|\langle \boldsymbol{a}, \tilde{f}(re^{i\theta})\rangle|} d\theta,$$

where $\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{\frac{1}{2}}$. Let $n_f(r, \boldsymbol{a})$ be the number of zeros of $\langle \boldsymbol{a}, \tilde{f}\rangle$ in the disk $|z| < r$, counting multiplicity. The counting function is then defined by

$$N_f(r, \boldsymbol{a}) = \int_0^r \frac{n_f(t, \boldsymbol{a}) - n_f(0, \boldsymbol{a})}{t} dt + n_f(0, \boldsymbol{a}) \log r.$$

The Poisson-Jensen formula implies:

The First Main Theorem (FMT).

$$T_f(r) = m_f(r, \boldsymbol{a}) + N_f(r, \boldsymbol{a}) + O(1),$$

where $O(1)$ is a constant independent of r.

We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} \frac{N_f(r, \boldsymbol{a})}{T_f(r)}$$

the defect of f with respect to $\boldsymbol{a}$.

Let $\mathcal{X}$ be a subset of vectors of $\mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$ in N-subgeneral position satisfying $\#X \geq 2N - n + 1$, where N is an integer such that $N \geq n$. That is to say, for any $N+1$ vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N+1}$ of $\mathcal{X}$, the linear span of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{N+1}$ is $\mathbf{C}^{n+1}$. Put $\mathcal{X}_0 = \{\boldsymbol{a} = (a_1, \ldots, a_{n+1}) \in \mathcal{X}; a_{n+1} = 0\}$. We say that the set $\mathcal{X}$ is in general position if $N = n$. The notations $\mathcal{X}$ and $\mathcal{X}_0$ were introduced by Toda, for more details see [16, 18].

Let $\nu(z)$ be the order of zero of $\langle \boldsymbol{a}, \tilde{f}\rangle$ at z. Suppose that k is a positive integer or ∞. We set

$$n_f^{(k)}(r, \boldsymbol{a}) = \sum_{|z|\leq r} \min\{\nu(z), k\},$$

$$N_f^{(k)}(r, \boldsymbol{a}) = \int_0^r \frac{n_f^{(k)}(t, \boldsymbol{a}) - n_f^{(k)}(0, \boldsymbol{a})}{t} dt + n_f^{(k)}(0, \boldsymbol{a}) \log r,$$

and

$$\delta_k(\boldsymbol{a}, f) = 1 - \limsup_{r\to\infty} \frac{N_f^{(k)}(r, \boldsymbol{a})}{T_f(r)}.$$

Thus, $\delta(\boldsymbol{a}, f) = \delta_\infty(\boldsymbol{a}, f)$ and it is easy to see that:

Remark 1.2. For any $0 < k \leq \infty$,

$$0 \leq \delta(\boldsymbol{a}, f) \leq \delta_k(\boldsymbol{a}, f) \leq 1.$$

It is well-known that the following defect relation is easily obtained from the fundamental inequality of H. Cartan [1] and E. Nochka [8].

The Truncated Defect Relation.

$$\sum_{\boldsymbol{a}\in\mathcal{X}} \delta_n(\boldsymbol{a}, f) \leq 2N - n + 1. \tag{1}$$

In [15], Toda introduced a sort of derivative to holomorphic curves which possess similar properties to the derivative of meromorphic functions. We call the holomorphic curve induced by the mapping

$$(f_1^{n+1}, \ldots, f_n^{n+1}, W(f_1, \ldots, f_{n+1})) : \mathbf{C} \to \mathbf{C}^{n+1}$$

the derived holomorphic curve of f and express it by f^*.

Remark 1.3 ([15, Proposition 1]). The definition of derived holomorphic curve f^* of f does not depend on the choice of a reduced representation of f. When $n = 1$, f^* corresponds exactly to the derivative of the meromorphic function f_2/f_1.

The main purpose of this paper is to investigate the properties of holomorphic curve f when f (or f^*) has maximum defect sum, that is, the equality in (1) holds for f (or f^*) .

2. Holomorphic curves with order of integer

The properties of derivative of a meromorphic function with maximum defect sum have been studied in depth, see [2–4, 10, 12, 19] and their references. For example, Edrei [4] and Weitsman [19] proved:

Theorem A. *Let f be a transcendental meromorphic function of finite order ρ such that*

$$\delta(\infty, f) = 1 \text{ and } \sum_{a\in\mathbf{C}} \delta(a, f) = 1.$$

Then f is of regular growth and ρ is a positive integer.

It is well known that there do exist meromorphic functions of order $\frac{p}{2}$ with an odd integer p and satisfying $\sum_{a\in\overline{\mathbf{C}}} \delta(a, f) = 2$. These interesting meromorphic functions offer a famous conjecture which is affirmed by D. Drasin [2]. M. Ozawa [10] considered the meromorphic function whose derivative has maximum defect sum and obtained:

Theorem B. *Let f be a meromorphic function of finite order ρ_f such that $\sum_{a\in\overline{\mathbf{C}}} \delta(a, f') = 2$. Then ρ_f is a positive integer.*

The first result to extend Theorem A to holomorphic curves is due to Mori [6]. See [7, 9, 14] for more results on holomorphic curves with maximal defect sum. Moreover, Toda [16] also gave a generalization of Theorem A to holomorphic curves. Recently, Toda [18] weakened the condition treated in [16] and obtained:

Theorem C. *Let f be a non-degenerate holomorphic curve. Suppose that $\rho_f < \infty$;*
(i) $\delta(\boldsymbol{e}_i, f) = 1$ $(i = 1, \dots, n)$ *and that*
(ii) $\sum_{\boldsymbol{a}\in\mathcal{X}} \delta(\boldsymbol{a}, f) = 2N - n + 1$.
Then f is of regular growth and ρ_f is a positive integer.

By the derived holomorphic curve introduced by Toda [15], we generalize Theorem B to holomorphic curves.

Theorem 2.1. *Let $\boldsymbol{e}_j, 1 \leq j \leq n+1$, be the standard basis of $\mathbf{C}^{n+1}$. Suppose that $\{\boldsymbol{e}_i\}_{i=1}^n \subset \mathcal{X}$. Let f be a holomorphic curve of finite order ρ_f such that*

$$\sum_{\boldsymbol{a}\in\mathcal{X}} \delta(\boldsymbol{a}, f^*) = 2N - n + 1. \tag{2}$$

Then f is of regular growth and ρ_f is a positive integer.

The following example shows that there exists a set $\mathcal{X}$ in general position which is uncountable.

Example 2.2. For each $\lambda \in \mathbf{C}$, we set

$$\boldsymbol{a}_\lambda = (1, \lambda, \lambda^2, \dots, \lambda^n).$$

It's easy to obtain that $\{\boldsymbol{a}_\lambda; \lambda \in \mathbf{C}\}$ is a family of vectors in general position. Indeed, take any $n+1$ distinct numbers $\lambda_1, \dots, \lambda_{n+1} \in \mathbf{C}$, we note that the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\ \vdots & & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_{n+1}^n \end{vmatrix} = \prod_{1\leq i<j\leq n+1} (\lambda_j - \lambda_i) \neq 0.$$

Therefore, the set $\mathcal{X} = \{\boldsymbol{a}_\lambda; \lambda \in \mathbf{C}\} \cup \{\boldsymbol{e}_j;\ 1 \leq j \leq n+1\}$ is uncountable.

We now prove our first main result.

Proof of Theorem 2.1. Suppose that $\tilde{f} = (f_1, \dots, f_{n+1}) : \mathbf{C} \to \mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$ is a reduced representation of f. Then there exists an entire function $d(z)$ such that $f_1^{n+1}/d, \dots, f_n^{n+1}/d$ and $W(f_1, \dots, f_{n+1})/d$ are entire functions without common zeros for all. Then

$$(f_1^{n+1}/d, \dots, f_n^{n+1}/d, W(f_1, \dots, f_{n+1})/d)$$

is a reduced representation of f^*.

We claim that all zeros of $\langle \boldsymbol{e}_1, f^*\rangle$ have multiplicity at least $n+1$, i.e., all zeros of f_1^{n+1}/d have multiplicity $\geq n+1$. To this purpose, let $z_0 \in \mathbf{C}$ be a zero of the holomorphic function f_1^{n+1}/d.

Suppose first that $d(z_0) \neq 0$. Then z_0 is also a zero of f_1^{n+1}. We see at once that z_0 is a zero of f_1^{n+1}/d whose multiplicity at least $n+1$.

Suppose now that $d(z_0) = 0$. Then z_0 is a common zero of the functions $f_1^{n+1}, \dots, f_n^{n+1}$ and $W(f_1, \dots, f_{n+1})$, and hence,

$$W(f_1, \dots, f_{n+1})(z_0) = f_1(z_0) = \cdots = f_n(z_0) = 0.$$

Assume that k_i $(i = 1, \dots, n)$ and k are the zero multiplicities at z_0 of f_i $(i = 1, \dots, n)$ and $W(f_1, \dots, f_{n+1})$, respectively. Then z_0 is a zero of d with multiplicity which equals

$$\min\{(n+1)k_1, \dots, (n+1)k_n, k\} = \min\{(n+1)k_{i_0}, k\},$$

where $k_{i_0} = \min\{k_1, \dots, k_n\}$. Thus, the zero multiplicity of d at z_0 is greater than or equal to $(n+1)k_1 - (n+1)k_{i_0} = (n+1)(k_1 - k_{i_0})$. It follows that if $k_1 = k_{i_0}$, $f_1^{n+1}/d(z_0) \neq 0$, and if $k_1 > k_{i_0}$, z_0 is a zero of f_1^{n+1}/d whose multiplicity at least is $(n+1)(k_1 - k_{i_0})$. The claim is valid.

Similarly, all zeros of $\langle \boldsymbol{e}_i, f^*\rangle$ have multiplicity at least $n+1$ $(i = 1, \dots, n)$. It follows that

$$N_{f^*}(r, \boldsymbol{e}_i) \geq \frac{n+1}{n} N_{f^*}^{(n)}(r, \boldsymbol{e}_i) = \big(1 + \frac{1}{n}\big) N_{f^*}^{(n)}(r, \boldsymbol{e}_i)$$

for $i = 1, \dots, n$. Hence, we get

$$\limsup_{r\to\infty} \frac{N_{f^*}(r, \boldsymbol{e}_i)}{T_{f^*}(r)} \geq \big(1 + \frac{1}{n}\big) \limsup_{r\to\infty} \frac{N_{f^*}^{(n)}(r, \boldsymbol{e}_i)}{T_{f^*}(r)}$$

for $i = 1, \dots, n$. This, together with the definition of $\delta(\boldsymbol{e}_i, f^*)$ and the definition of $\delta_n(\boldsymbol{e}_i, f^*)$, gives

$$1 - \delta(\boldsymbol{e}_i, f^*) \geq \big(1 + \frac{1}{n}\big)(1 - \delta_n(\boldsymbol{e}_i, f^*)) \tag{3}$$

for $i = 1, \dots, n$.

From the condition (2) and Remark 1.2 we have for each $\boldsymbol{a} \in \mathcal{X}$,

$$\delta(\boldsymbol{a}, f^*) = \delta_n(\boldsymbol{a}, f^*).$$

Then

$$0 \leq \delta(\boldsymbol{e}_i, f^*) = \delta_n(\boldsymbol{e}_i, f^*) \leq 1 \ (i = 1, \dots, n) \tag{4}$$

by virtue of the condition $\{\boldsymbol{e}_i\}_{i=1}^n \subset \mathcal{X}$. Thus, by (3) and (4) we obtain that

$$\delta(\boldsymbol{e}_i, f^*) = 1 \ (i = 1, \dots, n). \tag{5}$$

Combining (2) and (5) with Theorem C, we see that f^* is of regular growth and ρ_{f^*} is a positive integer. Note the properties of f^* that $\rho_{f^*} = \rho_f$ and $\mu_{f^*} = \mu_f$ (cf. [18, p. 74]), f is also of regular growth and ρ_f is also a positive integer. This completes the proof. □

3. Characteristic functions of derived holomorphic curves

For a non-constant meromorphic function f on $\mathbf{C}$, S. K. Singh and H. S. Gopalakrishna [13] gave the lower bound of the quantity $\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}$ by the truncated defect sum.

Theorem D. *If f is a non-constant meromorphic function of order ρ_f, then*

$$\liminf_{r\to\infty}\frac{T(r,f')}{T(r,f)}\geq\sum_{a\in\mathbf{C}}\delta_1(a,f),$$

where $r\to\infty$ without restriction if ρ_f is finite and $r\to\infty$ outside an exceptional set of finite measure if $\rho_f=\infty$.

In [12], S. M. Shah and S. K. Singh got the relations between the characteristics of a meromorphic function with maximum defect sum and its derivative.

Theorem E. *Let f be a meromorphic function of finite order and assume that $\sum_{a\in\mathbf{C}}\delta(a,f)=2$. Then*

$$T(r,f')\sim 2T(r,f)\ \text{as}\ r\to\infty.$$

Our second main result is to improve Theorem D to holomorphic curve case. As corollaries, we establish the relations between a holomorphic curve with maximum defect sum and its derived curve from the point of view of characteristics.

Theorem 3.1. *Let $f:\mathbf{C}\to\mathbf{P}^n(\mathbf{C})$ be a holomorphic curve of order ρ_f. Then*

$$\sum_{\boldsymbol{a}\in\mathcal{X}\setminus\mathcal{X}_0}\delta(\boldsymbol{a},f)\leq\frac{2N-n+1}{n+1}\liminf_{r\to\infty}\frac{T_{f^*}(r)}{T_f(r)}+(N-n),$$

where $r\to\infty$ without restriction if ρ_f is finite and $r\to\infty$ outside an exceptional set of finite measure if $\rho_f=\infty$.

For the proof, we need the following lemma.

Lemma 3.2 ([17, Lemma 5])**.** *For any $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_q$ $(2N-n+1<q<\infty)$ in $\mathcal{X}\setminus\mathcal{X}_0$ and for $r\geq 1$*

$$\sum_{j=1}^{q}m_f(r,\boldsymbol{a}_j)\leq\frac{2N-n+1}{n+1}m_{f^*}(r,\boldsymbol{e}_{n+1})+(N-n)T_f(r)+S(r,f).$$

Here and in the following, we denote by $S(r,f)$ any term such that if $\rho_f=\infty$, $S(r,f)=o(T_f(r))$ $(r\to\infty)$, possibly outside some set of finite measure, and if $\rho_f<\infty$, $S(r,f)=O(\log r)$ $(r\to\infty)$. In particular, if f is transcendental and of finite order, then we assume that $S(r,f)=o(T_f(r))$ as $r\to\infty$ without an exceptional set.

Proof of Theorem 3.1. By the argument in [20, p. 29], the set $\{\boldsymbol{a} \in \mathcal{X};\ \delta(\boldsymbol{a}, f) > 0\}$ is at most countable. Let $\{\boldsymbol{a}_j\}_{j=1}^{\infty}$ be an infinite sequence of distinct elements of $\mathcal{X}\backslash\mathcal{X}_0$, which includes every $\boldsymbol{a} \in \mathcal{X}\backslash\mathcal{X}_0$ for which $\delta(\boldsymbol{a}, f) > 0$. Then

$$\sum_{\boldsymbol{a}\in\mathcal{X}\backslash\mathcal{X}_0} \delta(\boldsymbol{a}, f) = \sum_{j=1}^{\infty} \delta(\boldsymbol{a}_j, f). \tag{6}$$

For any positive integer q, by Lemma 3.2 and FMT, we have

$$\begin{aligned}\sum_{j=1}^{q} m_f(r, \boldsymbol{a}_j) &\le \frac{2N-n+1}{n+1} m_{f^*}(r, \boldsymbol{e}_{n+1}) + (N-n)T_f(r) + S(r,f)\\ &\le \frac{2N-n+1}{n+1} T_{f^*}(r) + (N-n)T_f(r) + S(r,f).\end{aligned}$$

Adding $\sum_{j=1}^{q} N_f(r, \boldsymbol{a}_j)$ to both sides and using FMT again, we have

$$q\ T_f(r) \le \frac{2N-n+1}{n+1} T_{f^*}(r) + \sum_{j=1}^{q} N_f(r, \boldsymbol{a}_j) + (N-n)T_f(r) + S(r,f).$$

It follows that

$$q \le \frac{2N-n+1}{n+1}\frac{T_{f^*}(r)}{T_f(r)} + \sum_{j=1}^{q} \frac{N_f(r, \boldsymbol{a}_j)}{T_f(r)} + (N-n) + \frac{S(r,f)}{T_f(r)}.$$

Hence

$$\begin{aligned}q \le \frac{2N-n+1}{n+1} \liminf_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} + \sum_{j=1}^{q} \limsup_{r\to\infty} \frac{N_f(r, \boldsymbol{a}_j)}{T_f(r)}\\ + (N-n) + \limsup_{r\to\infty} \frac{S(r,f)}{T_f(r)}.\end{aligned} \tag{7}$$

From (7), the definition of $\delta(\boldsymbol{a}_j, f)$ and the definition of $S(r, f)$ we get

$$\sum_{j=1}^{q} \delta(\boldsymbol{a}_j, f) \le \frac{2N-n+1}{n+1} \liminf_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} + (N-n), \tag{8}$$

where $r \to \infty$ through all values if $\rho_f < \infty$, and $r \to \infty$ outside an exceptional set (which is independent of the $\boldsymbol{a}_j$ and q) of finite measure if $\rho_f = \infty$.

Letting $q \to \infty$ in (8) and noting (6), we obtain that

$$\sum_{\boldsymbol{a}\in\mathcal{X}\backslash\mathcal{X}_0} \delta(\boldsymbol{a}, f) \le \frac{2N-n+1}{n+1} \liminf_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} + (N-n),$$

where $r \to \infty$ through all values if $\rho_f < \infty$, and $r \to \infty$ outside an exceptional set of finite measure if $\rho_f = \infty$. □

In the case of $N = n$, we have the following corollary which is the extension of Theorem D to holomorphic curves.

Corollary 3.3. *Let $\mathcal{X}$ be a subset of vectors of $\mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve of order ρ_f. Then*

$$\liminf_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} \geq \sum_{\boldsymbol{a}\in\mathcal{X}\backslash\mathcal{X}_0} \delta(\boldsymbol{a}, f),$$

where $r \to \infty$ without restriction if ρ_f is finite and $r \to \infty$ outside an exceptional set of finite measure if $\rho_f = \infty$.

Corollary 3.3, together with the following lemma, improves Theorem E to holomorphic curves.

Lemma 3.4 ([15, Lemma 3])**.** *Let $\mathcal{X}$ be a subset of vectors of $\mathbf{C}^{n+1}\backslash\{\mathbf{0}\}$ in general position. If f is a non-degenerate holomorphic curve, then*

$$T_{f^*}(r) < (n+1)T_f(r) - N(r, 1/d) + S(r, f).$$

Corollary 3.5. *If $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is a holomorphic curve of finite order such that $\sum_{\boldsymbol{a}\in\mathcal{X}/\mathcal{X}_0} \delta(\boldsymbol{a}, f) = n + 1$, then*

$$T_{f^*}(r) \sim (n+1)T_f(r) \text{ as } r \to \infty.$$

Proof. We have $\sum_{\boldsymbol{a}\in\mathcal{X}\backslash\mathcal{X}_0} \delta(\boldsymbol{a}, f) = n + 1$. So, we have, by Corollary 3.3,

$$\liminf_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} \geq n + 1. \tag{9}$$

On the other hand, applying Lemma 3.4, we have

$$T_{f^*}(r) < (n+1)T_f(r) + S(r, f).$$

It follows that

$$\frac{T_{f^*}(r)}{T_f(r)} < n + 1 + \frac{S(r, f)}{T_f(r)}.$$

Hence

$$\limsup_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} \leq n + 1. \tag{10}$$

Therefore, we have

$$\lim_{r\to\infty} \frac{T_{f^*}(r)}{T_f(r)} = n + 1.$$

Thus Corollary 3.5 is proved. □

Acknowledgement. The authors are very grateful the anonymous referee for his valuable comments and suggestions to improve this paper.

References

[1] H. Cartan, *Sur les zéros des combinaisions linéarires de p fonctions holomorpes données*, Mathematica **7** (1933), 5–31.

[2] D. Drasin, *Proof of a conjecture of F. Nevanlinna concerning functions which have deficiency sum two*, Acta Math. **158** (1987), no. 1-2, 1–94. https://doi.org/10.1007/BF02392256

[3] D. Drasin and A. Weitsman, *Meromorphic functions with large sums of deficiencies*, Advances in Math. **15** (1975), 93–126. https://doi.org/10.1016/0001-8708(75)90126-7

[4] A. Edrei, *Sums of deficiencies of meromorphic functions. II*, J. Analyse Math. **19** (1967), 53–74. https://doi.org/10.1007/BF02788709

[5] K. Kodaira, Nevanlinna Theory, Springer, Tokyo, 2017. https://doi.org/10.1007/978-981-10-6787-7

[6] S. Mori, *Holomorphic curves with maximal deficiency sum*, Kodai Math. J. **2** (1979), no. 1, 116–122. http://projecteuclid.org/euclid.kmj/1138035969

[7] ———, *Defects of holomorphic curves into $P^n(\mathbb{C})$ for rational moving targets and a space of meromorphic mappings*, Complex Variables Theory Appl. **43** (2001), no. 3-4, 363–379. https://doi.org/10.1080/17476930108815326

[8] E. I. Nochka, *On the theory of meromorphic curves*, Dokl. Akad. Nauk SSSR **269** (1983), no. 3, 547–552.

[9] J. Noguchi, *A relation between order and defects of meromorphic mappings of mathbbC^n into $P^N(\mathbb{C})$*, Nagoya Math. J. **59** (1975), 97–106. http://projecteuclid.org/euclid.nmj/1118795558

[10] M. Ozawa, *On the dificiencies of meromorphic functions*, Kodai Math. Sem. Rep. **20** (1968), 385–388. http://projecteuclid.org/euclid.kmj/1138845742

[11] M. Ru, *Nevanlinna theory and its relation to Diophantine approximation*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. https://doi.org/10.1142/9789812810519

[12] S. M. Shah and S. K. Singh, *On the derivative of a meromorphic function with maximum defect*, Math. Z. **65** (1956), 171–174. https://doi.org/10.1007/BF01473876

[13] S. K. Singh and H. S. Gopalakrishna, *Exceptional values of entire and meromorphic functions*, Math. Ann. **191** (1971), 121–142. https://doi.org/10.1007/BF02330565

[14] W. Stoll, *An extension of the theorem of Steinmetz-Nevanlinna to holomorphic curves*, Math. Ann. **282** (1988), no. 2, 185–222. https://doi.org/10.1007/BF01456971

[15] N. Toda, *An extension of the derivative of meromorphic functions to holomorphic curves*, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), no. 6, 159–163. http://projecteuclid.org/euclid.pja/1195510977

[16] ———, *On the order of holomorphic curves with maximal deficiency sum*, Kodai Math. J. **18** (1995), no. 3, 451–474. https://doi.org/10.2996/kmj/1138043483

[17] ———, *On the second fundamental inequality for holomorphic curves*, Bull. Nagoya Inst. Tech. **50** (1998), 123–135 (1999).

[18] ———, *On the order of holomorphic curves with maximal deficiency sum, II*, Proc. Japan Acad. Ser. A Math. Sci. **94** (2018), no. 7, 71–75. https://doi.org/10.3792/pjaa.94.71

[19] A. Weitsman, *Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna*, Acta Math. **123** (1969), 115–139. https://doi.org/10.1007/BF02392387

[20] L. Yang, *Value distribution theory*, translated and revised from the 1982 Chinese original, Springer-Verlag, Berlin, 1993.

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