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FULLY PRIME MODULES AND FULLY SEMIPRIME MODULES

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Abstract. Fully prime rings (in which every proper ideal is prime) have been studied by Blair and Tsutsui, and fully semiprime rings (in which every proper ideal is semiprime) have been studied by Courter. For a given module M, we introduce the notions of a fully prime module and a fully semiprime module, and extend certain results of Blair, Tsutsui, and Courter to the category subgenerated by M. We also consider the relationship between the conditions (1) M is a fully prime (semiprime) module, and (2) the endomorphism ring of M is a fully prime (semiprime) ring.

1. Introduction

It will be assumed throughout that R is an associative ring with identity, and that M is a fixed nonzero left R-module. A module X in R–Mod, the category of unital left R-modules, is said to be M-*generated* if there exists an R-epimorphism from a direct sum of copies of M onto X. The category $\sigma[M]$ of modules *subgenerated* by M is defined to be the full subcategory of R–Mod that contains all modules ${}_RX$ such that X is isomorphic to a submodule of an M-generated module. The reader is referred to [9] and [16] for results on the category $\sigma[M]$. It is an abelian category, and in R–Mod it is closed under formation of homomorphic images, submodules, and direct sums.

The results in this paper concern the analog in $\sigma[M]$ of the notion of a prime ideal of the ring R. We recall that a proper ideal P of R is said to be *prime* if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for all ideals A, B of R, and it is said to be *semiprime* if $A^2 \subseteq P$ implies $A \subseteq P$ for all ideals A of R, or, equivalently, if P is an intersection of prime ideals of R.

A subfunctor τ of the identity on $\sigma[M]$ is called a *preradical* of $\sigma[M]$; it is called a *radical* if $\tau(X/\tau(X)) = (0)$ for all X in $\sigma[M]$. If ρ and τ are preradicals of $\sigma[M]$ such that $\rho(X) \subseteq \tau(X)$ for all modules ${}_RX$, then the notation $\rho \leq \tau$

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is used. If N is any submodule of M, we define the preradicals tr_N^M and rej_N^M as follows:

$$\mathrm{tr}_N^M(X) = \sum \{f(N) \mid f \in \mathrm{Hom}_R(M, X)\}$$

and

$$\mathrm{rej}_N^M(X) = \cap \left\{f^{-1}(N) \mid f \in \mathrm{Hom}_R(X, M)\right\}$$

for all X in R–Mod. We note that if N is a fully invariant submodule of M, then tr_N^M and rej_N^M are defined by the property that $\mathrm{tr}_N^M(M) = N = \mathrm{rej}_N^M(M)$, and if τ is any preradical with $\tau(M) = N$, then $\mathrm{tr}_N^M \leq \tau \leq \mathrm{rej}_N^M$.

For submodules N, L of M, we define a product as follows:

$$N_M L = \mathrm{tr}_N^M(L) = \sum\{f(N) \mid f \in \mathrm{Hom}_R(M, L)\}.$$

A proper submodule Q of M is called a *prime* submodule if Q is fully invariant in M, and $N_M L \subseteq Q$ implies $N \subseteq Q$ or $L \subseteq Q$ for all fully invariant submodules N, L of M. Similarly, Q is called a *semiprime* submodule if Q is fully invariant in M, and $N_M N \subseteq Q$ implies $N \subseteq Q$ for all fully invariant submodules N of M. The reader is referred to [3], [7], [13] and [14] for general properties of the product defined above and of prime and semiprime modules.

Finally, a submodule N of M is said to be *idempotent* if $N_M N = N$.

In the first section of the paper, we show that if $Q \subseteq M$ is a prime (resp. semiprime) submodule, then $\mathrm{ann}_R(M/Q)$ is a prime (resp. semiprime) ideal of R. We then give the definition of a fully prime (fully semiprime) module, and some preliminary lemmas.

In [8], Courter studies rings in which every proper ideal is semiprime. He gives sixteen equivalent conditions, including the statement that every proper ideal of a ring is semiprime if and only if every ideal of the ring is idempotent. He also notes that this condition holds in any von Neumann regular ring. In fact, [10, Corollary 1.18] states that the ring R is von Neumann regular if and only if every ideal of R is idempotent and R/P is von Neumann regular for all prime ideals P of R.

In Section 3, we show that twelve of Courter's characterizations can be extended to the category $\sigma[M]$. Our most basic example of a module all of whose proper fully invariant submodules are semiprime is that of a semisimple module. In fact, we show that for a left Artinian ring, these are the only examples. More generally, regular modules provide additional examples. We show that if every proper submodule of M is semiprime, then the same condition holds for any direct sum $M^{(I)}$ of copies of M.

In [4], Blair and Tsutsui studied rings in which every proper ideal is prime. They showed that every proper ideal of R is prime if and only if the ideals of R are totally ordered and idempotent. They note that in the commutative case the only such rings are fields.

In Section 4, we extend the above characterization to $\sigma[M]$. Using Gabriel's condition H for the module M, we show that if M is a fully prime finitely generated module over a commutative Noetherian ring, then $\mathrm{ann}_R(M)$ is a maximal ideal of R, and M is a finite dimensional vector space over $R/\operatorname{ann}_R(M)$. In general, the most basic example of a fully prime module is a homogeneous semisimple module, and a finitely generated module over a left Artinian ring is fully prime if and only if it is semisimple and homogeneous.

In Section 5, we show that the notion of a prime (semiprime) submodule is Morita invariant, and then the notion of a fully prime (fully semiprime) module is also a Morita invariant. Theorem 5.8 shows that if R is a fully prime ring and P is a finitely generated projective R-module, then $\operatorname{End}_R(P)$ is a fully prime ring. If M is a finitely generated quasi-projective module, then Theorem 5.12 (Theorem 5.4) shows that if M is a fully prime (fully semiprime) module, then $\operatorname{End}_R(M)$ is a fully prime (fully semiprime) ring. In Theorem 5.9 we give a partial converse: if every fully invariant submodule of M is M-generated and $\operatorname{End}_R(M)$ is a fully prime ring, then M is a fully prime module.

2. Definitions and preliminary results

Since a left ideal of R is fully invariant as a submodule if and only if it is an ideal of R, and the product $I_R J$ of two ideals I, J of R is the usual product IJ, it follows that the prime (resp. semiprime) submodules of ${}_R R$ are just the prime (resp. semiprime) ideals of R. We also note the following connection between prime submodules and prime ideals.

Proposition 2.1. *Let M be a left R-module.*

(a) *If Q is a prime submodule of M, then* $\operatorname{ann}_R(M/Q)$ *is a prime ideal of R.*

(b) *If Q is a semiprime submodule of M, then* $\operatorname{ann}_R(M/Q)$ *is a semiprime ideal of R.*

Proof. (a) Let A, B be ideals of R with $AB \subseteq \operatorname{ann}_R(M/Q)$. Then AM and BM are fully invariant submodules of M, so we have

$$\begin{aligned}(AM)_M(BM) &= \sum \{f(AM) \mid f \in \operatorname{Hom}_R(M, BM)\} \\ &= A\left(\sum \{f(M) \mid f \in \operatorname{Hom}_R(M, BM)\}\right) \\ &\subseteq A(BM) \subseteq Q.\end{aligned}$$

Since Q is a prime submodule, we have either $AM \subseteq Q$ or $BM \subseteq Q$. Thus either $A \subseteq \operatorname{ann}_R(M/Q)$ or $B \subseteq \operatorname{ann}_R(M/Q)$, showing that $\operatorname{ann}_R(M/Q)$ is a prime ideal.

(b) The proof is only a minor modification of the one given for part (a). □

Corollary 2.2. *Let M be a left module over a left Artinian ring R.*

(a) *If Q is a prime submodule of M, then M/Q is semisimple and homogeneous.*

(b) *If Q is a semiprime submodule of M, then M/Q is semisimple.*

Proof. (b) If Q is a semiprime submodule of M, then $\operatorname{ann}_R(M/Q)$ is a semiprime ideal of R by Proposition 2.1(b). Since M/Q is a module over the semisimple Artinian ring $R/\operatorname{ann}_R(M/Q)$, it follows that M/Q is a semisimple module.

(a) If Q is a prime submodule of M, then it follows from the proof of part (b) that M/Q is a semisimple module. Since $\operatorname{ann}_R(M/Q)$ is a prime ideal of R, the ring $R/\operatorname{ann}_R(M/Q)$ has only one isomorphism class of simple modules, so M/Q is homogeneous. □

Definition. The module ${}_RM$ is said to be a *fully prime module* (*resp. fully semiprime module*) if every proper fully invariant submodule $N \subseteq M$ is a prime (resp. semiprime) submodule.

The ring R is called *fully prime* (*resp. fully semiprime*) if ${}_RR$ is a fully prime (resp. semiprime) module.

The reader is cautioned that these definitions are intended to make sense in the category $\sigma[M]$. In the literature, there are several definitions of a prime submodule that seem appropriate in the category *R–Mod*. Of course, a fully prime module (resp. fully semiprime module) is a prime (resp. semiprime) module (in the sense of our definition), and it is a module over a prime (resp. semiprime) ring by Proposition 2.1.

Example 2.3. It follows immediately from Corollary 2.2 that any semiprime module over a left Artinian ring is fully semiprime. Similarly, any prime module over a left Artinian ring is fully prime.

Before investigating fully semiprime modules, we need some preliminary results.

Lemma 2.4. *Let N be a fully invariant submodule of ${}_RM$. If every fully invariant submodule of M is idempotent in M, then every fully invariant submodule of N is idempotent in N.*

Proof. Assume that every fully invariant submodule of M is idempotent, and let $L \subseteq N$ be a fully invariant submodule of N. Then L is easily seen to be fully invariant in M, so by assumption we have $L_M L = L$, but we must show that $L_N L = L$. By the definition of $L_M L$, for each element $y \in L$ there exist $f_i \in \operatorname{Hom}_R(M, L)$ and $x_i \in L$, for $1 \leq i \leq k$, such that $y = \sum_{i=1}^k f_i(x_i)$. For $1 \leq i \leq k$, let g_i be the restriction of f_i to N. Since $L \subseteq N$, we have $g_i(x_i) = f_i(x_i)$ for $1 \leq i \leq k$, and thus $y = \sum_{i=1}^k g_i(x_i)$, showing that $L \subseteq L_N L$. □

Remark 2.5. The proof of Lemma 2.4 can easily be modified to show that if every submodule of M is idempotent, then the same is true for any submodule $N \subseteq M$.

We give the following definition to simplify our earlier notation.

Definition. Let M be a left R-module, and let N, L be submodules of M. We use the following notation:

$$NL^{-1} = \{m \in M \mid f(m) \in N,\ \forall f \in \operatorname{Hom}_R(M, L)\}.$$

Since $NL^{-1} = \mathrm{rej}^{L}_{N\cap L}(M)$, it follows immediately that NL^{-1} is a fully invariant submodule of M.

Lemma 2.6. *The following conditions hold for submodules N, L of ${}_RM$:*
(a) $\left(NL^{-1}\right)_M L \subseteq N$;
(b) *if $K \subseteq M$ is any submodule such that $K_M L \subseteq N$, then $K \subseteq NL^{-1}$;*
(c) $(N \cap L)L^{-1} = NL^{-1}$;
(d) $N \subseteq (N_M L)\, L^{-1}$.
If N is a fully invariant submodule of M, then the following condition holds:
(e) $N \subseteq NL^{-1}$.

Proof. (a) This follows immediately from the definition of NL^{-1}.

(b) If $K_M L \subseteq N$, then $f(K) \subseteq N$ for all $f \in \mathrm{Hom}_R(M, L)$, and thus $K \subseteq NL^{-1}$.

(c) This follows immediately from the definition of $(N \cap L)L^{-1}$, since for $f \in \mathrm{Hom}_R(M, L)$ we have $f(m) \in N \cap L$ if and only if $f(m) \in N$.

(d) For any homomorphism $g \in \mathrm{Hom}_R(M, L)$, we have

$$g(N) \subseteq \sum\{f(N) \mid f \in \mathrm{Hom}_R(M, L)\} = N_M L\,,$$

and so it follows from the definition of $(N_M L)\, L^{-1}$ that $N \subseteq (N_M L)\, L^{-1}$.

(e) Since N is a fully invariant submodule, we have $N_M L \subseteq N \cap L$, so it follows from part (b) that $N \subseteq NL^{-1}$. □

Lemma 2.7. *The equality $((N_M L)L^{-1})_M L = N_M L$ holds for all submodules N, L of M.*

Proof. Since $N_M L \subseteq L$, we have $((N_M L)L^{-1})_M L \subseteq (N_M L) \cap L = N_M L$ by part (a) of Lemma 2.6. On the other hand, $N_M L \subseteq ((N_M L)L^{-1})_M L$ since $N \subseteq (N_M L)\, L^{-1}$ by part (d) of Lemma 2.6. □

Definition. For submodules N, L of ${}_RM$, let

$$N^{-1}L = \sum\{K \mid K \subseteq M \text{ and } N_M K = N_M L\}\,,$$

where K is any submodule of M.

Lemma 2.8. *If ${}_RM$ is projective in $\sigma[M]$, then $N_M\left(N^{-1}L\right) = N_M L$.*

Proof. Since M is projective in $\sigma[M]$, Lemma 2.1 of [6] shows that

$$\begin{aligned} &N_M\left(\sum\{K \mid K \subseteq M \text{ and } N_M K = N_M L\}\right) \\ &= \sum N_M\left(\{K \mid K \subseteq M \text{ and } N_M K = N_M L\}\right), \end{aligned}$$

and so by definition $N_M\left(N^{-1}L\right) = N_M L$. □

3. Fully semiprime modules

With the preliminary lemmas in hand, we are now able to extend results of Courter [8] to the category $\sigma[M]$.

Theorem 3.1. *The following conditions are equivalent for the module* ${}_RM$:

(1) M *is a fully semiprime module*;

(2) $N_M L = N \cap L$ *for all fully invariant submodules* N, L *of* M;

(3) $\left(NL^{-1}\right) \cap L = N \cap L$ *for all fully invariant submodules* N, L *of* M;

(4) $\left(NL^{-1}\right) \cap K = N \cap K$ *for all fully invariant submodules* N, L, K *of* M *such that* $L \supseteq K$;

(5) *every fully invariant submodule of* M *is idempotent.*

Proof. (1) $\Rightarrow$ (2): If N, L are fully invariant submodules of M, then $N_M L$ is fully invariant, and $N_M L \subseteq N \cap L$. If $N_M L = M$, then $N \cap L = M$. Thus we may assume that $N_M L \subset M$, so by assumption it is a semiprime submodule. Then

$$(N \cap L)_M (N \cap L) \subseteq N_M L$$

implies that $N \cap L \subseteq N_M L$.

(2) $\Rightarrow$ (3): Let N, L be fully invariant submodules of M. Since NL^{-1} is fully invariant in M, we have $\left(NL^{-1}\right) \cap L \subseteq \left(NL^{-1}\right)_M L$ by the condition (2). Then $\left(NL^{-1}\right)_M L \subseteq L$ always holds, and $\left(NL^{-1}\right)_M L \subseteq N$ by Lemma 2.6(a).

To show the reverse inclusion, note that since N is fully invariant in M we have $N \subseteq NL^{-1}$ by Lemma 2.6(e), and so $N \cap L \subseteq \left(NL^{-1}\right) \cap L$.

(3) $\Rightarrow$ (4): Let N, L, K be fully invariant submodules of M such that $L \supseteq K$. Then $K = L \cap K$, and so

$$\begin{aligned} \left(NL^{-1}\right) \cap K &= \left(NL^{-1}\right) \cap (L \cap K) = \left(\left(NL^{-1}\right) \cap L\right) \cap K) \\ &= (N \cap L) \cap K = N \cap (L \cap K) = N \cap K. \end{aligned}$$

(4) $\Rightarrow$ (5): Let L be a fully invariant submodule of M. By Lemma 2.6(d) we have $L \subseteq (L_M L)\, L^{-1}$, and so $L \subseteq \left((L_M L)\, L^{-1}\right) \cap L$. Assuming the condition (4) for $N = L_M L$ and $K = L$, we have $L \subseteq \left((L_M L)\, L^{-1}\right) \cap L = (L_M L) \cap L = L_M L$, and it follows that L is an idempotent submodule.

(5) $\Rightarrow$ (1): Assume that every fully invariant submodule of M is idempotent, and let N be a proper fully invariant submodule of M. If L is a fully invariant submodule of M such that $L_M L \subseteq N$, then $L \subseteq N$ since $L = L_M L$. ☐

Corollary 3.2. *The following conditions hold for any fully semiprime module* ${}_RM$:

(a) $N_M L = L_M N$ *for all fully invariant submodules* $N, L \subseteq M$;

(b) *every fully invariant submodule of* M *is* M*-generated*;

(c) *every fully invariant submodule of* M *is a fully semiprime module.*

Proof. (a) This follows immediately from Theorem 3.1(2).

(b) If N is a fully invariant submodule of M, then by part (a) we have $N_M M = M_M N$. Since $N \subseteq N_M M$, we have $N = M_M N$, which shows that N is M-generated.

(c) This follows immediately from Theorem 3.1(5) and Lemma 2.4. $\square$

If N is a direct summand of ${}_RM$, then the natural projection maps M onto N, and so it is clear that N is an idempotent submodule of M. Thus every proper submodule of a semisimple module is idempotent, and so the semisimple R-modules provide our most basic examples of fully semiprime modules. For a left Artinian ring, it follows from Corollary 2.2 that they are the only examples.

Example 3.3. The module ${}_RM$ is said to be *regular* if every cyclic submodule of M is a direct summand of M. It follows from [12, Proposition 2.2] that every regular module is fully idempotent (in the sense that $N_M N = N$ for every submodule N). Thus the class of regular modules provides examples of fully semiprime modules. In fact, [12, Corollary 2.4] shows that if M is a nonzero finitely generated quasi-projective module, then M is regular if and only if it is a fully semiprime module and every prime factor module of M is regular.

Given any module M and any indexing set I, it is known that every fully invariant submodule of $M^{(I)}$ has the form $N^{(I)}$ for some fully invariant submodule N of M. This implies that the lattice of fully invariant submodules of M is isomorphic to the lattice of fully invariant submodules of $M^{(I)}$.

Proposition 3.4. *Let M be a fully semiprime module. Then $M^{(I)}$ is fully semiprime for any index set I.*

Proof. Let $\eta_i : M \to M^{(I)}$ be the canonical inclusions. Let N be a fully invariant submodule of $M^{(I)}$. Since N is fully invariant, there is a fully invariant submodule L of M such that $N = L^{(I)}$. Let $(\ell_i)_{i\in I} \in N = L^{(I)}$. By hypothesis, $L_M L = L$. Then, for each $i \in I$

$$\ell_i = \textstyle\sum_{j=1}^{k_i} f_{ij}(a_{ij})$$

with $f_{ij} \in \mathrm{Hom}_R(M, L)$ and $a_{ij} \in L$. For each $1 \le j \le k$, there exists a unique homomorphism $\oplus_{i\in I} f_{ij} : M^{(I)} \to L^{(I)}$. Let $k = \max\{k_i \mid i \in I\}$. Adding zeros as needed, we have

$$\begin{aligned}(\ell_i)_{i\in I} &= \left(\textstyle\sum_{j=1}^{k_i} f_{ij}(a_{ij})\right) = \textstyle\sum_{j=1}^{k} \left(f_{ij}(a_{ij})\right)_{i\in I}\\ &= \textstyle\sum_{j=1}^{k} \left(\oplus_{i\in I} f_{ij}\right)(a_{ij})_{i\in I} \in \left(L^{(I)}\right)_M \left(L^{(I)}\right).\end{aligned}$$

Thus $N = N_{M^{(I)}} N$. By Theorem 3.1(6), $M^{(I)}$ is fully semiprime. $\square$

Theorem 2.7 of [8] implies that the direct product of two fully semiprime rings is fully semiprime.

Proposition 3.5. *Let ${}_RM$ and ${}_RN$ be fully semiprime modules. Then $M \oplus N$ is a fully semiprime module.*

Proof. If Q is any fully invariant submodule of $M\oplus N$, then it is invariant under the projection mappings onto $M \oplus (0)$ and $(0) \oplus N$. Therefore $Q = K \oplus L$ for $K = \{x \in M \mid (x,0) \in Q\}$ and $L = \{y \in N \mid (0,y) \in Q\}$, and K and L are fully invariant submodules of M and N, respectively. Then

$$\begin{aligned} Q_{M\oplus N}Q &= \sum \{f(K \oplus L) \mid f \in \mathrm{Hom}_R(M \oplus N, K \oplus L)\} \\ &\supseteq \sum \{f(K) \mid f \in \mathrm{Hom}_R(M,K)\} \oplus \sum \{g(L) \mid g \in \mathrm{Hom}_R(N,L)\} \\ &= (K_M K) \oplus (L_N L)\,. \end{aligned}$$

Thus Q is an idempotent submodule since by hypothesis K and L are idempotent submodules of M and N, respectively. It follows from Theorem 3.1(5) that $M \oplus N$ is a fully semiprime module. □

In [8], Courter gives sixteen equivalent conditions for a ring to be fully semiprime, labeled A - P. Our Theorem 3.1 gives the analogs in $\sigma[M]$ of conditions A, C, E, F and H. We have listed the analog of B as Corollary 3.2(c). The analogs of conditions D and G, which appear to require the assumption that M is projective in $\sigma[M]$, are given in Proposition 3.7. Courter's conditions I - L involve right ideals, and in $\sigma[M]$ we have no analog of a right ideal. In Proposition 3.8, we give a condition for R-submodules of M similar to Courter's condition I, but it is much stronger than the other conditions, since it is equivalent to the condition that every submodule of M is idempotent. Courter's conditions M - P involve left ideals, and the analogs of these conditions are given in the following proposition.

Proposition 3.6. *The following conditions are equivalent for the module* ${}_RM$:

(1) *M is a fully semiprime module*;

(6) *$X_M L \supseteq X \cap L$ for all submodules X, L of M such that L is fully invariant in M*;

(7) *$X_M L \supseteq X$ for all submodules $X \subseteq L$ of M such that L is fully invariant in M*;

(8) *$\left(XL^{-1}\right) \cap L \subseteq X \cap L$ for all submodules X, L of M such that L is fully invariant in M*;

(9) *$\left(XL^{-1}\right) \cap L \subseteq X$ for all submodules $X \subseteq L$ of M such that L is fully invariant in M.*

Proof. (1) ⇒ (6): Let X, L be submodules of M, such that L is a fully invariant submodule of M. We have $X \cap L \subseteq \left((X_M L)\, L^{-1}\right) \cap L$ since $X \subseteq (X_M L)\, L^{-1}$ by Lemma 2.6(d). Note that $X_M L$ is a fully invariant submodule of L, and so it is a fully invariant submodule of M. Since M is a fully prime module, by Theorem 3.1(3) we have $\left((X_M L)\, L^{-1}\right) \cap L = (X_M L) \cap L = X_M L$, and it follows that $X \cap L \subseteq X_M L$.

(6) ⇒ (7): Since $X \cap L = X$, it is clear that the condition (7) is a special case of the condition (6).

(7) $\Rightarrow$ (1): Given any fully invariant submodule L of M, that $L \subseteq L_M L$ follows immediately upon taking $X = L$ in the condition (7). Thus the condition (5) of Theorem 3.1 holds, and M is a fully prime module.

(6) $\Rightarrow$ (8): Since Lemma 2.6(a) holds for all submodules, the first part of the proof that (2) implies (3) in Theorem 3.1 remains valid, showing that (6) implies (8).

(8) $\Rightarrow$ (1): It follows from Lemma 2.6(e) that $X \cap L \subseteq \left(XL^{-1}\right) \cap L$, and so condition (3) of Theorem 3.1 is a special case of the condition (8).

(8) $\Leftrightarrow$ (9): If the condition (8) holds and $X \cap L = X$, then the condition (9) holds. On the other hand, if the condition (9) holds, then for any submodule X we have $\left(XL^{-1}\right) \cap L = \left((X \cap L)L^{-1}\right) \cap L \subseteq X \cap L$ by Lemma 2.6(d). □

Proposition 3.7. *If M is projective in $\sigma[M]$, then the following conditions are equivalent for the module ${}_RM$:*

(1) *M is a fully semiprime module*;

(10) *$N \cap \left(N^{-1}L\right) = N \cap L$ for all fully invariant submodules N, L of M*;

(11) *$K \cap \left(N^{-1}L\right) = K \cap L$ for all fully invariant submodules K, N, L of M such that $N \supseteq K$.*

Proof. (1) $\Rightarrow$ (10): It is clear from the definition of $N^{-1}L$ that $L \subseteq N^{-1}L$, and so we always have $N \cap L \subseteq N \cap \left(N^{-1}L\right)$. Lemma 2.8 shows that $N_M\left(N^{-1}L\right) = N_M L$, since M is projective in $\sigma[M]$. Thus

$$N_M L = N_M\left(N^{-1}L\right) \subseteq N \cap \left(N^{-1}L\right) .$$

Since M is fully semiprime, by the condition (2) of Theorem 3.1, we have $N \cap L = N_M L$, and therefore $N \cap L \subseteq N \cap \left(N^{-1}L\right)$.

(10) $\Rightarrow$ (1): Let N, L be fully invariant submodules of M, and assume that the condition (10) holds. Since N is fully invariant, we have $N_M L \subseteq N \cap L$. Since $N_M L$ is a fully invariant submodule of M, we can substitute $N_M L$ for L, and so

$$N \cap \left(N^{-1}\left(N_M L\right)\right) = N \cap \left(N_M L\right) = N_M L \,.$$

It follows from the definition that $L \subseteq N^{-1}\left(N_M L\right)$, and thus

$$N \cap L \subseteq N \cap \left(N^{-1}\left(N_M L\right)\right) = N_M L \,,$$

so the condition (2) of Theorem 3.1 holds.

(10) $\Leftrightarrow$ (11): It is clear that (10) is a special case of (11), and the proof that (10) implies (11) is similar to the proof that (3) implies (4) in Theorem 3.1. □

Proposition 3.8. *For the module ${}_RM$, we have $N_M X \supseteq N \cap X$ for all submodules N, X of M such that N is fully invariant in M if and only if every submodule of M is idempotent.*

Proof. Let X be any submodule of M, and suppose that $N_M X \supseteq N \cap X$ for all submodules N of M such that N is fully invariant in M. Then $X_M X =$

$\left((X_M X)\, X^{-1}\right)_M X$ by Lemma 2.7, and $(X_M X)\, X^{-1} \supseteq X$ by Lemma 2.6(d). Since $(X_M X)\, X^{-1}$ is fully invariant in M, by assumption we have

$$X_M X = \left((X_M X)\, X^{-1}\right)_M X \supseteq \left((X_M X)\, X^{-1}\right) \cap X \supseteq X\,,$$

and so X is an idempotent submodule of M.

Conversely, if every submodule of M is idempotent, and X, Y are any submodules of M, then $X \cap Y = (X \cap Y)_M (X \cap Y) \subseteq X_M Y$, since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. □

4. Fully prime modules

It ${}_R M$ is semisimple and homogeneous, it is clear that its only fully invariant submodules are M and (0). This is the most elementary fully prime module, and following the characterization of fully prime modules in the next theorem we will investigate conditions under which a fully prime module must be semisimple and homogeneous.

Theorem 4.1. *The module ${}_R M$ is a fully prime module if and only if the fully invariant submodules of M are idempotent and totally ordered.*

Proof. First assume that M is a fully prime module. Then in particular it is a fully semiprime module, and so every fully invariant submodule of M is idempotent by Theorem 3.1. Next, let N, L be proper fully invariant submodules of M. Then $N \cap L$ is also fully invariant, and by assumption it is a prime submodule. Since $N_M L \subseteq N \cap L$, it follows that either $N \subseteq N \cap L$, and therefore $N \subseteq L$, or $L \subseteq N \cap L$, which implies that $L \subseteq N$.

To show the converse, assume that the fully invariant submodules of M are totally ordered and idempotent, and let Q be a proper fully invariant submodule of M. Suppose that N, L are fully invariant submodules of M such that $N_M L \subseteq Q$. Since the fully invariant submodules are totally ordered, we have either $N \subseteq L$ or $L \subseteq N$. In the first case, $N_M N \subseteq N_M L$, so $N = N_M N \subseteq Q$ since N is idempotent. In the second case, we $L_M L \subseteq N_M L$, so $L = L_M L \subseteq Q$ since L is idempotent. □

Proposition 4.2. *Let M be a nonzero fully prime module, and let N be a fully invariant submodule of M. Then*

(a) *N is a fully prime module*;

(b) *if M is a quasi-projective module, then M/N is a fully prime module.*

Proof. (a) By Theorem 4.1 the fully invariant submodules of M are totally ordered and idempotent. Since each fully invariant submodule of N is also fully invariant in M, it follows immediately that the fully invariant submodules of N are totally ordered, and they are idempotent by Lemma 2.4.

(b) Let L/N be a proper fully invariant submodule of M/N. By [13, Lemma 17], L is fully invariant in M and so $M/L \cong (M/N)/(L/N)$ is a prime module. Furthermore, M/N is quasi-projective since M is quasi-projective. It follows from [13, Proposition 18] that L/N is a prime submodule of M/N. □

Proposition 4.3. *Let M be a fully prime module. Then $M^{(I)}$ is a fully prime module for any index set I.*

Proof. Since M is fully prime, the fully invariant submodules of M are totally ordered. Therefore, the fully invariant submodules of $M^{(I)}$ are totally ordered. It follows from Proposition 3.4 that every fully invariant submodule of $M^{(I)}$ is idempotent. Thus $M^{(I)}$ is fully prime by Theorem 4.1. □

It is shown in [4] that a left fully bounded left Noetherian ring that is fully prime is simple Artinian, generalizing the fact that a fully prime commutative ring must be a field. We recall that a left Noetherian ring is left fully bounded if and only if every finitely generated left module is finitely annihilated. To extend this notion to $\sigma[M]$, the annihilator in M of a module X is defined as follows:

$$\operatorname{ann}_M(X) = \operatorname{rej}^X_{(0)}(M) = \cap \{\ker(f) \mid f \in \operatorname{Hom}_R(M, X)\} .$$

We say that ${}_RX$ is *finitely M-generated* if there exists an epimorphism from a finite direct sum of copies of M onto X, and that ${}_RX$ is *finitely M-annihilated* if there exists an embedding of $M/\operatorname{ann}_M(X)$ into a finite direct sum of copies of X. Then M is said to satisfy *Gabriel's condition H* if every finitely M-generated module is finitely M-annihilated [1].

The Jacobson radical $J(M)$ of the module M is defined to be the intersection of the maximal submodules of M. If $\mathcal{C}$ is the class of simple modules in $\sigma[M]$, then $J(M) = \operatorname{rad}_{\mathcal{C}}(M)$, where $\operatorname{rad}_{\mathcal{C}}(X)$ is the radical defined as the intersection of the kernels of all homomorphisms from X into a module in $\mathcal{C}$.

Theorem 4.4. *Let M be a nonzero Noetherian module that satisfies Gabriel's condition H. If M is fully prime, then M is a homogeneous semisimple module.*

Proof. First suppose that $J(M) \neq (0)$. Then since M is Noetherian, $J(M)$ is finitely generated, and so it has a maximal submodule. For $f \in \operatorname{Hom}_R(M, J(M))$ we therefore have

$$f(J(M)) = f(\operatorname{rad}_{\mathcal{C}}(J(M)) \subseteq \operatorname{rad}_{\mathcal{C}}(J(M)) \subset J(M) .$$

Since $J(M)$ is a fully invariant submodule of M, this contradicts the assumption that $J(M)$ is an idempotent submodule.

Since M is Noetherian, there exists a maximal submodule $M_1 \subset M$, and then $\operatorname{ann}_M(M/M_1)$ is a fully invariant submodule of M. Since M satisfies Gabriel's condition H and M/M_1 is finitely M-generated, it follows that M/M_1 is finitely M-annihilated, so there exists an embedding of $M/\operatorname{ann}_M(M/M_1)$ into a direct sum $(M/M_1)^n$ of copies of M/M_1. It follows that $M/\operatorname{ann}_M(M/M_1)$ is a homogeneous semisimple module.

Suppose that $M_2 \subset M$ is a maximal submodule of M such that M/M_2 is not isomorphic to M/M_1. Then $\operatorname{ann}_M(M/M_2)$ is a fully invariant submodule of M, so by assumption either $\operatorname{ann}_M(M/M_2) \subseteq \operatorname{ann}_M(M/M_1)$ or $\operatorname{ann}_M(M/M_1) \subseteq \operatorname{ann}_M(M/M_2)$. If $\operatorname{ann}_M(M/M_2) \subseteq \operatorname{ann}_M(M/M_1)$, then the projection mapping is a nonzero homomorphism from $M/\operatorname{ann}_M(M/M_2)$ onto $M/\operatorname{ann}_M(M/M_1)$. This is a contradiction since both modules are homogeneous semisimple and M/M_2 is not isomorphic to M/M_1. We conclude that the intersection of all maximal submodules of M is $\operatorname{ann}_M(M/M_1)$, so $J(M) = \operatorname{ann}_M(M/M_1)$, and thus $M = M/J(M)$ is a homogeneous semisimple module. □

As noted in [4], a commutative ring is fully prime if and only if it is a field. The following corollary shows, in particular, that an analogous result holds for finitely generated modules over a commutative Noetherian ring: in this case ${}_RM$ is a fully prime module if and only if $\operatorname{ann}_R(M)$ is a maximal ideal and M is a finite dimensional vector space over $R/\operatorname{ann}_R(M)$.

It is proved in [2] that if R is finitely generated as a module over a Noetherian subring S of its center, then any finitely generated module satisfies Gabriel's condition H. For the reader's convenience, we include an outline of the proof. Suppose that ${}_RM$ is finitely generated, and ${}_RN$ is a finitely M-generated module. Then ${}_SN$ is Noetherian since ${}_SR$ is Noetherian, and it can be shown that $\operatorname{Hom}_R(M, N)$ is a finitely generated S-module. A brief argument then shows that N is finitely M-annihilated.

Corollary 4.5. *Let R be a ring that is finitely generated as a module over a Noetherian subring of its center, and let M be a finitely generated R-module. Then M is a fully prime module if and only if $\operatorname{ann}_R(M)$ is a maximal ideal and M is a finitely generated homogeneous semisimple module over $R/\operatorname{ann}_R(M)$.*

Proof. Since M satisfies condition H by the result from [2] quoted above, it follows from Theorem 4.4 that M is a homogeneous semisimple module. Therefore $R/\operatorname{ann}_R(M)$ is a simple Artinian ring, since R satisfies condition H, and so $\operatorname{ann}_R(M)$ is a maximal ideal of R. The converse is clear. □

The following example shows that in Corollary 4.5 the hypothesis that M is finitely generated is necessary.

Example 4.6. Over the ring of integers $\mathbb{Z}$, consider the group $\mathbb{Q}$ of rational numbers. Given nonzero elements $a, b \in \mathbb{Q}$, it is easy to construct an automorphism of $\mathbb{Q}$ that maps a to b. It follows that $\mathbb{Q}$ has no proper nontrivial fully invariant submodules, and so $\mathbb{Q}$ is a fully prime $\mathbb{Z}$-module that is not semisimple.

Definition. A proper fully invariant submodule P of M is said to be *primitive* if $P = \operatorname{ann}_M(S)$ for some simple module S.

Proposition 4.7. *Let M be a quasi-projective Noetherian module. If M is a fully prime module, then every proper fully invariant submodule of M is an intersection of primitive submodules.*

Proof. Since M is Noetherian and fully prime, as in the proof of Theorem 4.4 we have $J(M) = 0$. That is, 0 is an intersection of primitive submodules. Given a fully invariant submodule N of M, since M/N is Noetherian and quasi-projective, we have $J(M/N) = 0$, and so M/N is semiprimitive. This implies that N is an intersection of primitive submodules of M. $\square$

Corollary 4.8. *Let R be a left Noetherian ring. If R is fully prime, then every ideal of R is semiprimitive.*

We note that, in particular, any left Noetherian fully prime ring is a Jacobson ring (i.e., every prime ideal of R is semiprimitive).

5. Endomorphism rings, and Morita invariance

Proposition 5.1. *Let $F : R\text{–}Mod \to S\text{–}Mod$ be a category equivalence, and let M be an R-module. If N, L are fully invariant submodules of M, then $F(N_M L) = F(N)_{F(M)} F(L)$.*

Proof. The equivalence F induces a natural isomorphism

$$\mathrm{Hom}_S(F(M), F(L)) \cong \mathrm{Hom}_R(M, L)\,.$$

Since F preserves direct sums and images, we have the following calculation:

$$\begin{aligned} F(N)_{F(M)} F(L) &= \sum \{g(F(N)) \mid g \in \mathrm{Hom}_S(F(M), F(L))\} \\ &= \sum \{F(f)(F(N)) \mid f \in \mathrm{Hom}_R(M, L)\} \\ &= \sum \{F(f(N)) \mid f \in \mathrm{Hom}_R(M, L)\} \\ &= F\left(\sum \{f(N) \mid f \in \mathrm{Hom}_R(M, L)\}\right) \\ &= F\left(N_M L\right), \end{aligned}$$

which completes the proof. $\square$

Corollary 5.2. *Let $F : R\text{–}Mod \to S\text{–}Mod$ be an equivalence, and let M be a left R-module. If P is a prime (semiprime) submodule of M, then $F(P)$ is a prime (semiprime) submodule of $F(M)$.*

Proposition 5.3. *Let $F : R\text{–}Mod \to S\text{–}Mod$ be an equivalence, and let M be an R-module. If M is a fully prime (fully semiprime) left R-module, then $F(M)$ is a fully prime (fully semiprime) left S-module.*

Proof. There is a bijection between the fully invariant submodules of M and the fully invariant submodules of $F(M)$. It follows from Corollary 5.2 that if M is a fully prime or fully semiprime module, then so is $F(M)$. $\square$

Theorem 5.4. *Let M be a finitely generated quasi-projective module. If M is a fully semiprime module, then $S = \mathrm{End}_R(M)$ is a fully semiprime ring.*

Proof. Since M is finitely generated and quasi-projective, $\mathrm{Hom}_R(M, IM) = I$ for every ideal of S by [15, 18.4]. Hence

$$(IM)_M(IM) = \mathrm{Hom}_R(M, IM)(IM) = I^2M$$

for any ideal I of S. This implies that

$$I = \mathrm{Hom}_R(M, IM) = \mathrm{Hom}_R(M, (IM)_M(IM)) = \mathrm{Hom}_R(M, I^2M) = I^2$$

for any ideal of S. □

Corollary 5.5. *Let R be a fully semiprime ring. Then $M_n(R)$ is a fully semiprime ring for any $n > 0$.*

Proposition 5.6. *Let e be an idempotent element of a ring R. If R is a fully semiprime ring, then so is eRe.*

Proof. Let $e \in R$ be idempotent, and let I be an ideal of eRe. By [11, 21.11], we have $I = e(RIR)e$. Therefore

$$I = e(RIR)e = e(RIRRIR)e = e(RI^2R)e = I^2\,,$$

completing the proof. □

Corollary 5.7. *The property of being a fully semiprime ring is a Morita invariant property.*

We note that in [4, Theorem 2.1, Theorem 2.3] it is proved that if R is a fully prime ring, then $M_n(R)$ is a fully prime ring for all $n > 0$, and if $e \in R$ is an idempotent element, then eRe is fully prime.

Theorem 5.8. *Let R be a fully prime ring, and let P be a finitely generated projective R-module. Then $\operatorname{End}_R(P)$ is a fully prime ring.*

Proof. There is a positive integer n such that $R^n = P \oplus Q$. It follows that

$$M_n(R) \cong \begin{bmatrix} \operatorname{End}_R(P) & \operatorname{Hom}_R(Q,P) \\ \operatorname{Hom}_R(P,Q) & \operatorname{End}_R(Q) \end{bmatrix}.$$

Then there exists an idempotent $e \in M_n(R)$ such that $eM_n(R)e \cong \operatorname{End}_R(P)$. Therefore $\operatorname{End}_R(P)$ is a fully prime ring by [4, Theorem 2.1, Theorem 2.3]. □

Theorem 5.9. *Let M be an R-module and let $S = \operatorname{End}_R(M)$. Suppose that every fully invariant submodule of M is M-generated. If S is a fully prime ring, then M is a fully prime module.*

Proof. Let N be a proper fully invariant submodule of M. By hypothesis, $\operatorname{Hom}_R(M,N)$ is a prime ideal of S. It follows from [5, proposition 1.8] that N is a prime submodule. Thus M is fully prime. □

Example 5.10. Let R be a fully prime ring, and let ${}_RP$ be a finitely generated projective generator. Then every submodule of P is P-generated, and so it follows from Theorem 5.9 that P is a fully prime module, because $\operatorname{End}_R(P)$ is a fully prime ring by Theorem 5.8.

On the other hand, it need not be true that *every* finitely generated left R-module is fully prime. In fact, if R is a simple ring with two non-isomorphic simple modules S_1 and S_2, then R is a fully prime ring but $S_1 \oplus S_2$ is not fully prime since the fully invariant submodules $S_1 \oplus (0)$ and $(0) \oplus S_2$ are incomparable.

The following example shows that if M does not generate its fully invariant submodules, then the conclusion of Theorem 5.9 may not be true.

Example 5.11. Consider the ring $R = \left[\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{smallmatrix}\right]$. We have the following decomposition of R, as left a R-module:

$$R = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}.$$

Set $M = \left[\begin{smallmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{smallmatrix}\right]$. Then $\mathrm{End}_R(M) \cong \mathbb{Z}_2$. Hence $\mathrm{End}_R(M)$ is a fully prime ring. On the other hand,

$$\begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}_M \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} = 0.$$

Thus M is not fully prime.

The following proposition is a partial converse of Theorem 5.9.

Theorem 5.12. *Let M be a finitely generated quasi-projective module. If M is fully prime, then $S = \mathrm{End}_R(M)$ is a fully prime ring.*

Proof. Since M is finitely generated and quasi-projective, it is shown in [15, 18.4] that $\mathrm{Hom}_R(M, IM) = I$ for every ideal of S. Let I and J be ideals of S. Then

$$(JM)_M(IM) = \mathrm{Hom}_R(M, IM)JM = IJM\,.$$

We claim that if N is a prime submodule of M, then $\mathrm{Hom}_R(M, N)$ is a prime ideal of S. Let I and J be ideals of S such that $IJ \subseteq \mathrm{Hom}_R(M, N)$. Then $IJM \subseteq \mathrm{Hom}_R(M, N)M \subseteq N$. Since $(JM)_M(IM) = IJM$, either $IM \subseteq N$ or $JM \subseteq N$. Therefore $J = \mathrm{Hom}_R(M, JM) \subseteq \mathrm{Hom}_R(M, N)$ or $I = \mathrm{Hom}_R(M, IM) \subseteq \mathrm{Hom}_R(M, N)$, proving the claim. Given an ideal I of S, it follows that IM is a prime submodule of M, and hence $I = \mathrm{Hom}_R(M, IM)$ is a prime ideal of S. $\square$

Corollary 5.13 ([4, Theorem 2.1])**.** *Let R be a fully prime ring. Then $M_n(R)$ is a fully prime ring for any $n > 0$.*

Proof. By Proposition 4.3, R^n is a fully prime module. It follows from Theorem 5.12 that $\mathrm{End}_R(R^n) \cong M_n(R)$ is a fully prime ring. $\square$

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