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REMARKS ON LIOUVILLE TYPE THEOREMS FOR THE 3D STATIONARY MHD EQUATIONS

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Abstract. The aim of this paper is to establish Liouville type results for the stationary MHD equations. In particular, we show that the velocity and magnetic field, belonging to some Lorentz spaces, must be zero. Moreover, we also obtain Liouville type theorem for the case of axially symmetric MHD equations. Our results generalize previous works by Schulz [14] and Seregin-Wang [18].

1. Introduction

MHD equations are a combination of Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In this paper, we investigate the following three-dimensional steady-state incompressible MHD equations

$$\begin{cases} \boldsymbol{u}\cdot\nabla\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla p=\boldsymbol{H}\cdot\nabla\boldsymbol{H}, \\ \boldsymbol{u}\cdot\nabla\boldsymbol{H}-\boldsymbol{H}\cdot\nabla\boldsymbol{u}=\Delta\boldsymbol{H}, \\ \operatorname{div}\boldsymbol{u}=0, \\ \operatorname{div}\boldsymbol{H}=0. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u}(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$, $p(x,t)$ denote the velocity and pressure of the fluid respectively, and $\boldsymbol{H}(x,t)=(H_1(x,t),H_2(x,t),H_3(x,t))$ is the magnetic field vector. The MHD equations play an important role in astrophysics, geophysics and cosmology, for more details see [3,13].

Recently, there are many works on the well-posedness theory for the classical MHD equations. We refer the reader to interesting papers [7,10] and references therein.

Liouville type theorems for partial differential equations have drawn much attention. Actually, Liouville type theorems naturally appear when considering the regularity of solutions to partial differential equations. When $\boldsymbol{H}=\mathbf{0}$, (1.1)

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reduces to the stationary incompressible Navier-Stokes equations and Galdi in [4] proved that the condition

$$\boldsymbol{u} \in L^{\frac{9}{2}}(\mathbb{R}^3)$$

implies $\boldsymbol{u} \equiv 0$. In [2], Chae-wolf showed a logarithmic improvement of Galdi's result under the assumption

$$N(u) := \int_{\mathbb{R}^3} |u|^{\frac{9}{2}} \left\{ \log\left(2 + \frac{1}{|u|}\right) \right\}^{-1} dx < \infty.$$

Moreover, the authors in [8, 18] extend the result of [4] to the Lorentz spaces. Since the convective terms are similar in the Navier-Stokes equations and MHD equations, many researchers also consider the Liouville type theorems for (1.1). Chae and Weng [1] showed that the smooth solution $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$ to (1.1) provided

$$(\boldsymbol{u}, \boldsymbol{H}) \in L^3(\mathbb{R}^3) \quad \text{and} \quad (\nabla \boldsymbol{u}, \nabla \boldsymbol{H}) \in L^2(\mathbb{R}^3).$$

Another interesting result (see [14]) pointed out that the condition

$$(\boldsymbol{u}, \boldsymbol{H}) \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$$

implies $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$ as well. More references, we recommend [9, 15, 17].

Motivated by [18], we consider the velocity and magnetic field belongs to some Lorentz spaces. It is a natural way to extend the space widely and improve the previous results. Compared with the result in [14], we relax the restriction that $(\boldsymbol{u}, \boldsymbol{H}) \in L^6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$. For the case of axially symmetric MHD equations, we also show that the condition $|(\boldsymbol{u}(x), \boldsymbol{H}(x))| \leqslant \frac{C}{(1+|x'|)^{\mu}}$ with $x' = (x_1, x_2)$ and $\mu \approx 0.67$ implies $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$.

Define

$$M_{\gamma,q,\ell}((\boldsymbol{u}, \boldsymbol{H}), R) := R^{\gamma - \frac{3}{q}} \|(\boldsymbol{u}, \boldsymbol{H})\|_{L^{q,\ell}(B(R) \setminus B(\frac{R}{2}))},$$

where $B(R)$ is a ball of radius R centered at the origin. Our main results can be stated as follows.

Theorem 1.1. *Let $(\boldsymbol{u}, \boldsymbol{H})$ be a smooth solution to* (1.1).

(i) *For $q > 3$, $3 \leqslant \ell \leqslant \infty$, (or $q = \ell = 3$), if $\gamma > \frac{2}{3}$, assume*

$$\liminf_{R \to \infty} M_{\gamma,\,q,\,\ell}((\boldsymbol{u}, \boldsymbol{H}), R) < \infty, \tag{1.2}$$

then $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$; If $\gamma = \frac{2}{3}$, under (1.2) *and for some $0 < \delta < \frac{1}{C(q,\ell)}$,*

$$\liminf_{R \to \infty} M^3_{\frac{2}{3},q,\ell}((\boldsymbol{u}, \boldsymbol{H}), R) \leqslant \delta \int_{\mathbb{R}^3} |\nabla(\boldsymbol{u} + \boldsymbol{H})|^2 \, dx, \tag{1.3}$$

then $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$.

(ii) *For $\frac{12}{5} < q < 3$, $1 \leqslant \ell \leqslant \infty$, $\gamma > \frac{1}{3} + \frac{1}{q}$, assume that*

$$\liminf_{R \to \infty} M_{\gamma,\,q,\,\ell}((\boldsymbol{u}, \boldsymbol{H}), R) < \infty, \tag{1.4}$$

then $(\boldsymbol{u}, \boldsymbol{H}) \equiv \boldsymbol{0}$.

Theorem 1.2. *Let* $(\boldsymbol{u}, \boldsymbol{H})$ *be an axially symmetric smooth solution to* (1.1) *and satisfy*

$$|(\boldsymbol{u}(x),\, \boldsymbol{H}(x))| \leqslant \frac{C}{(1+|x'|)^{\mu}} \tag{1.5}$$

for any $x=(x',\, x_3)$, $\mu > \frac{2}{3}$. *Then* $(\boldsymbol{u},\, \boldsymbol{H}) \equiv \boldsymbol{0}$.

Remark 1.1. If let $q=\ell=3$ and assume $(\boldsymbol{u},\, \boldsymbol{H}) \in L^3(\mathbb{R}^3)$, we see that $M_{\gamma,\,3,\,3} \to 0$ as $R\to\infty$ for every $\gamma \in [\frac{2}{3}, 1]$. Hence, Chae and Weng's result in [1] follows from Theorem 1.1.

Remark 1.2. For the case of Navier–Stokes equations, there are also some results similar to Theorem 1.2, see [16, 19].

The rest of this paper is organized as follows. In Section 2, we recall the definition of Lorentz spaces and the related inequalities. In Section 3, we obtain the Caccioppoli type inequality, which is the key of our proof. Finally, we will give the complicate proof of Theorems 1.1–1.2, respectively in Sections 4–5.

2. Preliminaries

For convenience of the readers, we will describe some basic function spaces and useful inequalities which will be used later.

Let us recall the definition of Lorentz spaces. Let $1 \le p < \infty$, $1 \le q \le \infty$. We say that a measurable function f belongs to the Lorentz spaces $L^{p,q}(\mathbb{R}^3)$ if $\|f\|_{L^{p,q}(\mathbb{R}^3)} < +\infty$, where

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} := \begin{cases} \left(\int_0^\infty t^{q-1}|\{x\in\mathbb{R}^3 : |f(x)|>t\}|^{\frac{q}{p}}\,dt\right)^{\frac{1}{q}}, & \text{if} \quad q<+\infty,\\ \sup\limits_{t>0} t|\{x\in\mathbb{R}^3 : |f(x)|>t\}|^{\frac{1}{p}}, & \text{if} \quad q=+\infty. \end{cases}$$

It is well known that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm, namely, $\|\cdot\|_{L^{p,q}}$ do not satisfy the usual triangle inequality. Instead, we have

$$\|f+g\|_{L^{p,q}} \le C(p,q)(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}}),$$

where $C(p,q) = 2^{1/p}\max(1, 2^{(1-q)/q})$. For details see [11].

We also need the Hölder inequality and the Calderón-Zygmund inequality in Lorentz spaces.

Lemma 2.1 (see [12]). *Let* $f \in L^{p_1,q_1}(\mathbb{R}^3)$ *and* $g\in L^{p_2,q_2}(\mathbb{R}^3)$ *with* $1\le p_1, p_2 \le \infty$, $1\le q_1, q_2\le\infty$. *Then* $fg \in L^{p,q}(\mathbb{R}^3)$ *with* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{q}\le\frac{1}{q_1}+\frac{1}{q_2}$ *and the Hölder inequality in Lorentz spaces*

$$\|fg\|_{L^{p,q}(\mathbb{R}^3)} \le C\|f\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|g\|_{L^{p_2,q_2}(\mathbb{R}^3)} \tag{2.1}$$

holds true for a constant $C>0$.

Lemma 2.2 (See [6]). *Let Ω be a bounded domain in $\mathbb{R}^n$, $f \in L^p(\Omega)$, $1 < p < \infty$, and w the Newtonian potential of f. Then $w \in W^{2,p}(\Omega)$, $\Delta w = f$ a.e. Ω, and*

$$\int_\Omega |\nabla^2 w|^p dx \leqslant C \int_\Omega |f|^p dx,$$

where constant $C > 0$ only dependents on n and p.

3. Caccioppoli type inequalities

We begin with an auxiliary proposition about Caccioppoli type inequality, which is the key of our proof.

Proposition 3.1. *Let $(\boldsymbol{u}, \boldsymbol{H})$ be a smooth solution to (1.1) and $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{H}$, $\boldsymbol{T} = \boldsymbol{u} - \boldsymbol{H}$. Then the following Caccioppoli type inequalities hold*:

(i) *for $q > 3$, $3 \leqslant \ell \leqslant \infty$,*

$$\int_{B(\frac{R}{2})} |\nabla \boldsymbol{v}|^2 \, dx \leqslant CR^{-2} \int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{v}|^2 \, dx + D_1, \tag{3.1}$$

where $D_1 := C(q,\ell) R^{2-\frac{9}{q}} \|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))} \|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}$;

(ii) *for $0 < \delta \leqslant 1$, $6\,\frac{3-\delta}{6-\delta} < q < 3$,*

$$\int_{B(\frac{R}{2})} |\nabla \boldsymbol{v}|^2 \, dx \leqslant CR^{-2} \int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{v}|^2 \, dx + D_2, \tag{3.2}$$

where $D_2 := C(\delta)\Big(R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}} \|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))} \|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}$.

Proof. Given $R > 0$, fix two numbers ϱ and r so that $\frac{3R}{4} \leqslant \varrho < r \leqslant R$. Now, choose a cut–off function $\varphi \in C_0^\infty(B(R))$ satisfying the following conditions:

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in B(\varrho), \\ 0, & \text{if } x \in B(r)^c, \end{cases}$$

$0 \leqslant \varphi \leqslant 1$ and $|\nabla\varphi(x)| \leqslant \frac{c}{(r-\varrho)}$.

We consider the following Dirichlet problem

$$\begin{cases} \Delta\psi = \operatorname{div}(\varphi \boldsymbol{v}) & \text{in } B(R) \setminus B(\frac{2r}{3}), \\ \psi = 0 & \text{on } \partial B(R) \cup \partial B(\frac{2r}{3}). \end{cases}$$

From the standard elliptic equations theory, there exists a unique solution $\psi \in W_0^{1,s}(B(R)\setminus B(\frac{2r}{3})) \cap W^{2,s}(B(R)\setminus B(\frac{2r}{3}))$. Therefore $\boldsymbol{w} = \nabla\psi \in W_0^{1,s}(B(R)\setminus B(\frac{2r}{3}))$ such that $\operatorname{div} \boldsymbol{w} = \operatorname{div}(\varphi\boldsymbol{v}) = \nabla\varphi \cdot \boldsymbol{v}$. Applying Lemma 2.2, we can deduce

$$\begin{aligned}\int_{B(R)\setminus B(\frac{2r}{3})} |\nabla \boldsymbol{w}|^s\,dx &= \int_{B(R)\setminus B(\frac{2r}{3})} |\nabla^2\psi|^s\,dx\\ &\leqslant C\int_{B(R)\setminus B(\frac{2r}{3})} |\Delta\psi|^s\,dx\\ &= C\int_{B(R)\setminus B(\frac{2r}{3})} |\nabla\varphi\cdot\boldsymbol{v}|^s\,dx\\ &\leqslant \frac{C}{(r-\varrho)^s}\int_{B(R)\setminus B(\frac{2r}{3})} |\boldsymbol{v}|^s\,dx,\end{aligned}\tag{3.3}$$

where C is independent of R and only depends on s $(1 < s < \infty)$.

Based on the general Marcinkiewicz interpolation theorem [12], it infers

$$\|\nabla \boldsymbol{w}\|_{L^{q,\ell}(B(r)\setminus B(\frac{2r}{3}))} \leq C(q)\|\nabla\varphi\cdot\boldsymbol{v}\|_{L^{q,\ell}(B(r)\setminus B(\frac{2r}{3}))}\tag{3.4}$$

for any $1 < q < \infty$ and $1 \leq \ell \leq \infty$.

Adding the equations $(1.1)_1$ and $(1.1)_2$, $(1.1)_3$ and $(1.1)_4$ respectively, we obtain

$$\begin{cases}\boldsymbol{T}\cdot\nabla\boldsymbol{v} - \Delta\boldsymbol{v} = -\nabla p,\\ \operatorname{div}\boldsymbol{v} = 0.\end{cases}\tag{3.5}$$

Multiplying $(3.5)_1$ by $(\varphi\boldsymbol{v} - \boldsymbol{w})$, and integrating by parts over $B(r)$, it follows

$$\begin{aligned}\int_{B(r)} \varphi|\nabla\boldsymbol{v}|^2\,dx = &-\int_{B(r)} \nabla\boldsymbol{v} : (\nabla\varphi\otimes\boldsymbol{v})\,dx + \int_{B(r)} \nabla\boldsymbol{w} : \nabla\boldsymbol{v}\,dx\\ &-\int_{B(r)} (\boldsymbol{T}\cdot\nabla\boldsymbol{v})\cdot\varphi\boldsymbol{v}\,dx + \int_{B(r)} (\boldsymbol{T}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{w}\,dx\\ =& \sum_{i=1}^{4} I_i.\end{aligned}\tag{3.6}$$

Noticing that $R \geqslant r > \varrho \geqslant \frac{3R}{4} > \frac{2r}{3} > \frac{R}{2}$, Hölder's inequality gives

$$\begin{aligned}|I_1| &= \Big|\int_{B(r)} \nabla\boldsymbol{v} : (\nabla\varphi\otimes\boldsymbol{v})\,dx\Big|\\ &\leqslant C\Big(\int_{B(r)} |\nabla\boldsymbol{v}|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{B(r)\setminus B(\varrho)} |\nabla\varphi\otimes\boldsymbol{v}|^2\,dx\Big)^{\frac{1}{2}}\\ &\leqslant \frac{C}{r-\varrho}\Big(\int_{B(r)} |\nabla\boldsymbol{v}|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{v}|^2\,dx\Big)^{\frac{1}{2}}.\end{aligned}\tag{3.7}$$

By (3.3), we deduce

$$
\begin{aligned}
|I_2| &= \Big| \int_{B(r)} \nabla \boldsymbol{w} : \nabla \boldsymbol{v} \, dx \Big| \\
&\leqslant C\Big(\int_{B(r)} |\nabla \boldsymbol{v}|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{B(r)\setminus B(\frac{2r}{3})} |\nabla \boldsymbol{w}|^2 \, dx \Big)^{\frac{1}{2}} \qquad (3.8) \\
&\leqslant \frac{C}{r-\varrho} \Big(\int_{B(r)} |\nabla \boldsymbol{v}|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{v}|^2 \, dx \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Let us prove (3.1). To estimate I_3 and I_4, we are going to use the fact that $\operatorname{div} \boldsymbol{T} = \operatorname{div} \boldsymbol{u} - \operatorname{div} \boldsymbol{H} = 0$. Integration by parts gives

$$
\begin{aligned}
I_3 = - \int_{B(r)} (\boldsymbol{T} \cdot \nabla \boldsymbol{v}) \cdot \varphi \boldsymbol{v} \, dx &= -\frac{1}{2} \int_{B(r)} \varphi \, \operatorname{div}(\boldsymbol{T} |\boldsymbol{v}|^2) \, dx \\
&= \frac{1}{2} \int_{B(r)} (\boldsymbol{T} |\boldsymbol{v}|^2) \cdot \nabla \varphi \, dx.
\end{aligned}
$$

Using Lemma 2.1, assuming that $q > 3$ and $\ell \geqslant 3$, we have

$$
\begin{aligned}
|I_3| &= \Big| \frac{1}{2} \int_{B(r)} (\boldsymbol{T} |\boldsymbol{v}|^2) \cdot \nabla \varphi \, dx \Big| \\
&\leqslant \frac{C}{(r-\varrho)} \int_{B(r)\setminus B(\varrho)} |\boldsymbol{T}| |\boldsymbol{v}|^2 \, dx \qquad (3.9) \\
&\leqslant \frac{C}{(r-\varrho)} \|\boldsymbol{T}\|_{L^{q,\ell}(B(r)\setminus B(\varrho))} \|\boldsymbol{v}\|^2_{L^{q,\ell}(B(r)\setminus B(\varrho))} \|1\|_{L^{\frac{q}{q-3}, \frac{\ell}{\ell-3}}(B(R)\setminus B(\frac{R}{2}))} \\
&\leqslant \frac{C(q,\ell)}{(r-\varrho)} R^{3-\frac{9}{q}} \|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))} \|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}.
\end{aligned}
$$

Thanks to (3.4), I_4 can be evaluated as follow:

$$
\begin{aligned}
|I_4| &= \Big| \int_{B(r)} (\boldsymbol{T} \cdot \nabla \boldsymbol{v}) \cdot \boldsymbol{w} \, dx \Big| = \Big| \int_{B(r)\setminus B(\frac{2}{3}r)} (\boldsymbol{T} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} \, dx \Big| \\
&\leqslant C \|\boldsymbol{T}\|_{L^{q,\ell}(B(r)\setminus B(\varrho))} \|\boldsymbol{v}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))} \|\nabla \boldsymbol{w}\|_{L^{q,\ell}(B(r)\setminus B(\frac{2}{3}r))} \\
&\qquad \times \|1\|_{L^{\frac{q}{q-3}, \frac{\ell}{\ell-3}}(B(R)\setminus B(\frac{R}{2}))} \qquad (3.10) \\
&\leqslant \frac{C}{(r-\varrho)} \|\boldsymbol{T}\|_{L^{q,\ell}(B(r)\setminus B(\varrho))} \|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))} \|1\|_{L^{\frac{q}{q-3}, \frac{\ell}{\ell-3}}(B(R)\setminus B(\frac{R}{2}))} \\
&\leqslant \frac{C(q,\ell)}{(r-\varrho)} R^{3-\frac{9}{q}} \|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))} \|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}.
\end{aligned}
$$

Thus, inserting (3.7)-(3.10) into (3.6) leads to

$$
\begin{aligned}
\int_{B(\varrho)} |\nabla \boldsymbol{v}|^2 \, dx &\leqslant \int_{B(r)} \varphi |\nabla \boldsymbol{v}|^2 \, dx \\
&\leqslant \frac{C}{r-\varrho} \Big(\int_{B(r)} |\nabla \boldsymbol{v}|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{v}|^2 \, dx \Big)^{\frac{1}{2}} \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C(q,\ell)}{(r-\varrho)}R^{3-\frac{9}{q}}\|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\\
\leqslant{}& \frac{1}{2}\int_{B(r)}|\nabla\boldsymbol{v}|^2\,dx+\frac{C}{(r-\varrho)^2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\\
&+\frac{C(q,\ell)}{(r-\varrho)}R^{3-\frac{9}{q}}\|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}.
\end{aligned}
$$

Using the standard iteration argument (see [5], Lemma 3.1), for reader's convenience, we give a sketch. Let us first define $\rho_0 := \frac{R}{2}$, $\rho_{k+1} := \rho_k + (1-\tau)\tau^k\frac{R}{2}$, where $k = 0, 1, 2, \dots$ and $\tau \in (\frac{\sqrt{2}}{2}, 1)$. Applying (3.11), we get

$$
\begin{aligned}
\int_{B(\varrho_k)}|\nabla\boldsymbol{v}|^2\,dx \leqslant{}& \frac{1}{2}\int_{B(\rho_{k+1})}|\nabla\boldsymbol{v}|^2\,dx+\frac{4C}{[(1-\tau)\tau^kR]^2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\\
&+\frac{2C(q,\ell)}{(1-\tau)\tau^kR}R^{3-\frac{9}{q}}\|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}.
\end{aligned}
\tag{3.12}
$$

By iteration from (3.12), we have

$$
\begin{aligned}
&\int_{B(\varrho_0)}|\nabla\boldsymbol{v}|^2\,dx\\
\leqslant{}& (\frac{1}{2})^k\int_{B(\rho_k)}|\nabla\boldsymbol{v}|^2\,dx+\frac{4C}{(1-\tau)^2}\sum_{m=0}^{k-1}(\frac{1}{2\tau^2})^m\frac{1}{R^2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\\
&+\frac{2C(q,\ell)}{(1-\tau)}\sum_{m=0}^{k-1}(\frac{1}{2\tau})^mR^{2-\frac{9}{q}}\|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}.
\end{aligned}
$$

Letting $k\to\infty$, we arrive at

$$
\int_{B(\frac{R}{2})}|\nabla\boldsymbol{v}|^2\,dx\leqslant CR^{-2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx+D_1,
$$

where $D_1 := C(q,\ell)R^{2-\frac{9}{q}}\|\boldsymbol{T}\|_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}$. This completes the proof of (3.1).

Next, let us prove (3.2). We only need to re-estimate I_3 and I_4. To the end, we introduce $\overline{\boldsymbol{v}} = \boldsymbol{v} - [\boldsymbol{v}]_{B(r)\setminus B(\frac{2r}{3})}$, where $[\boldsymbol{v}]_\Omega$ is the mean value of $\boldsymbol{v}$ over a domain Ω. Thanks to the integration by parts, we know

$$
\begin{aligned}
I_3 &= -\frac{1}{2}\int_{B(r)}(\boldsymbol{T}\cdot\nabla|\boldsymbol{v}|^2)\,\varphi\,dx\\
&= -\frac{1}{2}\int_{B(r)}\big(\boldsymbol{T}\cdot\nabla(|\boldsymbol{v}|^2-|[\boldsymbol{v}]_{B(r)\setminus B(\frac{2r}{3})}|^2)\big)\,\varphi\,dx\\
&= -\frac{1}{2}\int_{B(r)}\varphi\,\mathrm{div}\,\big(\boldsymbol{T}\,(|\boldsymbol{v}|^2-|[\boldsymbol{v}]_{B(r)\setminus B(\frac{2r}{3})}|^2)\big)\,dx\\
&= \frac{1}{2}\int_{B(r)\setminus B(\varrho)}(\boldsymbol{T}\cdot\nabla\varphi)(|\boldsymbol{v}|^2-|[\boldsymbol{v}]_{B(r)\setminus B(\frac{2r}{3})}|^2)\,dx
\end{aligned}
\tag{3.13}
$$

and, since $\frac{R}{2} < \frac{2r}{3} < \frac{3R}{4} \leqslant \varrho$,

$$|I_3| \leqslant \frac{C}{r-\varrho}\int_{B(r)\setminus B(\frac{2r}{3})} \big|\boldsymbol{T}\big|\,\big|\overline{\boldsymbol{v}}\big|\,\big|\boldsymbol{v}+[\boldsymbol{v}]_{B(r)\setminus B(\frac{2r}{3})}\big|\,dx. \tag{3.14}$$

By the assumptions $0<\delta\leqslant 1$, $\frac{6(3-\delta)}{6-\delta} < q < 3$, it implies

$$0<\beta := 1-\frac{3-\delta}{q}-\frac{\delta}{6} < 1.$$

In virtue of Lemma 2.1, we show

$$\begin{aligned}
&|I_3|\\
&\leqslant \frac{C}{(r-\varrho)}\int_{(B(r)\setminus B(\frac{2r}{3}))} |\boldsymbol{T}||\overline{\boldsymbol{v}}|^{1-\delta}|\overline{\boldsymbol{v}}|^{\delta}|\boldsymbol{v}+[\boldsymbol{v}]_{(B(r)\setminus B(\frac{2r}{3}))}|\,dx\\
&\leqslant \frac{C}{(r-\varrho)}\|\boldsymbol{T}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\||\overline{\boldsymbol{v}}|^{1-\delta}\|_{L^{\frac{q}{1-\delta},\infty}(B(r)\setminus B(\frac{2r}{3}))}\||\overline{\boldsymbol{v}}|^{\delta}\|_{L^{\frac{6}{\delta}}(B(r)\setminus B(\frac{2r}{3}))}\\
&\quad\times\|1\|_{L^{\frac{1}{\beta},\frac{6}{6-\delta}}(B(r)\setminus B(\frac{2r}{3}))}\|\boldsymbol{v}+[\boldsymbol{v}]_{(B(r)\setminus B(\frac{2r}{3}))}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\\
&\leqslant \frac{C}{(r-\varrho)}R^{3\beta}\|\boldsymbol{T}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\|\overline{\boldsymbol{v}}\|^{1-\delta}_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\|\overline{\boldsymbol{v}}\|^{\delta}_{L^{6}(B(r)\setminus B(\frac{2r}{3}))}\\
&\quad\times\|\boldsymbol{v}+[\boldsymbol{v}]_{(B(r)\setminus B(\frac{2r}{3}))}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}.
\end{aligned}$$

Using

$$\|\boldsymbol{v}+[\boldsymbol{v}]_{(B(r)\setminus B(\frac{2r}{3}))}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\leqslant C\|\boldsymbol{v}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}$$

and Galiardo-Nireberg-Sobolev inequality, it gives

$$\begin{aligned}
&|I_3|\\
&\leqslant \frac{C}{(r-\varrho)}R^{3\beta}\|\boldsymbol{T}\|_{L^{q,\infty}(B(r)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(r)\setminus B(\frac{R}{2}))}\|\nabla\boldsymbol{v}\|^{\delta}_{L^{2}(B(r)\setminus B(\frac{2r}{3}))}\\
&\leqslant \frac{1}{8}\int_{B(r)\setminus B(\frac{2r}{3})}|\nabla\boldsymbol{v}|^2\,dx + C(\delta)\Big(\frac{R^{3\beta}}{r-\varrho}\|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}.
\end{aligned}$$

By (3.4), we get

$$\begin{aligned}
|I_4| &= \Big|\int_{B(r)\setminus B(\frac{2r}{3})}(\boldsymbol{T}\cdot\nabla\overline{\boldsymbol{v}})\cdot\boldsymbol{w}\,dx\Big|\\
&= \Big|-\int_{B(r)\setminus B(\frac{2r}{3})}(\boldsymbol{T}\otimes\overline{\boldsymbol{v}}):\nabla\boldsymbol{w}\,dx\Big|\\
&\leqslant R^{3\beta}\|\boldsymbol{T}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\|\overline{\boldsymbol{v}}\|^{1-\delta}_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\|\overline{\boldsymbol{v}}\|^{\delta}_{L^{6}(B(r)\setminus B(\frac{2r}{3}))}\\
&\quad\|\nabla\boldsymbol{w}\|_{L^{q,\infty}(B(r)\setminus B(\frac{2r}{3}))}\\
&\leqslant \frac{C}{r-\varrho}R^{3\beta}\|\boldsymbol{T}\|_{L^{q,\infty}(B(r)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(r)\setminus B(\frac{R}{2}))}\|\nabla\boldsymbol{v}\|^{\delta}_{L^{2}(B(r)\setminus B(\frac{2r}{3}))}\\
&\leqslant \frac{1}{8}\int_{B(r)\setminus B(\frac{2r}{3})}|\nabla\boldsymbol{v}|^2\,dx
\end{aligned}$$

$$+ C(\delta)\Big(\frac{R^{3\beta}}{r-\varrho}\|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}.$$

Therefore, from (3.6),

$$\begin{aligned}\int_{B(\varrho)}|\nabla\boldsymbol{v}|^2\,dx &\leqslant \frac{C}{r-\varrho}\Big(\int_{B(r)}|\nabla\boldsymbol{v}|^2\,dx\Big)^{\frac12}\Big(\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\Big)^{\frac12}\\ &\quad+\frac18\int_{B(r)}|\nabla\boldsymbol{v}|^2\,dx\\ &\quad+C(\delta)\Big(\frac{R^{3\beta}}{r-\varrho}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}\\ &\leqslant \frac12\int_{B(r)}|\nabla\boldsymbol{v}|^2\,dx+\frac{C}{(r-\varrho)^2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\\ &\quad+C(\delta)\Big(\frac{R^{3\beta}}{r-\varrho}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}\end{aligned}$$

for any $\frac{3R}{4}\leqslant\varrho<r\leqslant R$. The following inequality can be obtained by the standard iterative arguments

$$\int_{B(\frac{R}{2})}|\nabla\boldsymbol{v}|^2\,dx\leqslant\frac{C}{R^2}\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx+D_2,$$

where $D_2:=C(\delta)\Big(R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}}\|\boldsymbol{T}\|_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(B(R)\setminus B(\frac{R}{2}))}\Big)^{\frac{2}{2-\delta}}$. It completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.1

With Proposition 3.1 in hand, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1(i). It is easy to check that, for $q>2$,

$$\begin{aligned}R^{-2}\Big(\int_{B(R)\setminus B(\frac{R}{2})}|\boldsymbol{v}|^2\,dx\Big)&\leqslant R^{-2}\|1\|_{L^{\frac{q}{q-2},\frac{\ell}{\ell-2}}(B(R)\setminus B(\frac{R}{2}))}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\\ &\leqslant C(q,\ell)R^{1-\frac{6}{q}}\|\boldsymbol{v}\|^2_{L^{q,\ell}(B(R)\setminus B(\frac{R}{2}))}\\ &=C(q,\ell)R^{1-2\gamma}M^2_{\gamma,q,\ell}((\boldsymbol{u},\boldsymbol{H}),R).\end{aligned}$$

Note that

$$D_1\leqslant C(q,\ell)R^{2-3\gamma}M^3_{\gamma,q,\ell}((\boldsymbol{u},\boldsymbol{H}),R).$$

By conditions (1.2) and (1.3), we derive

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla\boldsymbol{v}|^2\,dx\\ \leqslant&\begin{cases}C(q,\ell)\liminf_{R\to\infty}\big(R^{2-3\gamma}M^3_{\gamma,q,\ell}((\boldsymbol{u},\boldsymbol{H}),R)\big), & \text{if}\quad \gamma>\frac23,\\ C(q,\ell)\liminf_{R\to\infty}M^3_{\frac23,q,\ell}((\boldsymbol{u},\boldsymbol{H}),R)<\int_{\mathbb{R}^3}|\nabla\boldsymbol{v}|^2\,dx, & \text{if}\quad\gamma=\frac23.\end{cases}\end{aligned}$$

Then $\boldsymbol{v}\equiv\boldsymbol{0}$. Hence, $\boldsymbol{u}\equiv-\boldsymbol{H}$.

Substituting $\boldsymbol{u} \equiv -\boldsymbol{H}$ into $(1.1)_1$ and $(1.1)_3$, we know

$$\begin{cases} \Delta \boldsymbol{u} = 0, \\ \operatorname{div} \boldsymbol{u} = 0. \end{cases} \tag{4.1}$$

As before, we can also find a $\boldsymbol{w} \in W_0^{1,s}(B(r)\backslash B(\frac{2r}{3}))$ such that $\operatorname{div} \boldsymbol{w} = \operatorname{div}(\varphi \boldsymbol{u})$, where φ is the same cut–off function used in the proof of Proposition 3.1. Testing $(4.1)_1$ with $\varphi \boldsymbol{u} - \boldsymbol{w}$, and the integrating by parts allows

$$\int_{B(r)} \varphi |\nabla \boldsymbol{u}|^2 \, dx = -\int_{B(r)} \nabla \boldsymbol{u} : (\nabla \varphi \otimes \boldsymbol{u}) \, dx + \int_{B(r)} \nabla \boldsymbol{u} : \nabla \boldsymbol{w} \, dx.$$

Thus, once again we obtain

$$\begin{aligned} \int_{B(\frac{R}{2})} |\nabla \boldsymbol{u}|^2 \, dx &\leqslant CR^{-2} \int_{B(R)\backslash B(\frac{R}{2})} |\boldsymbol{u}|^2 \, dx \\ &\leqslant CR^{-2} \|1\|_{L^{\frac{q}{q-2}, \frac{\ell}{\ell-2}}(B(R)\backslash B(\frac{R}{2}))} \|\boldsymbol{u}\|^2_{L^{q,\ell}(B(R)\backslash B(\frac{R}{2}))} \\ &\leqslant C(q, \ell) R^{1-\frac{6}{q}} \|\boldsymbol{u}\|^2_{L^{q,\ell}(B(R)\backslash B(\frac{R}{2}))}. \end{aligned} \tag{4.2}$$

As a result, let $R \to \infty$, we have $\boldsymbol{u} \equiv \boldsymbol{0}$. The proof of Theorem 1.1(i) is ended.

Proof of Theorem 1.1(ii). Based on Proposition 3.1(ii), using Lemma 2.1, we have

$$\begin{aligned} R^{-2} \int_{B(R)\backslash B(\frac{R}{2})} |\boldsymbol{v}|^2 \, dx &\leqslant R^{-2} \|\boldsymbol{v}\|^2_{L^{q,\infty}(B(R)\backslash B(\frac{R}{2}))} \|1\|_{L^{\frac{q}{q-2},1}(B(R)\backslash B(\frac{R}{2}))} \\ &\leqslant CR^{\frac{1}{3}-\frac{2}{q}} (R^{\frac{1}{3}+\frac{1}{q}-\frac{3}{q}} \|(\boldsymbol{u}, \boldsymbol{H})\|_{L^{q,\infty}(B(R)\backslash B(\frac{R}{2}))})^2 \\ &= CR^{\frac{1}{3}-\frac{2}{q}} (R^{\frac{1}{3}+\frac{1}{q}-\gamma})^2 M^2_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) \end{aligned}$$

and

$$\begin{aligned} D_2 &\leqslant C(\delta) \Big(M^{2-\delta}_{\gamma,q,\infty}(\boldsymbol{v}, R) M_{\gamma,q,\infty}(\boldsymbol{T}, R) R^{3\beta-1-(\gamma-\frac{3}{q})(3-\delta)} \Big)^{\frac{2}{2-\delta}} \\ &\leqslant C(\delta) \Big(M^{2-\delta}_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) M_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) R^{2-\frac{\delta}{2}-\gamma(3-\delta)} \Big)^{\frac{2}{2-\delta}} \\ &\leqslant C(\delta) \Big(M^{3-\delta}_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) R^{2-\frac{\delta}{2}-\gamma(3-\delta)} \Big)^{\frac{2}{2-\delta}}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{B(\frac{R}{2})} |\nabla \boldsymbol{v}|^2 \, dx \leqslant{}& CR^{1-2\gamma} M^2_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) \\ &+ C(\delta) \Big(R^{2-\frac{\delta}{2}-\gamma(3-\delta)} M^{3-\delta}_{\gamma,q,\infty}((\boldsymbol{u}, \boldsymbol{H}), R) \Big)^{\frac{2}{2-\delta}}. \end{aligned}$$

Now, for any $q \in (\frac{12}{5}, 3)$, we can find q_1 satisfying the following relationship

$$q > q_1 > \frac{12}{5}, \quad \gamma > \frac{1}{3} + \frac{1}{q_1} > \frac{1}{3} + \frac{1}{q},$$

and then there is a real number $\delta \in (0,\, 1)$ such that

$$q_1 = \frac{6(3-\delta)}{6-\delta} < q.$$

Noticing that

$$\begin{aligned} 2 - \frac{\delta}{2} - \gamma(3-\delta) &= 2 - \frac{3(3-q_1)}{6-q_1} - \gamma\big(3 - \frac{6(3-q_1)}{6-q_1}\big) \\ &= \frac{3+q_1-3q_1\gamma}{6-q_1} = \frac{3q_1(\frac{1}{3}+\frac{1}{q_1}-\gamma)}{6-q_1} < 0, \end{aligned}$$

we have $\boldsymbol{v} \equiv \mathbf{0}$ via letting $R \to \infty$. Hence, once again we deduce $\boldsymbol{u} \equiv -\boldsymbol{H}$.

Now, our goal is to prove $\boldsymbol{u} \equiv \mathbf{0}$. Using the relation $\boldsymbol{u} \equiv -\boldsymbol{H}$ as we did in the proof Theorem 1.1(i) we have (4.1) and the Caccioppoli inequality

$$\int_{B(\frac{R}{2})} |\nabla \boldsymbol{u}|^2 \, dx \leqslant \frac{C}{R^2} \int_{B(R)\setminus B(\frac{R}{2})} |\boldsymbol{u}|^2 \, dx.$$

Therefore

$$\int_{B(\frac{R}{2})} |\nabla \boldsymbol{u}|^2 \, dx \leqslant C R^{1-2\gamma} M^2_{\gamma,\,q,\,\infty}(\boldsymbol{u},\, R) \leqslant C(q,\ell) R^{1-2\gamma} M^2_{\gamma,\,q,\,\ell}(\boldsymbol{u},\, R).$$

Considering (1.4), $\boldsymbol{u} \equiv \mathbf{0}$ can be yielded by passing $R \to \infty$. Now we complete the proof of Theorem 1.1. □

5. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $C(R) := \{x = (x',\, x_3) \in \mathbb{R}^3 \mid |x'| \leqslant R, |x_3| \leqslant R\}$, which is the cylindrical region.

In terms of (3.2), we have

$$\begin{aligned} &\int_{C(\frac{\sqrt{2}R}{4})} |\nabla \boldsymbol{v}|^2 \, dx \\ &\leqslant C R^{-2} \int_{C(R)\setminus C(\frac{\sqrt{2}R}{4})} |\boldsymbol{v}|^2 \, dx \\ &\quad + C(\delta)\Big(R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}} \|\boldsymbol{T}\|_{L^{q,\infty}(C(R)\setminus C(\frac{\sqrt{2}R}{4}))} \|\boldsymbol{v}\|^{2-\delta}_{L^{q,\infty}(C(R)\setminus C(\frac{\sqrt{2}R}{4}))}\Big)^{\frac{2}{2-\delta}} \\ &\leqslant C R^{1-\frac{6}{q}} \|\boldsymbol{v}\|^2_{L^q(C(R))} + C(\delta,\, q)(R^{2-\frac{9-3\delta}{q}-\frac{\delta}{2}} \|\boldsymbol{T}\|_{L^q(C(R))} \|\boldsymbol{v}\|^{2-\delta}_{L^q(C(R))}))^{\frac{2}{2-\delta}} \end{aligned}$$

for $\frac{12}{5} < q < 3$, where we used the fact that $B(R) \subset C(R) \subset B(\sqrt{2}R)$ and the property

$$\|\boldsymbol{v}\|_{L^{q,\infty}(\Omega)} \leqslant C(q,\ell) \|\boldsymbol{v}\|_{L^{q,\ell}(\Omega)}.$$

Introducing the polar coordinates, the decay assumption (1.5) yields

$$\|\boldsymbol{v}\|_{L^q(C(R))} = \Big(\int_{-R}^{R} \int_{|x'|<R} |\boldsymbol{v}(x)|^q \, dx' dx_3 \Big)^{\frac{1}{q}}$$

$$\leqslant C\Big(\int_{-R}^{R}\int_{|x'|<R}\frac{1}{(1+|x'|)^{\mu q}}\,dx'dx_3\Big)^{\frac{1}{q}}$$
$$= C(2R)^{\frac{1}{q}}\Big(\int_0^{2\pi}\int_0^R\frac{1}{(1+\rho)^{\mu q}}\,\rho\,d\rho d\theta\Big)^{\frac{1}{q}}.$$

For $\mu q > 2$, we have

$$\|\boldsymbol{v}\|_{L^q(C(R))} \leqslant C(4\pi R)^{\frac{1}{q}}\Big(\int_0^R (1+\rho)^{1-\mu q}\,d\rho\Big)^{\frac{1}{q}} \leqslant C(\mu,\,q)R^{\frac{1}{q}}.$$

Similarly,

$$\|\boldsymbol{T}\|_{L^q(C(R))} \leqslant C(\mu,\,q)R^{\frac{1}{q}}.$$

Combining the above estimates together, it follows

$$\int_{C(\frac{\sqrt{2}}{4}R)}|\nabla\boldsymbol{v}|^2dx \leqslant C(\mu,\,q)R^{1-\frac{4}{q}} + C(\delta,\,\mu,\,q)R^{(2-\frac{\delta}{2}-\frac{6-2\delta}{q})\frac{2}{2-\delta}}. \tag{5.1}$$

For fixed $\mu > \frac{2}{3}$, there is a constant $\delta \in (0,\,1)$ such that

$$\frac{2}{\mu} < 4\,\frac{3-\delta}{4-\delta}.$$

Since $\delta > 0$, we know $6\,(4-\delta) < 4\,(6-\delta)$. Then let

$$q := \frac{1}{2}\Big(\max\big\{6\,\frac{3-\delta}{6-\delta},\,\frac{2}{\mu}\big\} + 4\,\frac{3-\delta}{4-\delta}\Big).$$

It is easily to see

$$\frac{12}{5} < \max\big\{6\,\frac{3-\delta}{6-\delta},\,\frac{2}{\mu}\big\} < q < 4\,\frac{3-\delta}{4-\delta} < 3 \quad \text{and} \quad \mu q > 2.$$

Then

$$2 - \frac{\delta}{2} - \frac{6-2\delta}{q} < 0.$$

Passing $R \to \infty$, it follows from (5.1) that

$$\int_{\mathbb{R}^3}|\nabla\boldsymbol{v}|^2\,dx = 0,$$

which implies $\boldsymbol{v} \equiv 0$. Hence $\boldsymbol{u} \equiv -\boldsymbol{H}$.

Similar to the proof of Theorem 1.1, repeating above arguments yield $\boldsymbol{u} \equiv \boldsymbol{H} \equiv 0$. ☐

References

[1] D. Chae and S. Weng, *Liouville type theorems for the steady axially symmetric Navier-Stokes and magnetohydrodynamic equations*, Discrete Contin. Dyn. Syst. **36** (2016), no. 10, 5267–5285. https://doi.org/10.3934/dcds.2016031

[2] D. Chae and J. Wolf, *On Liouville type theorems for the steady Navier-Stokes equations in* $\mathbb{R}^3$, J. Differential Equations **261** (2016), no. 10, 5541–5560. https://doi.org/10.1016/j.jde.2016.08.014

[3] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001. https://doi.org/10.1017/CBO9780511626333

[4] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, second edition, Springer Monographs in Mathematics, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-09620-9

[5] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, **105**, Princeton University Press, Princeton, NJ, 1983.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[7] G. Gui, *Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity*, J. Funct. Anal. **267** (2014), no. 5, 1488–1539. https://doi.org/10.1016/j.jfa.2014.06.002

[8] H. Kozono, Y. Terasawa, and Y. Wakasugi, *A remark on Liouville-type theorems for the stationary Navier-Stokes equations in three space dimensions*, J. Funct. Anal. **272** (2017), no. 2, 804–818. https://doi.org/10.1016/j.jfa.2016.06.019

[9] Z. Lei and Q. S. Zhang, *A Liouville theorem for the axially-symmetric Navier-Stokes equations*, J. Funct. Anal. **261** (2011), no. 8, 2323–2345. https://doi.org/10.1016/j.jfa.2011.06.016

[10] F. Lin, L. Xu, and P. Zhang, *Global small solutions of 2-D incompressible MHD system*, J. Differential Equations **259** (2015), no. 10, 5440–5485. https://doi.org/10.1016/j.jde.2015.06.034

[11] G. G. Lorentz, *Some new functional spaces*, Ann. of Math. (2) **51** (1950), 37–55. https://doi.org/10.2307/1969496

[12] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. **30** (1963), 129–142. http://projecteuclid.org/euclid.dmj/1077374532

[13] E. Priest and T. Forbes, *Magnetic Reconnection*, Cambridge University Press, Cambridge, 2000. https://doi.org/10.1017/CBO9780511525087

[14] S. Schulz, *Liouville type theorem for the stationary equations of magneto-hydrodynamics*, Acta Math. Sci. Ser. B (Engl. Ed.) **39** (2019), no. 2, 491–497. https://doi.org/10.1007/s10473-019-0213-7

[15] G. Seregin, *Liouville type theorem for stationary Navier-Stokes equations*, Nonlinearity **29** (2016), no. 8, 2191–2195. https://doi.org/10.1088/0951-7715/29/8/2191

[16] ———, *Remarks on Liouville type theorems for steady-state Navier-Stokes equations*, St. Petersburg Math. J. **30** (2019), no. 2, 321–328; translated from Algebra i Analiz **30** (2018), no. 2, 238–248. https://doi.org/10.1090/spmj/1544

[17] ———, *A Liouville type theorem for steady-state Navier-Stokes equations*, arXiv: 1611.01563

[18] G. Seregin and W. Wang, *Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations*, St. Petersburg Math. J. **31** (2020), no. 2, 387–393; translated from Algebra i Analiz **31** (2019), no. 2, 269–278. https://doi.org/10.1090/spmj/1603

[19] W. Wang, *Remarks on Liouville type theorems for the 3D steady axially symmetric Navier-Stokes equations*, J. Differential Equations **266** (2019), no. 10, 6507–6524. https://doi.org/10.1016/j.jde.2018.11.014

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