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SOBOLEV TRACE INEQUALITY ON $W^{s,q}(\mathbb{R}^n)$

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Abstract. Sobolev trace inequalities on nonhomogeneous fractional Sobolev spaces are established.

1. Trace inequalities on fractional Sobolev spaces

Sobolev trace inequality on nonhomogeneous Sobolev spaces $H^s(\mathbb{R}^n)$ is given by: for $1 \le m < n$ and $s > \frac{m}{2}$,

$$\|\tau u\|_{H^{s-\frac{m}{2}}(\mathbb{R}^{n-m})} \le C_{s,m}\|u\|_{H^s(\mathbb{R}^n)}, \tag{1}$$

where $\tau u \in H^{s-\frac{m}{2}}(\mathbb{R}^{n-m})$ is the trace of $u \in H^s(\mathbb{R}^n)$ restricted to the $(n-m)$-dimensional subspace $\{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_{n-m}, 0, 0, \ldots, 0)\}$ of $\mathbb{R}^n$. The best constant for this inequality is presented by H. Pak and Y. Park [6] as

$$C_{s,m} = \left(\frac{\Gamma(s-\frac{m}{2})}{(4\pi)^{\frac{m}{2}}\Gamma(s)} \right)^{1/2}$$

together with various forms of extremal functions. A homogeneous version of trace inequality (1) is obtained by A. Einav and M. Loss with the same sharp constant [4]. Even though the trace of a given function is addressed as its fundamental importance in the theory of boundary value problems of partial differential equations, the continuity of trace operators has not been reported yet even on general fractional Sobolev spaces $W^{s,q}(\mathbb{R}^n)$.

This paper establishes fractional Sobolev trace inequalities on the nonhomogeneous fractional Sobolev spaces $W^{s,q}(\mathbb{R}^n)$, $1 \le q \le 2$. The main result can be summarized as follows:

Theorem 1.1 (Fractional Sobolev trace inequality on $W^{s,q}(\mathbb{R}^n)$). *Let p, q be extended real numbers of $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le q \le 2$ and let s, t be real numbers with*

$$s - n\left(\frac{1}{q} - \frac{1}{p}\right) > t \ge \frac{m}{p}. \tag{2}$$

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Then for $u \in W^{s,q}(\mathbb{R}^n)$ with the trace τu on $\mathbb{R}^{n-m}$, we have

$$\|\tau u\|_{W^{t-\frac{m}{p},p}(\mathbb{R}^{n-m})} \le C_{q,s,t,m} \|u\|_{W^{s,q}(\mathbb{R}^n)}, \tag{3}$$

where

$$C_{q,s,t,m} = \frac{\pi^{\frac{n-m}{2q}-\frac{n}{2p}}}{2^{\frac{m}{2}}} \frac{q^{\frac{2n-m}{2q}}}{p^{\frac{2n-m}{2p}}} \left[\frac{\Gamma(\frac{sq-m}{2})}{\Gamma(\frac{sq}{2})}\right]^{\frac{1}{q}} \left[\frac{\Gamma(\frac{(s-t)p}{2(p-2)}-\frac{n}{2})}{\Gamma(\frac{(s-t)p}{2(p-2)}-\frac{m}{2})}\right]^{\frac{1}{q}-\frac{1}{p}}. \tag{4}$$

The main difficulty of the proof of the theorem arises from the absence of the isometry, and even from the lack of the continuity of the Fourier transform on $L^p(\mathbb{R}^n)$ with $p \neq 2$.

The classical Sobolev trace inequalities on $\mathbb{R}^{n+1}$ are given by

$$\|\tau u\|_{L^p(\mathbb{R}^n)} \le A_{q,n} \|\nabla u\|_{L^q(\mathbb{R}^{n+1})} \tag{5}$$

with

$$1 - n\left(\frac{1}{q} - \frac{1}{p}\right) = \frac{1}{q}$$

for some positive constant $A_{q,n}$ which is independent of the function u. It can be noticed that the range $L^p(\mathbb{R}^n)$ of the trace map is *too big* to constitute a proper container of all the traces. Theorem 1.1 illustrates that all the traces of the functions in $W^{1,q}(\mathbb{R}^{n+1})$ are included at least in the space

$$\bigcap W^{t-\frac{1}{p},p}(\mathbb{R}^n),$$

where the intersection is taken over all indices t satisfying (2).

The best constant of (5) is still open except for the cases $q = 1$ and $q = 2$. A conjectured extremal function is the function of the form

$$\tau u(x) = \frac{1}{(1+|x|^2)^{\frac{n+1-p}{2(p-1)}}}.$$

J. Escobar first identified the best constant for the case $q = 2$ of (5) by exploiting the conformal invariance of this inequality and using characteristics of an Einstein metric [5]. W. Beckner independently achieved the sharp constant by inverting the inequality to a fractional integral on the dual space and using the sharp Hardy-Littlewood-Sobolev inequality [3]. The limiting case $p = 1$ is investigated by Y. Park [7]. We present an upper bound for the sharp constant of the fractional Sobolev trace inequality and observe that $C_{q,s,t,m}$ blows up to infinity as t approaches to $s - n\left(\frac{1}{q} - \frac{1}{p}\right)$. We also note that $C_{q,s,t,m}$ converges to 0 as s goes to infinity.

Some basic notations are listed. The fractional Sobolev spaces $W^{s,q}(\mathbb{R}^n)$ of functions with $s \in \mathbb{R}$ are defined as

$$W^{s,q}(\mathbb{R}^n) := \left\{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}_n^{-1}\left((1+|\xi|^2)^{s/2}\widehat{u}\right) \in L^q(\mathbb{R}^n)\right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the set of all tempered distributions on $\mathbb{R}^n$ and the Fourier transform $\widehat{u} = \mathcal{F}_n(u)$ on $\mathbb{R}^n$ of the function $u \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\widehat{u}(\xi) = \mathcal{F}_n(u)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, dx.$$

The nonhomogeneous Sobolev space $W^{s,q}(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|_{W^{s,q}} := \left(\int_{\mathbb{R}^n} \left| \mathcal{F}_n^{-1} \left((1+|\xi|^2)^{s/2} \widehat{u} \right)(x) \right|^q dx \right)^{1/q}.$$

2. Proof of Theorem 1.1

The nonhomogeneous Sobolev space $W^{s,q}(\mathbb{R}^n)$ is the completion of Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Hence, by the continuous extension argument, it suffices to show that for $u \in S(\mathbb{R}^n)$, we have

$$\|\tau u\|_{W^{t-\frac{m}{p},p}(\mathbb{R}^{n-m})} \le C_{q,s,t,m} \|u\|_{W^{s,q}(\mathbb{R}^n)}, \tag{6}$$

where $C_{q,s,t,m}$ is the constant defined at (4). Here the trace τu of u is defined by $\tau u(x') = u(x', 0, \dots, 0)$ for $x' \in \mathbb{R}^{n-m}$.

To accomplish it, take $u \in \mathcal{S}(\mathbb{R}^n)$ and set $f := \tau u$ to have

$$\begin{aligned} \widehat{f}(\xi') = \mathcal{F}_{n-m}(f)(\xi') &= \frac{1}{(2\pi)^{\frac{n-m}{2}}} \int_{\mathbb{R}^{n-m}} f(x') e^{-ix'\cdot\xi'} dx' \\ &= \frac{1}{(2\pi)^{\frac{n-m}{2}}} \int_{\mathbb{R}^{n-m}} u(x', 0) e^{-ix'\cdot\xi'} dx' \end{aligned}$$

for $\xi' \in \mathbb{R}^{n-m}$. Apply the Fourier inversion formula in the ξ''-variable to get

$$u(x', 0) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathcal{F}_m(u(x', \cdot))(\xi'') d\xi''$$

for $x' \in \mathbb{R}^{n-m}$, where $\mathcal{F}_m$ represents the Fourier transform with respect to x''-variable for $(x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Then Fubini's theorem gives

$$\begin{aligned} \widehat{f}(\xi') &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-m}} \left(\int_{\mathbb{R}^m} \mathcal{F}_m(u(x', \cdot))(\xi'')\, d\xi'' \right) e^{-ix'\cdot\,\xi'} dx' \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^m}\!\int_{\mathbb{R}^{n-m}} \left(\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} u(x', x'') e^{-ix''\cdot\xi''} dx'' \right) e^{-ix'\cdot\,\xi'} dx' d\xi'' \\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \widehat{u}(\xi', \xi'')\, d\xi'', \end{aligned} \tag{7}$$

where $(\xi', \xi''), (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Let ϕ be a function in $S(\mathbb{R}^{n-m})$ with $\|\phi\|_{L^q(\mathbb{R}^{n-m})} = 1$. Multiply both sides of (7) by $(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}} \widehat{\phi}(-\xi')$ and integrate with respect to ξ' to get

$$\int_{\mathbb{R}^{n-m}} \widehat{f}(\xi')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}} \widehat{\phi}(-\xi') d\xi'$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^n} \widehat{u}(\xi', \xi'')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}} \widehat{\overline{\phi}}(-\xi') d\xi \tag{8}$$

for $\xi = (\xi', \xi'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Here the bar $\overline{z}$ indicates the complex conjugate of z. The left hand side of (8) becomes

$$\begin{aligned} &\int_{\mathbb{R}^{n-m}} \widehat{f}(\xi')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}} \widehat{\overline{\phi}}(-\xi') d\xi' \\ &= \int_{\mathbb{R}^{n-m}} \Big[\widehat{f}(\xi')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}}\Big]^{\vee}(x')\, \overline{\phi(x')}\, dx', \end{aligned} \tag{9}$$

where $g^{\vee}$ represents the Fourier inversion $\mathcal{F}_n^{-1}(g)$ of the function g. Hölder's inequality on the right hand side of (8) yields that

$$\begin{aligned} &\left| \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^n} \widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \overline{\phi}^{\vee}(\xi') \frac{(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2p}}}{(1+|\xi|^2)^{\frac{s}{2}}} d\xi \right| \\ &\le \frac{1}{(2\pi)^{\frac{m}{2}}} \left(\int_{\mathbb{R}^n} \left|\widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}}\right|^p d\xi \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\overline{\phi}^{\vee}(\xi')|^q \frac{(1+|\xi'|^2)^{\frac{tq}{2}-\frac{mq}{2p}}}{(1+|\xi|^2)^{\frac{sq}{2}}} d\xi \right)^{\frac{1}{q}} \end{aligned} \tag{10}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1. *For $\alpha > \frac{m}{2}$, we have*

$$\int_{\mathbb{R}^m} \frac{(1+|\xi'|^2)^{\alpha-\frac{m}{2}}}{(1+|\xi'|^2+|\xi''|^2)^{\alpha}} d\xi'' = \pi^{\frac{m}{2}} \frac{\Gamma(\alpha - \frac{m}{2})}{\Gamma(\alpha)}. \tag{11}$$

Proof. A direct computation reveals that for $\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{(1+|\xi'|^2+\xi_n^2)^{\alpha}} d\xi_n &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(1+|\xi'|^2)^{\alpha-\frac{1}{2}}} (1+\tan^2\theta)^{-\alpha+1} d\theta \\ &= \frac{1}{(1+|\xi'|^2)^{\alpha-\frac{1}{2}}} \left(2\int_0^{\frac{\pi}{2}} \cos^{2\alpha-2}\theta d\theta \right) \\ &= \sqrt{\pi}\, \frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)} \frac{1}{(1+|\xi'|^2)^{\alpha-\frac{1}{2}}}. \end{aligned}$$

An induction argument gives that for fixed $\xi' \in \mathbb{R}^{n-m}$,

$$\int_{\mathbb{R}^m} \frac{1}{(1+|\xi'|^2+|\xi''|^2)^{\alpha}} d\xi'' = \pi^{\frac{m}{2}} \frac{\Gamma(\alpha-\frac{m}{2})}{\Gamma(\alpha)} \frac{1}{(1+|\xi'|^2)^{\alpha-\frac{m}{2}}}.$$

This implies the identity (11). □

Lemma 2.1 and Hölder's inequality imply that

$$\int_{\mathbb{R}^n} |\overline{\phi}^{\vee}(\xi')|^q \frac{(1+|\xi'|^2)^{\frac{tq}{2}-\frac{mq}{2p}}}{(1+|\xi|^2)^{\frac{sq}{2}}} d\xi$$

$$= \pi^{\frac{m}{2}} \frac{\Gamma(\frac{sq}{2} - \frac{m}{2})}{\Gamma(\frac{sq}{2})} \int_{\mathbb{R}^{n-m}} \frac{|\overline{\phi}^{\vee}(\xi')|^q}{(1+|\xi'|^2)^{\frac{(s-t)q}{2} + \frac{m(q-2)}{2}}} d\xi'$$

$$\le \pi^{\frac{m}{2}} \frac{\Gamma(\frac{sq}{2} - \frac{m}{2})}{\Gamma(\frac{sq}{2})} \left(\int_{\mathbb{R}^{n-m}} |\overline{\phi}^{\vee}(\xi')|^p d\xi' \right)^{\frac{q}{p}}$$

$$\times \left(\int_{\mathbb{R}^{n-m}} \frac{1}{(1+|\xi'|^2)^{\frac{(s-t)p}{2(p-2)} - \frac{m}{2}}} d\xi' \right)^{\frac{q}{p}(p-2)}$$

$$= \pi^{\frac{m}{2}} \frac{\Gamma(\frac{sq}{2} - \frac{m}{2})}{\Gamma(\frac{sq}{2})} \left[\frac{\pi^{\frac{n-m}{2}} \Gamma(\frac{(s-t)p}{2(p-2)} - \frac{n}{2})}{\Gamma(\frac{(s-t)p}{2(p-2)} - \frac{m}{2})} \right]^{\frac{q}{p}(p-2)} \left(\int_{\mathbb{R}^{n-m}} |\overline{\phi}^{\vee}(\xi')|^p d\xi' \right)^{\frac{q}{p}}. \tag{12}$$

By virtue of the Babenko-Beckner's inequality [1,2], we have

$$\left(\int_{\mathbb{R}^{n-m}} |\overline{\phi}^{\vee}(\xi')|^p d\xi' \right)^{\frac{1}{p}} \le \frac{q^{\frac{n-m}{2q}}}{p^{\frac{n-m}{2p}}} \left(\int_{\mathbb{R}^{n-m}} |\overline{\phi(\xi')}|^q d\xi' \right)^{\frac{1}{q}} = \frac{q^{\frac{n-m}{2q}}}{p^{\frac{n-m}{2p}}} \tag{13}$$

and

$$\left(\int_{\mathbb{R}^n} \left| \widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \right|^p d\xi \right)^{1/p} = \left(\int_{\mathbb{R}^n} \left| \mathcal{F}_n \circ \mathcal{F}_n^{-1} \left(\widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \right) \right|^p d\xi \right)^{1/p}$$

$$\le \frac{q^{\frac{n}{2q}}}{p^{\frac{n}{2p}}} \left(\int_{\mathbb{R}^n} \left| \left(\widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}} \right)^{\vee} (x) \right|^q dx \right)^{1/q}. \tag{14}$$

Collecting the estimates (8), (9), (10), (12), (13) and (14), we establish the inequality (6). The proof is now completed. □

Remark 2.2. Let p, q be extended real numbers of $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le q \le 2$ and let s, t be real numbers with $t - n\left(\frac{1}{q} - \frac{1}{p}\right) > s > \frac{t}{2} + \frac{m}{p}$. Then for any $g \in W^{t-\frac{m}{q},q}(\mathbb{R}^{n-m})$, there is a function $u \in W^{s,p}(\mathbb{R}^n)$ such that

$$\tau u = g.$$

In fact, for $g \in W^{t-\frac{m}{q},q}(\mathbb{R}^{n-m})$, we consider the function

$$u(x) := 2^{m/2} \frac{\Gamma(s)}{\Gamma(s-\frac{m}{2})} \left(\widehat{g}(\xi') \frac{(1+|\xi'|^2)^{s-\frac{m}{2}}}{(1+|\xi|^2)^s} \right)^{\vee} (x).$$

Then by the identity (7) together with Lemma 2.1, we observe that

$$\widehat{\tau u}(\xi') = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \widehat{u}(\xi', \xi'')\, d\xi'' = \widehat{g}(\xi').$$

In order to demonstrate that u belongs to $W^{s,p}(\mathbb{R}^n)$, we first note that:

Lemma 2.3. *For $\alpha > \frac{m}{p}$ and $\beta \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}^n} \left| \widehat{g}(\xi') \frac{(1+|\xi'|^2)^{\beta-\frac{m}{2}}}{(1+|\xi|^2)^{\alpha}} \right|^p d\xi = C \int_{\mathbb{R}^{n-m}} \left| \widehat{g}(\xi')(1+|\xi'|^2)^{\beta-\alpha-\frac{m}{2q}} \right|^p d\xi'$$

for some constant C.

Proof. The result follows from a direct computation and Lemma 2.1. $\square$

The same arguments used in the proof of Theorem 1.1 yields: for any $\|\phi\|_{L^q(\mathbb{R}^n)} = 1$

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \left(\widehat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}}\right)^{\vee} \overline{\phi(x)}\, dx\right| &= C_1 \left|\int_{\mathbb{R}^n} \widehat{g}(\xi') \frac{(1+|\xi'|^2)^{s-\frac{m}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \overline{\phi}^{\vee}(\xi)\, d\xi\right| \\
&\lesssim \left(\int_{\mathbb{R}^n} \left|\widehat{g}(\xi') \frac{(1+|\xi'|^2)^{s-\frac{m}{2}}}{(1+|\xi|^2)^{s-\frac{t}{2}}}\right|^p d\xi\right)^{1/p} \\
&\quad \times \left(\int_{\mathbb{R}^n} \left|\frac{\overline{\phi}^{\vee}(\xi)}{(1+|\xi|^2)^{\frac{t-s}{2}}}\right|^q d\xi\right)^{1/q} \\
&= C_2 \left(\int_{\mathbb{R}^{n-m}} \left|\left(\widehat{g}(\xi')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2q}}\right)\right|^p d\xi'\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}^{n-m}} \left|\left(\widehat{g}(\xi')(1+|\xi'|^2)^{\frac{t}{2}-\frac{m}{2q}}\right)^{\vee}(x')\right|^q dx'\right)^{1/q}
\end{aligned}$$

for some positive constants C_1 and C_2. Hence we can see that u belongs to $W^{s,p}(\mathbb{R}^n)$ and $\tau u = g$.

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