

A GENERALIZED HURWITZ METRIC

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ABSTRACT. In 2016, the Hurwitz metric was introduced by D. Minda in arbitrary proper subdomains of the complex plane and he proved that this metric coincides with the Poincaré’s hyperbolic metric when the domains are simply connected. In this paper, we provide an alternate definition of the Hurwitz metric through which we could define a generalized Hurwitz metric in arbitrary subdomains of the complex plane. This paper mainly highlights various important properties of the Hurwitz metric and the generalized metric including the situations where they coincide with each other.

1. Introduction and preliminaries

In 19th century, the notion of hyperbolic metric was first introduced. As pointed out, for instance in [5, p. 132] and [7], the hyperbolic density on a hyperbolic domain Ω can be understood through the extremal problem of maximizing $|f'(0)|$ over all holomorphic functions f that map the unit disk into Ω . In 1981, Hahn [2] introduced a pseudo-differential metric for complex manifolds by means of an extremal problem. Two years later, Minda [6] reconsidered the Hahn metric in Riemann surfaces. Recently, Minda considered an extremal problem of Hurwitz [3] and introduced a new conformal metric, namely, the Hurwitz metric [7] in any proper subdomain of the complex plane \mathbb{C} . Our objective in this paper is to investigate further properties of the Hurwitz metric and their applications.

In 2007, Keen and Lakic [4] defined some new densities in arbitrary plane domains that generalize the hyperbolic density. They are namely the generalized Kobayashi density (see [4, Definition 2]) and the generalized Carathéodory density (see [5, Definition 9.2, p. 166]). The Kobayashi density is defined by pushing forward the hyperbolic density from the unit disk to a plane domain by a holomorphic function, whereas the Carathéodory density is defined by

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pulling back the hyperbolic density from a plane domain to the unit disk by a holomorphic function. This paper deals with a generalized Hurwitz metric in the sense of Kobayashi. We are considering a generalized Hurwitz metric in the sense of Carathéodory in our next paper. Since holomorphic functions are infinitesimal contractions in the hyperbolic metric, it is easy to see that the generalized Kobayashi density exceeds over hyperbolic density on hyperbolic domains. Furthermore, the hyperbolic and the generalized Kobayashi densities coincide whenever there is a regular holomorphic covering map [5, p. 125] from the source domain to the range domain. Similar to the case of hyperbolic distance the generalized Kobayashi distance, in association with the generalized Kobayashi density, between two points can be defined by taking infimum of the generalized Kobayashi length of all rectifiable paths joining the points. In fact, with this definition, it becomes a complete metric space.

As an analogue of the generalized Kobayashi density, we shall generalize the Hurwitz metric by pushing forward the Hurwitz density from a proper subdomain of the complex plane to an arbitrary domain by holomorphic functions with some specific properties. We call this new density *the generalized Hurwitz density*. Note that, on hyperbolic domains the Hurwitz density exceeds the hyperbolic density. Furthermore, in this work we prove that the generalized Hurwitz density is always greater than the Hurwitz density.

Throughout this article, our notations are relatively standard and we are working mainly on the complex plane \mathbb{C} . First we denote the open unit disk by $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. The classical Hyperbolic density [5, p. 33] in \mathbb{D} is defined as

$$\lambda_{\mathbb{D}}(w) = \frac{2}{1 - |w|^2}$$

for $w \in \mathbb{D}$. Note that we consider the hyperbolic metric with constant curvature -1 . The hyperbolic distance between two points w_1, w_2 in \mathbb{D} is

$$\lambda_{\mathbb{D}}(w_1, w_2) = \inf_{\gamma} \int_{\gamma} \lambda_{\mathbb{D}}(w) |dw|,$$

where the infimum is taken over all paths γ joining w_1 and w_2 in \mathbb{D} . Since the hyperbolic metric is conformal invariant, by the Riemann Mapping Theorem one can easily define it on the proper simply connected domains of \mathbb{C} . However, this metric is also defined in more general domains so-called hyperbolic domains. A plane domain Ω is called hyperbolic if $\mathbb{C} \setminus \Omega$ contains at least two points. On a hyperbolic domain Ω , the hyperbolic density λ_{Ω} [5, p. 124] is

$$\lambda_{\Omega}(w) = \frac{\lambda_{\mathbb{D}}(t)}{|\pi'(t)|},$$

where $\pi : \mathbb{D} \rightarrow \Omega$ is a universal covering map with $\pi(t) = w$. Analogue to the case of the unit disk, the hyperbolic distance between two points w_1 and w_2 in Ω is defined by

$$\lambda_{\Omega}(w_1, w_2) = \inf_{\gamma} \int_{\gamma} \lambda_{\Omega}(w) |dw|,$$

where the infimum is taken over all paths γ joining w_1 and w_2 in Ω .

We now define some notations that are used in the definition of the Hurwitz density defined in [7].

Throughout this paper we denote by $\mathcal{H}(Y, \Omega)$ for the set of all holomorphic functions from a domain Y to another domain Ω . For a fixed point $s \in \mathbb{D}$, let $\mathcal{H}_s(w, \Omega)$ be the family of all holomorphic functions h from \mathbb{D} into Ω with $h(s) = w$, $h(t) \neq w$ for all $t \in \mathbb{D} \setminus \{s\}$ and $h'(s) > 0$. For a point $w \in \Omega$, we write $\mathcal{H}(w, \Omega) := \mathcal{H}_0(w, \Omega)$. The *Hurwitz density* [7] in Ω is defined as

$$\eta_{\Omega}(w) = \frac{2}{G'(0)} = \frac{2}{r_{\Omega}(w)},$$

where $G'(0) = \max\{h'(0) : h \in \mathcal{H}(w, \Omega)\} =: r_{\Omega}(w)$. If $\Omega \subsetneq \mathbb{C}$ is a domain and $b \in \Omega$ is any point, then the covering map $G : \mathbb{D} \setminus \{0\} \rightarrow \Omega \setminus \{b\}$ extends to a holomorphic function $G_b : \mathbb{D} \rightarrow \Omega$ with $G_b(0) = b$, $G'_b(0) > 0$. The extended holomorphic function G_b is called the *Hurwitz covering* [7] of $(\mathbb{D}, 0)$ onto (Ω, b) .

The distance decreasing property, which is stated below, of the Hurwitz density for the holomorphic function plays a crucial role to prove our results in this article.

Theorem A (Distance decreasing property of the Hurwitz density) [7]. *Suppose that Ω and Δ are proper subdomains of \mathbb{C} , $a \in \Omega$ and $b \in \Delta$. If h is a holomorphic function of Ω into Δ with $h(a) = b$ and $h(w) \neq b$ for $w \in \Omega \setminus \{a\}$, then*

$$\eta_{\Delta}(b)|h'(a)| \leq \eta_{\Omega}(a).$$

Moreover, equality holds if and only if h is a covering of $\Omega \setminus \{a\}$ onto $\Delta \setminus \{b\}$ that extends to a holomorphic map of Ω onto Δ with $h(a) = b$ and $h'(a) \neq 0$.

The structure of this document is organized as follows. In Section 2, certain basic properties of the Hurwitz density are studied leading to the concept of Hurwitz distance which produces the completeness property of the metric space. Furthermore, Section 3 deals with some basic properties of the generalized Hurwitz density. Finally, in Section 4, we relate the Hurwitz and the generalized Hurwitz densities over some specific plane domains having certain geometric properties.

2. The Hurwitz metric

Firstly, we present here a characterization of the Hurwitz density which gives us ideas to introduce the notion of generalized Hurwitz density in the next section. Let F be the extremal function for the extremal problem $\max\{h'(s) : h \in \mathcal{H}_s(w, \Omega)\}$. Then, we have

$$\begin{aligned} F'(s) &= \max\{h'(s) : h \in \mathcal{H}_s(w, \Omega)\} \\ &= \max\{h'(s) = (f \circ T)'(s) : f \in \mathcal{H}(w, \Omega) \text{ and } T \text{ is the Möbius} \\ &\quad \text{transformation of } \mathbb{D} \text{ onto itself with } T(s) = 0 \text{ and } T'(s) > 0\}. \end{aligned}$$

Since $T(z) = (z - s)/(1 - \bar{s}z)$, it follows that

$$F'(s) = \max \left\{ \frac{f'(0)}{1 - |s|^2} : f \in \mathcal{H}(w, \Omega) \right\}.$$

Since the hyperbolic density on \mathbb{D} is given by $\lambda_{\mathbb{D}}(s) = 2/(1 - |s|^2)$, by the notations defined in the previous section, we have

$$F'(s) = \frac{\lambda_{\mathbb{D}}(s)}{\eta_{\Omega}(w)}.$$

By using this argument, we provide here an alternate definition of the Hurwitz density as follows:

Definition 1. The *Hurwitz density* on a proper subdomain Ω of \mathbb{C} is defined as

$$(2.1) \quad \eta_{\Omega}(w) = \frac{\eta_{\mathbb{D}}(s)}{g'(s)},$$

where $g = h \circ T$ such that T is the Möbius transformation from \mathbb{D} onto \mathbb{D} with $T(s) = 0$, $T'(s) > 0$ and h is the Hurwitz covering map from \mathbb{D} onto Ω with $h(0) = w$.

To define the Hurwitz distance between any two points in Ω we integrate the density η_{Ω} and obtain the following definition:

Definition 2 (Hurwitz distance). For w_1, w_2 in Ω , we define

$$\eta_{\Omega}(w_1, w_2) = \inf_{\gamma} \int_{\gamma} \eta_{\Omega}(w) |dw|,$$

where infimum is taken over all rectifiable paths γ in Ω joining w_1 and w_2 .

Note that we are using the same notation for the Hurwitz density as well as the Hurwitz distance between any two points where the distinction can be observed by seeing the number of parameters. However, now onward, for simplicity, we sometimes use the notation η_{Ω} for $\eta_{\Omega}(w_1, w_2)$. To justify our above definition we indeed prove that η_{Ω} defines a metric when the domain Ω is assumed to be hyperbolic.

Theorem 2.1. *If Ω is a hyperbolic domain, then (Ω, η_{Ω}) is a complete metric space.*

Proof. By the definition of η_{Ω} , symmetry and triangle inequality follow directly. Therefore to prove that (Ω, η_{Ω}) is a metric space, we need to prove strictly positivity of the Hurwitz distance between any two distinct points. Let w_1, w_2 be any two distinct points in Ω . Since $\eta_{\Omega}(w_1, w_2)$ is the infimum of the Hurwitz length of all rectifiable curves joining w_1 and w_2 in Ω , for any $\epsilon > 0$ there exists a rectifiable path γ such that

$$\eta_{\Omega}(w_1, w_2) \geq \int_{\gamma} \eta_{\Omega}(w) |dw| - \epsilon.$$

Note that in a hyperbolic domain Ω , the inequality $\eta_\Omega \geq \lambda_\Omega$ is well-known; see [7]. Then we have

$$\eta_\Omega(w_1, w_2) \geq \int_\gamma \lambda_\Omega(w) |dw| - \epsilon \geq \lambda_\Omega(w_1, w_2) - \epsilon.$$

Letting $\epsilon \rightarrow 0$, we obtain

$$(2.2) \quad \eta_\Omega(w_1, w_2) \geq \lambda_\Omega(w_1, w_2) > 0.$$

To prove the completeness, we use the following fact (see [1, Theorem 2.5.28, p. 52]): *a locally compact length (metric) space X is complete if and only if every closed disc in X is compact (see also [1, p. 28])*. Because λ_Ω is complete, each closed hyperbolic disk $\overline{D}_{\lambda_\Omega}(a, r) = \{w \in \Omega : \lambda_\Omega(a, w) \leq r\}$ is compact. Because $\lambda_\Omega \leq \eta_\Omega$, $\overline{D}_{\eta_\Omega}(a, r) \subset \overline{D}_{\lambda_\Omega}(a, r)$. A closed subset of a compact set is compact, so $\overline{D}_{\eta_\Omega}(a, r)$ is compact. \square

The following remark assures that there exists a non-hyperbolic domain for which Theorem 2.1 still satisfies.

Remark 2.2. Let $\Omega = \mathbb{C} \setminus \{0\}$. Then, the Hurwitz density has the elementary formula $\eta_\Omega(w) = 1/8|w|$. This is nothing but a scalar multiplication of the classical quasihyperbolic metric of Ω . The completeness property now follows from the fact that the quasihyperbolic metric space is complete.

We know that the holomorphic functions are global as well as infinitesimal contraction functions with respect to the hyperbolic metric. In analogy to this we now prove that the one-to-one holomorphic functions are global contraction functions for the Hurwitz metric as well.

Proposition 2.3. *Let Ω and Y be proper subdomains of \mathbb{C} and h be an injective holomorphic function from Ω to Y . Then we have the inequality*

$$\eta_Y(h(w_1), h(w_2)) \leq \eta_\Omega(w_1, w_2)$$

for all w_1, w_2 in Ω . Equality holds in the above inequality if h is a conformal homeomorphism.

Proof. By definition of $\eta_\Omega(w_1, w_2)$, for any $\epsilon > 0$ there exists a path γ joining w_1 and w_2 in Ω such that

$$\int_\gamma \eta_\Omega(w) |dw| \leq \eta_\Omega(w_1, w_2) + \epsilon.$$

By Definition 2, it follows clearly that

$$(2.3) \quad \eta_Y(h(w_1), h(w_2)) \leq \int_{h(\gamma)} \eta_Y(z) |dz| = \int_\gamma \eta_Y(h(w)) |h'(w)| |dw|.$$

Since h is a one-to-one holomorphic function, by Theorem A, we have

$$(2.4) \quad \eta_Y(h(w)) |h'(w)| \leq \eta_\Omega(w)$$

for every w in Ω . Combining (2.3) and (2.4), we obtain

$$\eta_Y(h(w_1), h(w_2)) \leq \int_{\gamma} \eta_{\Omega}(w) |dw| \leq \eta_{\Omega}(w_1, w_2) + \epsilon.$$

Letting $\epsilon \rightarrow 0$, we conclude what we wanted to prove. \square

3. The generalized Hurwitz metric

In Section 2 we discussed the alternate definition of the Hurwitz density. By adopting the idea of generalized Kobayashi density we are going to define and study the generalized Hurwitz density in this section. The distance decreasing property of the Hurwitz density implies that for any holomorphic function h from \mathbb{D} to Ω with $h(s) = w$, $h(t) \neq w$ for all t in $\mathbb{D} \setminus \{s\}$ and $h'(s) \neq 0$, we have the inequalities

$$\eta_{\Omega}(h(s)) |h'(s)| \leq \eta_{\mathbb{D}}(s),$$

and

$$\eta_{\Omega}(h(s)) \leq \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|}.$$

Since the formula (2.1) provides an existence of a holomorphic function h for which the equality holds, we have

$$\eta_{\Omega}(w) = \inf \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|},$$

where the infimum is taken over all holomorphic functions h from \mathbb{D} to Ω with $h(s) = w$, $h(t) \neq w$ for all $t \in \mathbb{D} \setminus \{s\}$, $h'(s) \neq 0$. This leads to the notion of introducing generalized Hurwitz density for an arbitrary domain Ω .

Definition 3. For any domain $\Omega \subset \mathbb{C}$, the *generalized Hurwitz density* is defined as

$$\eta_{\Omega}^{\mathbb{D}}(w) = \inf \frac{\eta_{\mathbb{D}}(s)}{|h'(s)|},$$

where the infimum is taken over all $h \in \mathcal{H}(\mathbb{D}, \Omega)$ with $h(s) = w$, $h(t) \neq w$ for all $t \in \mathbb{D} \setminus \{s\}$, $h'(s) \neq 0$, and all s in \mathbb{D} .

Remark 3.1. If $\Omega \subsetneq \mathbb{C}$ and $a \in \Omega$, then there exists a Hurwitz covering map g from \mathbb{D} to Ω which realizes the infimum; thus $\eta_{\Omega}(a) = \eta_{\Omega}^{\mathbb{D}}(a)$ for all $a \in \Omega$.

Note that, in Definition 3 it is not required to choose the domain Ω to be a proper subdomain of the complex plane \mathbb{C} . In the following theorem, we calculate $\eta_{\Omega}^{\mathbb{D}}$, when $\Omega = \mathbb{C}$.

Theorem 3.2. Suppose that Ω is the whole complex plane. Then the generalized Hurwitz density $\eta_{\Omega}^{\mathbb{D}}(w)$ is identically equal to zero for all elements w in Ω .

Proof. Let $w \in \Omega$ be an arbitrary element and n be any positive integer. Setting $h_n(s) = (s - t)n + w$. Clearly, h_n is a sequence of holomorphic functions from \mathbb{D} into \mathbb{C} with $h_n(t) = w$ and $h_n(s) \neq w$ for all $s \in \mathbb{D} \setminus \{t\}$. By Definition 3, we have

$$\eta_{\Omega}^{\mathbb{D}}(w) \leq \frac{\eta_{\mathbb{D}}(t)}{|h'_n(t)|}.$$

Since the Hurwitz and the hyperbolic densities coincide on simply connected domains, we have

$$\eta_{\Omega}^{\mathbb{D}}(w) \leq \frac{\lambda_{\mathbb{D}}(t)}{n}.$$

Letting n goes to infinity, we obtain that $\eta_{\Omega}^{\mathbb{D}}(w) = 0$. \square

The definition of the generalized Hurwitz density can further be generalized by changing the fixed domain \mathbb{D} to an arbitrary proper subdomain Y of \mathbb{C} , that is, by pushing forward the Hurwitz density on Y to Ω by a holomorphic function having some special property. Here, we call Y as the *basepoint domain*. This idea leads to the following definition.

Definition 4. Let $\Omega \subset \mathbb{C}$ be arbitrary. For all $s \in Y$, the *generalized Hurwitz density* η_{Ω}^Y for the basepoint domain Y is defined as

$$\eta_{\Omega}^Y(w) = \inf \frac{\eta_Y(s)}{|h'(s)|},$$

where η_Y is the Hurwitz density on Y and the infimum is taken over all holomorphic functions h from Y to Ω with $h(s) = w$, $h(t) \neq w$ for all $t \in Y \setminus \{s\}$, $h'(s) \neq 0$.

In view of the nature of Definition 4, it is here appropriate to remark that η_{Ω}^Y can be $+\infty$ at some points, or even at every point.

We will now prove some expected elementary properties of η_{Ω}^Y . We start by comparing the Hurwitz and the generalized Hurwitz densities on proper subdomains of the complex plane.

Proposition 3.3. Let $Y \subset \mathbb{C}$ be a domain and Ω be a proper subdomain of \mathbb{C} . Then for every point w in Ω , we have

$$\eta_{\Omega}^Y(w) \geq \eta_{\Omega}(w).$$

Proof. Let $a \in Y$ and h be any holomorphic function from Y to Ω with $h(a) = b$, $h(s) \neq b$ for all $s \in Y \setminus \{a\}$ and $h'(a) \neq 0$. Then by distance decreasing property of the Hurwitz density, we have

$$\eta_{\Omega}(h(a))|h'(a)| \leq \eta_Y(a),$$

and

$$\eta_{\Omega}(b) \leq \frac{\eta_Y(a)}{|h'(a)|}.$$

Taking the infimum on both sides over $h \in \mathcal{H}(Y, \Omega)$ with $h(a) = b$, $h(s) \neq b$ for all $s \in Y \setminus \{a\}$, $h'(a) \neq 0$, we have

$$\eta_{\Omega}^Y(b) \geq \eta_{\Omega}(b).$$

Since $b \in \Omega$ is arbitrary, the above inequality holds true for every $b \in \Omega$. \square

One naturally asks the comparison of the classical generalized Kobayashi density with the generalized Hurwitz density. Recall the definition of the generalized Kobayashi density.

Definition 5. Let Ω be a domain in the complex plane. For every $w \in \Omega$, the *generalized Kobayashi density* is given by

$$\kappa_{\Omega}^Y(w) = \inf \frac{\lambda_Y(t)}{|f'(t)|},$$

where λ_Y is the hyperbolic density on a hyperbolic domain $Y \subset \mathbb{C}$ and the infimum is taken over all $f \in \mathcal{H}(Y, \Omega)$ and all points $t \in Y$ such that $f(t) = w$.

We immediately have:

Corollary 3.4. If Y and Ω are hyperbolic domains, then $\eta_{\Omega}^Y \geq \kappa_{\Omega}^Y$.

Corollary 3.5. For proper subdomains Ω and Y of \mathbb{C} , we have $\eta_{\Omega}^Y(w) \geq 0$ for all w in Ω .

There are certain situations where the classical generalized Kobayashi density agrees with the hyperbolic density, see for instance [4]. This motivates us to investigate the situations under which the generalized Hurwitz density coincides with the Hurwitz density. The following proposition justifies one such case and a few more situations will be covered in the next section.

Proposition 3.6. Let Ω and Y be proper subdomains of \mathbb{C} . If for every $b \in \Omega$, there exists a holomorphic covering map h_b from $Y \setminus \{a\}$ onto $\Omega \setminus \{b\}$ that extends to a holomorphic map of Y onto Ω with $h_b(a) = b$ and $h'_b(a) \neq 0$, then

$$\eta_{\Omega}^Y(w) = \eta_{\Omega}(w)$$

for all w in Ω . In particular, we also have

$$\eta_{\Omega}^{\Omega}(w) = \eta_{\Omega}(w)$$

for every w in Ω .

Proof. Since h_b is a holomorphic map from Y to Ω with $h_b(a) = b$, $h_b(w) \neq b$ for all w in $Y \setminus \{a\}$ and $h'_b(a) \neq 0$, by the definition of generalized Hurwitz density, we have

$$(3.1) \quad \eta_{\Omega}^Y(b) \leq \frac{\eta_Y(a)}{|h'_b(a)|}.$$

In addition, by Theorem A, we obtain

$$(3.2) \quad \eta_{\Omega}(b)|h'_b(a)| = \eta_Y(a).$$

Combining (3.1) and (3.2), we obtain

$$\eta_{\Omega}^Y(b) \leq \eta_{\Omega}(b).$$

Since b is an arbitrary point, it follows that $\eta_{\Omega}^Y(w) \leq \eta_{\Omega}(w)$ for all $w \in \Omega$. On the other hand, by Proposition 3.3 it follows that $\eta_{\Omega}^Y(w) \geq \eta_{\Omega}(w)$. Hence the proof is complete. \square

Proposition 3.6 is stronger, because for non-simply connected domains Y and Ω the proposition certainly holds (see for instance Example 3.9). To demonstrate this, we use the distance decreasing property of the generalized Hurwitz density, which is proved below (see Theorem 3.8). However, for simply connected domains we have the following special situation.

Corollary 3.7. *If $Y \subsetneq \mathbb{C}$ is a simply connected domain and $\Omega \subsetneq \mathbb{C}$ is any domain, then*

$$\eta_{\Omega}^{\mathbb{D}} \equiv \eta_{\Omega}^Y \equiv \eta_{\Omega}.$$

Proof. Since $Y \subsetneq \mathbb{C}$ is a simply connected domain, by Riemann Mapping Theorem there exists a conformal homeomorphism T from Y onto \mathbb{D} . Furthermore, $\Omega \subsetneq \mathbb{C}$ implies that for every point $w \in \Omega$ there is a Hurwitz covering map g_w from \mathbb{D} onto Ω with $g_w(0) = w$. Hence, by using the composed map $g \circ T$ from Y onto Ω in Proposition 3.6, we conclude our result. \square

It is well-known that both the Hurwitz and the hyperbolic metrics as well the generalized Kobayashi metric κ have distance decreasing properties. The following result provides a similar property for the generalized Hurwitz metric.

Theorem 3.8 (Distance decreasing property of the generalized Hurwitz density). *Let Ω and Δ be any subdomains of \mathbb{C} and $Y \subsetneq \mathbb{C}$ be a domain. If h is a holomorphic function from Ω to Δ with $h(a) = b$, $h'(a) \neq 0$ and $h(w) \neq b$ for all $w \in \Omega \setminus \{a\}$, then*

$$\eta_{\Delta}^Y(h(a))|h'(a)| \leq \eta_{\Omega}^Y(a).$$

Proof. By the definition of generalized Hurwitz density, for every $\epsilon > 0$ there exist a point $c \in Y$ and a holomorphic map g from Y to Ω with $g(c) = a$, $g(s) \neq a$ for all $s \in Y \setminus \{c\}$, $g'(c) \neq 0$ and

$$(3.3) \quad \eta_{\Omega}^Y(a) \geq \frac{\eta_Y(c)}{|g'(c)|} - \epsilon.$$

Note that, $h \circ g$ maps Y to Δ such that $(h \circ g)(c) = b$, $(h \circ g)(s) \neq b$ for all $s \in Y \setminus \{c\}$ and $(h \circ g)'(c) = h'(g(c))g'(c) = h'(a)g'(c) \neq 0$. Therefore, using $h \circ g$ in the definition of $\eta_{\Delta}^Y(h(a))$, we have

$$\eta_{\Delta}^Y(h(a)) \leq \frac{\eta_Y(c)}{|(h \circ g)'(c)|}.$$

By (3.3) and using the chain rule, it follows that

$$\eta_{\Delta}^Y(h \circ g)(c)|h'(a)| \leq \frac{\eta_Y(c)}{|g'(c)|} \leq \eta_{\Omega}^Y(a) + \epsilon.$$

Letting ϵ goes to zero, we obtain

$$\eta_{\Delta}^Y(h(a))|h'(a)| \leq \eta_{\Omega}^Y(a).$$

This completes the proof. \square

We now provide an example which demonstrate Proposition 3.6 in non-simply connected domains.

Example 3.9. Let $Y = \mathbb{D}^* := \mathbb{D} \setminus \{0\}$, the punctured unit disk and $\Omega = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the punctured plane. We shall prove that for all $w \in \mathbb{C}^*$

$$\eta_{\mathbb{C}^*}^{\mathbb{D}^*}(w) = \eta_{\mathbb{C}^*}(w).$$

As stated in [8, p. 322] (see also [7, (4.1)]), the Hurwitz covering map from \mathbb{D} onto $\mathbb{C} \setminus \{1\}$ is obtained by the infinite product representation

$$(3.4) \quad g(w) = 16w \prod_{n=1}^{\infty} \left(\frac{1+w^{2n}}{1+w^{2n-1}} \right)^8, \quad |w| < 1.$$

For $s \in \mathbb{D}$, it is well known that the map $T(z) = (z-s)/(1-\bar{s}z)$ defines a Möbius transformation of \mathbb{D} onto itself. Clearly,

$$T(s) = 0, \quad T'(s) = (1+|s|^2)/(1-|s|^2) > 0.$$

Since g is a holomorphic covering map and T is a one-one holomorphic map on \mathbb{D} , the composition $g \circ T$ is also a holomorphic covering map satisfying $(g \circ T)(s) = 0$ and $(g \circ T)(t) \neq 0$ for all $t \in \mathbb{D} \setminus \{s\}$. Furthermore, $(g \circ T)'(s) > 0$ which follows from the chain rule and the fact that $g'(0) = 16 > 0$ and $T'(s) > 0$. By the distance decreasing property we have

$$(3.5) \quad \eta_{\mathbb{C} \setminus \{1\}}((g \circ T)(s))(g \circ T)'(s) = \eta_{\mathbb{D}}(s).$$

Restricting the function $g \circ T$ onto \mathbb{D}^* and plugging it in the definition of $\eta_{\mathbb{C} \setminus \{1\}}^{\mathbb{D}^*}(0)$ we obtain

$$\eta_{\mathbb{C} \setminus \{1\}}^{\mathbb{D}^*}(0) \leq \frac{\eta_{\mathbb{D}^*}(s)}{(g \circ T)'(s)}.$$

Now, we choose a sequence $s_n \in \mathbb{D}^*$ such that $|s_n| \rightarrow 1$. By using the same argument as above, we can find Möbius transformations T_n from \mathbb{D} onto itself with $T(s_n) = 0$ and $T'(s_n) > 0$. Therefore, it follows from (3.5) that

$$\eta_{\mathbb{C} \setminus \{1\}}^{\mathbb{D}^*}(0) \leq \frac{\eta_{\mathbb{D}^*}(s_n)}{(g \circ T_n)'(s_n)} = \frac{\eta_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)} \eta_{\mathbb{C} \setminus \{1\}}(0),$$

since $(g \circ T_n)(s_n) = 0$. We notice from [7, Section 2] that the Hahn density of the punctured unit disk obtained by (see [6, (2)])

$$S_{\mathbb{D}^*}(s_n) = \frac{1 + |s_n|}{4|s_n|(1 - |s_n|)}$$

exceeds the Hurwitz density. Thus, we obtain

$$\begin{aligned} \eta_{\mathbb{C} \setminus \{1\}}^{\mathbb{D}^*}(0) &\leq \frac{\eta_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)} \eta_{\mathbb{C} \setminus \{1\}}(0) \leq \frac{S_{\mathbb{D}^*}(s_n)}{\eta_{\mathbb{D}}(s_n)} \eta_{\mathbb{C} \setminus \{1\}}(0) \\ &= \frac{1 + |s_n|}{4|s_n|(1 - |s_n|)} (1 - |s_n|^2) \eta_{\mathbb{C} \setminus \{1\}}(0). \end{aligned}$$

Now, letting $|s_n| \rightarrow 1$ we have $\eta_{\mathbb{C} \setminus \{1\}}^{\mathbb{D}^*}(0) \leq \eta_{\mathbb{C} \setminus \{1\}}(0)$. The reverse inequality is followed by Proposition 3.3. Now, by using the holomorphic functions $f(w) = 1 - w$ and $h(w) = bw$ (for some complex constant b) in the distance decreasing property for the generalized Hurwitz density, it follows that, both metrics coincide on \mathbb{C}^* . That is, $\eta_{\mathbb{C}^*}^{\mathbb{D}^*}(w) = \eta_{\mathbb{C}^*}(w)$ for all $w \in \mathbb{C}^*$.

Next we define the generalized Hurwitz distance between two points in a domain.

Definition 6. Let $\Omega \subset \mathbb{C}$ and $Y \subsetneq \mathbb{C}$ be domains. For $w_1, w_2 \in \Omega$, define

$$\eta_{\Omega}^Y(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \eta_{\Omega}^Y(w) |dw|,$$

where the infimum is taken over all rectifiable paths γ in Ω joining z_1 to z_2 .

Proof of the following theorem is similar to that of Theorem 2.1.

Theorem 3.10. Let $Y \subsetneq \mathbb{C}$ be a domain. If Ω is a hyperbolic domain, then $(\Omega, \eta_{\Omega}^Y)$ is a complete metric space.

We do have also the distance decreasing property in the global sense whose proof follows the steps of the proof of Proposition 2.3.

Theorem 3.11. Let Y be a proper subdomain of \mathbb{C} and Ω, Δ be any subdomain of \mathbb{C} . If h is a one-to-one holomorphic map from Ω to Δ , then

$$\eta_{\Delta}^Y(h(w_1), h(w_2)) \leq \eta_{\Omega}^Y(w_1, w_2)$$

for all w_1, w_2 in Ω .

Note that, till now we have derived all the results of the generalized Hurwitz density η_{Ω}^Y for a base domain Y . In the next theorem we will see the comparison between generalized Hurwitz densities when the range domain is fixed while the source domain is varying.

Theorem 3.12. Let Y_1, Y_2 be proper subdomains of \mathbb{C} and Ω be any subdomain of \mathbb{C} . If for every point $b \in Y_2$, there exist a point $a \in Y_1$ and a holomorphic covering map h_b from $Y_1 \setminus \{a\}$ onto $Y_2 \setminus \{b\}$ which extends to a holomorphic

map from Y_1 onto Y_2 with $h_b(a) = b$, $h_b(w) \neq b$ for any $w \in Y_1 \setminus \{a\}$ and $h'_b(a) \neq 0$, then

$$\eta_{\Omega}^{Y_1}(\zeta) \leq \eta_{\Omega}^{Y_2}(\zeta)$$

for all ζ in Ω .

Proof. Let ζ be any arbitrary point in Ω and ϵ be a positive real number. By definition of $\eta_{\Omega}^{Y_2}$, there exist a point b in Y_2 and a holomorphic function g from Y_2 to Ω with $g(b) = \zeta$, $g(s) \neq \zeta$ for any $s \in Y_2 \setminus \{b\}$, $g'(b) \neq 0$ such that

$$(3.6) \quad \eta_{\Omega}^{Y_2}(\zeta) \geq \frac{\eta_{Y_2}(b)}{|g'(b)|} - \epsilon.$$

Since for every point $b \in Y_2$, there exist a point a in Y_1 and a holomorphic covering h_b from Y_1 onto Y_2 with $h_b(a) = b$, $h_b(w) \neq b$ for any $w \in Y_1 \setminus \{a\}$, $g'(a) \neq 0$, by Theorem A we have

$$(3.7) \quad \eta_{Y_2}(b)|h'_b(a)| = \eta_{Y_1}(a).$$

Note that, the composition $g \circ h_b$ is a holomorphic function from Y_1 to Ω with $(g \circ h_b)(a) = \zeta$, $(g \circ h_b)(w) \neq \zeta$ for any w in Y_1 and $(g \circ h_b)'(a) = g'(b)h'_b(a) \neq 0$. Therefore, by the definition of $\eta_{\Omega}^{Y_1}$, we obtain

$$(3.8) \quad \eta_{\Omega}^{Y_1}(a) \leq \frac{\eta_{Y_1}(a)}{|(g \circ h_b)'(a)|} = \frac{\eta_{Y_1}(a)}{|g'(b)h'_b(a)|}.$$

By (3.6), (3.7), (3.8), it follows that

$$\eta_{\Omega}^{Y_2}(\zeta) \geq \frac{\eta_{Y_2}(b)}{|g'(b)|} - \epsilon = \frac{\eta_{Y_1}(a)}{|g'(b)||h'_b(a)|} - \epsilon \geq \eta_{\Omega}^{Y_1}(\zeta) - \epsilon.$$

Letting ϵ goes to zero, we have $\eta_{\Omega}^{Y_2}(\zeta) \geq \eta_{\Omega}^{Y_1}(\zeta)$, which completes the proof of our result. \square

We look forward for the existence of non-simply connected domains Y_1 and Y_2 validating the statement of Theorem 3.12, however, they remain open due to their non-trivial nature.

Corollary 3.13. *If $Y_1 \subsetneq \mathbb{C}$ is a simply connected domain, then for all proper subdomains Y_2 and Ω of \mathbb{C} , we have*

$$\eta_{\Omega}(w) = \eta_{\Omega}^{\mathbb{D}}(w) = \eta_{\Omega}^{Y_1}(w) \leq \eta_{\Omega}^{Y_2}(w)$$

for all w in Ω .

Proof. Since $Y_1 \subsetneq \mathbb{C}$ is a simply connected domain, by Riemann Mapping Theorem, there exists a conformal homeomorphism f from Y_1 onto \mathbb{D} . Furthermore, there exists a Hurwitz covering map T_b from \mathbb{D} onto Y_2 for every b in Y_2 . Thus, the composed map $T_b \circ f$ is a holomorphic covering from $Y_1 \setminus \{f^{-1}(0)\}$ onto $Y_2 \setminus \{b\}$, which extends to a holomorphic function from Y_1 to Y_2 with $(T_b \circ f)(f^{-1}(0)) = b$, $(T_b \circ f)'(f^{-1}(0)) \neq 0$ and $(T_b \circ f)(s) \neq b$ for all s in $Y_1 \setminus \{f^{-1}(0)\}$. Taking $h_b = T_b \circ f$ in Theorem 3.12, we obtain the desired result. \square

Two subdomains Y_1 and Y_2 of \mathbb{C} are conformally equivalent if there exists a holomorphic bijection f from Y_1 to Y_2 .

Corollary 3.14. *Let $\Omega \subset \mathbb{C}$ be any arbitrary domain. If $Y_1 \subsetneq \mathbb{C}$ and $Y_2 \subsetneq \mathbb{C}$ are conformally equivalent domains, then*

$$\eta_{\Omega}^{Y_1}(w) = \eta_{\Omega}^{Y_2}(w)$$

for all w in Ω .

4. Lipschitz domain

In this section, one of our main objectives is to study the situations, in terms of the Hurwitz non-Lipschitz domains, when the Hurwitz density coincides with the generalized Hurwitz density. The following notations are useful in the definition of Hurwitz Lipschitz domains. Let Y be a hyperbolic domain and Ω be a subdomain of Y . If i is the inclusion map from Ω to Y , the global contraction constant $gl_{\eta}(\Omega, Y)$ is defined by

$$gl_{\eta}(\Omega, Y) := \sup_{w_1, w_2 \in \Omega, w_1 \neq w_2} \frac{\eta_Y(w_1, w_2)}{\eta_{\Omega}(w_1, w_2)}.$$

If Y is any proper subdomain of \mathbb{C} , then the infinitesimal contraction constant is defined as

$$l_{\eta}(\Omega, Y) := \sup_{w \in \Omega} \frac{\eta_Y(w)}{\eta_{\Omega}(w)}.$$

Since the inclusion map is an *injective* holomorphic function from Ω to Y , by the distance decreasing property of Hurwitz density, we have $\eta_Y(w) \leq \eta_{\Omega}(w)$ for every w in Ω . Furthermore, by Proposition 2.3 it follows that $\eta_Y(w_1, w_2) \leq \eta_{\Omega}(w_1, w_2)$ for all w_1 and w_2 in Ω . Thus, both infinitesimal and global contraction constants are less than or equal to 1.

Theorem 4.1. *Let $Y \subsetneq \mathbb{C}$ be a domain. If Ω is a subdomain of Y , then $gl_{\eta}(\Omega, Y) \leq l_{\eta}(\Omega, Y) \leq 1$. Furthermore, the inclusion map i from Ω to Y is a strict infinitesimal contraction map, whenever Ω is a proper subdomain of Y .*

Proof. Let w_1, w_2 be any two points in Ω and $\gamma \subset \Omega$ be any path joining w_1 and w_2 such that

$$\eta_{\Omega}(w_1, w_2) = \int_{\gamma} \eta_{\Omega}(w) |dw|.$$

By the definition of $\eta_Y(w_1, w_2)$, it follows that

$$\eta_Y(w_1, w_2) \leq \int_{\gamma} \frac{\eta_Y(w)}{\eta_{\Omega}(w)} \eta_{\Omega}(w) |dw| \leq l_{\eta}(\Omega, Y) \eta_{\Omega}(w_1, w_2).$$

Thus, $gl_{\eta}(\Omega, Y) \leq l_{\eta}(\Omega, Y)$.

The proof of the second part follows from [7, Theorem 6.1]. \square

Definition 7. Let Ω be a subdomain of a domain Y in \mathbb{C} . Then Ω is called a *Hurwitz Lipschitz subdomain* of Y , if the inclusion map from Ω to Y is a strict infinitesimal contraction. That is, the infinitesimal contraction constant l_Ω is strictly less than 1.

By Proposition 2.3, for any proper subdomain Ω of \mathbb{C} , we have $\eta_\Omega^Y \geq \eta_\Omega$. However, in the following theorem, we find a condition on Y so that for every proper subdomain Ω of \mathbb{C} , the Hurwitz and the generalized Hurwitz densities coincide. We adopt the proof technique from [9, Theorem 2.1] where the author compares the hyperbolic density with the Kobayashi density.

Theorem 4.2. *If Ω is any proper subdomain of \mathbb{C} and Y is a Hurwitz non-Lipschitz subdomain of \mathbb{D} , then we have*

$$\eta_\Omega^Y(w) = \eta_\Omega(w)$$

for every w in Ω .

Proof. To show that $\eta_\Omega^Y = \eta_\Omega$, we only need to show that $\eta_\Omega^Y \leq \eta_\Omega$, as the relation $\eta_\Omega^Y \geq \eta_\Omega$ always holds. Since $\Omega \subsetneq \mathbb{C}$, for every point $w \in \Omega$ there exists a Hurwitz covering map g from \mathbb{D} onto Ω with $g(0) = w$, $g(s) \neq w$ for all $s \in \mathbb{D} \setminus \{0\}$ and $g'(0) \neq 0$. We now pre-compose g with a Möbius transformation T of \mathbb{D} that maps s in Y to the origin. Therefore $g \circ T$ is a holomorphic covering of Ω from $\mathbb{D} \setminus \{s\}$ onto $\Omega \setminus \{w\}$, which extends to a holomorphic function from \mathbb{D} to Ω with $(g \circ T)(s) = w$, $(g \circ T)(t) \neq w$ for all t in $\mathbb{D} \setminus \{s\}$ and $(g \circ T)'(s) = g'(0)T'(s) \neq 0$. Thus, by Theorem A, we have

$$(4.1) \quad \eta_\Omega(w)|(g \circ T)'(s)| = \eta_{\mathbb{D}}(s).$$

Now, let h be the restriction of $g \circ T$ on Y . By the definition of η_Ω^Y , we obtain

$$\eta_\Omega^Y(w) \leq \frac{\eta_Y(s)}{|h'(s)|} = \frac{\eta_Y(s)}{|(g \circ T)'(s)|}.$$

On the other hand, by the help of (4.1), we have

$$\eta_\Omega^Y(w) \leq \frac{\eta_Y(s)}{\eta_{\mathbb{D}}(s)} \eta_\Omega(w).$$

Since Y is a non-Lipschitz Hurwitz subdomain of \mathbb{D} , by choosing s in Y appropriately, $\eta_Y(s)/\eta_{\mathbb{D}}(s)$ can be made as close to 1 as we wish. Thus, we can say that

$$\eta_\Omega^Y(w) \leq \eta_\Omega(w).$$

Since $w \in \Omega$ is an arbitrary element, therefore we have $\eta_\Omega^Y(w) = \eta_\Omega(w)$ for all w in Ω . \square

In order to generalize Theorem 4.2 for a broader class of domains in \mathbb{C} , we now discuss the notion of quasi-bounded domains as follows.

Definition 8. A domain Y in \mathbb{C} is said to be *quasi-bounded* if the smallest simply connected plane domain containing Y is a proper subset of \mathbb{C} . We denote the smallest simply connected domain by \hat{Y} .

Example 4.3. All bounded domains in \mathbb{C} are quasi-bounded.

The following theorem is an analog of [9, Theorem 2.2] from the notion of hyperbolic density to the notion of Hurwitz density.

Theorem 4.4. *If Ω is any proper subdomain of the complex plane and Y is quasi-bounded, non-Lipschitz Hurwitz subdomain of \hat{Y} , then*

$$\eta_{\Omega}^Y(w) = \eta_{\Omega}(w)$$

for all w in Ω .

Proof. Since \hat{Y} is simply connected, by Riemann Mapping Theorem there exists a conformal homeomorphism h from \hat{Y} onto \mathbb{D} . Note that the conformal mappings are isometries for the Hurwitz metric (see [7, Corollary 6.2]), and thus we have

$$(4.2) \quad \eta_{\mathbb{D}}(h(s))|h'(s)| = \eta_{\hat{Y}}(s)$$

for all s in \hat{Y} . Furthermore, the restriction of h to Y , resulting a conformal homeomorphism from Y onto $h(Y)$. Therefore, we have

$$(4.3) \quad \eta_{h(Y)}(s)|h'(s)| = \eta_Y(s).$$

By using (4.2), (4.3) and the definition of $l_{\eta}(Y, \hat{Y})$, we obtain

$$l_{\eta}(Y, \hat{Y}) = \sup_{z \in Y} \frac{\eta_Y(z)}{\eta_{\hat{Y}}(z)} = \sup_{z \in h(Y)} \frac{\eta_{h(Y)}(z)}{\eta_{\mathbb{D}}(z)} = l_{\eta}(h(Y), \mathbb{D}).$$

Thus, Y is a non-Lipschitz Hurwitz subdomain of \hat{Y} if and only if $h(Y)$ is a non-Lipschitz Hurwitz subdomain of \mathbb{D} . By Theorem 4.2, it follows that

$$(4.4) \quad \eta_{\Omega}^{h(Y)}(w) = \eta_{\Omega}(w)$$

for all w in Ω . Since Y and $h(Y)$ are conformally homeomorphic by the map h , by Corollary 3.14, we have

$$(4.5) \quad \eta_{\Omega}^Y(w) = \eta_{\Omega}^{h(Y)}(w)$$

for all w in Ω . Combining (4.4) and (4.5), we obtain

$$\eta_{\Omega}^Y(w) = \eta_{\Omega}(w),$$

as desired. \square

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