

## ON THE RATES OF CONVERGENCE IN WEAK LIMIT THEOREMS FOR NORMALIZED GEOMETRIC SUMS

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ABSTRACT. The main purpose of this paper is to establish the rates of convergence in weak limit theorems for normalized geometric sums of independent identically distributed random variables via Zolotarev's probability metric.

### 1. Introduction

Throughout the paper let  $\{X_j, j \geq 1\}$  be a sequence of independent, identically distributed (i.i.d.) random variables. Let  $\nu_p$  be a geometric random variable with parameter  $p \in (0, 1)$ , denoted by  $\nu_p \sim \text{Geo}(p)$ , whose probability mass function given by

$$\mathbb{P}(\nu_p = r) = p(1-p)^{r-1}, \quad r = 1, 2, \dots$$

Moreover, assume that  $\nu_p$  is independent for all  $X_j$  for  $j \geq 1$ . Set a summation up to random variable  $\nu_p$  of i.i.d. random variables

$$S_{\nu_p} = X_1 + X_2 + \dots + X_{\nu_p}.$$

Random sum  $S_{\nu_p}$  is said to be *geometric sum*. For  $p \in (0, 1)$ , let us denote by  $c(p)$  the positive constant depending on  $p \in (0, 1)$ , such that  $c(p) \rightarrow 0$  as  $p \downarrow 0^+$ . Asymptotic behavior of distribution of normalized geometric sum  $c(p)S_{\nu_p}$  is a main research object in this paper.

During the last several decades, weak limit theorems for geometric sums have become one of most important problems in applied probability and related topics. There are many applications in insurance risk theory, stochastic finance and queuing theory, etc. A number of known results related to geometric sums and their applications have been investigated by Rényi (1957), Korolev and Kruglov (1990), Gnedenko and Kruglov (1996), Kalashnikov (1997), Sandhya (1999), Kozubowski (2000, 2017), Kotz et al. (2001), Klebanov et

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al. (1984, 2003), Asmussen (2003, 2010), Bon (2002), Grandell (2002), Korolev et al. (2016, 2017), Hung (2013), etc. (see [1, 2, 4–6, 9–19, 22] and the references given there).

Asymptotic behavior of geometric sums with convergence rates in weak limit theorems is a matter of concern in probability theory and in applied areas. Up to the present, an estimation on convergence rate in well-known Rényi's limit theorem (1957) for geometric sum is established based on method of probability metrics, but the proof is omitted, (see comments in [20], Theorem 8.1.6, page 246). In recent years, some rates of convergence in limit theorems for geometric sums of i.i.d. random variables is given by Hung (2013) in [9], using a linear operator–method originated by Trotter (1959) in [23]. However, in several situations it is very hard to establish convergence rates in weak limit theorems for geometric sums. Some classic methods are not satisfying strict requirements of considered distributions. Perhaps this is a main reason that applications of geometric summations are still not widely available. Therefore, establishment of convergence rates for distributions of normalized geometric sums via Zolotarev's probability metric is main aim of this paper.

It is worth pointing out that the Zolotarev's probability metric used in our paper since its simpleness and ideality. Furthermore, Zolotarev's probability metric may be compared with well-known metrics like Kolmogorov metric, total variation metric, Lévy-Prokhorov metric and the metric based on Trotter operator, etc. (see [3, 8, 24–26]).

The article is organized as follows. Definitions and properties of Zolotarev's probability metric will be recalled in Section 2. Moreover, a class of geometrically strictly stable distributions and some special distributions belonging to this class will be recalled in this section. Our main results are presented in Section 3.

In this paper,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{R} = (-\infty, +\infty)$  are denoted by set of natural numbers and set of real numbers, respectively. Symbols  $\stackrel{D}{=}$  and  $\xrightarrow{D}$  are denoted equality in distribution and convergence in distribution, respectively.

## 2. Preliminaries

Let us denote by  $\mathfrak{X}$  the set of random variables defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and by  $C_B(\mathbb{R})$  the set of all real-valued, bounded, uniformly continuous functions defined on  $\mathbb{R}$ , with norm  $\|f\| = \sup_{w \in \mathbb{R}} |f(w)|$ . Furthermore, for any  $m \in \mathbb{N}$ ,  $m < s \leq m + 1$  and  $\beta = s - m$ , let us set

$$C_B^m(\mathbb{R}) = \left\{ f \in C_B(\mathbb{R}) : f^{(k)} \in C_B(\mathbb{R}), 1 \leq k \leq m \right\},$$

where  $f^{(k)}$  is a derivative function of order  $k \in \mathbb{N}$ . Moreover, we denote

$$\mathfrak{D}_s = \left\{ f \in C_B^m(\mathbb{R}) : |f^{(m)}(x) - f^{(m)}(y)| \leq |x - y|^\beta \right\}.$$

**Definition** ([24], [25] and [26]). *Zolotarev's probability metric* on  $\mathfrak{X}$  between two arbitrary random variables  $X, Y \in \mathfrak{X}$ , is defined by

$$\zeta_s(X, Y) = \sup_{f \in \mathfrak{D}_s} \left| \mathbb{E}[f(X) - f(Y)] \right|.$$

Interesting cases of Zolotarev's probability metric of order  $s = 2$  and  $s = 3$  are presented as follows.

- For  $m = 1$  and  $s = 2$ , Zolotarev's probability metric of order 2 is defined by

$$\zeta_2(X, Y) = \sup_{f \in \mathfrak{D}_2} \left| \mathbb{E}[f(X) - f(Y)] \right|,$$

where  $X, Y \in \mathfrak{X}$ , and

$$\mathfrak{D}_2 = \left\{ f \in C_B^1(\mathbb{R}) : |f'(x) - f'(y)| \leq |x - y| \right\}.$$

- For  $m = 2$  and  $s = 3$ , Zolotarev's probability metric of order 3 is given by

$$\zeta_3(X, Y) = \sup_{f \in \mathfrak{D}_3} \left| \mathbb{E}[f(X) - f(Y)] \right|,$$

where  $X, Y \in \mathfrak{X}$ , and

$$\mathfrak{D}_3 = \left\{ f \in C_B^2(\mathbb{R}) : |f''(x) - f''(y)| \leq |x - y| \right\}.$$

*Remark 2.1* ([3], [24], [25] and [26]).

- (1) Zolotarev's probability metric  $\zeta_s$  is an ideal metric of order  $s$ , i.e., for any  $c \neq 0$ , and for random variables  $X, Y, Z \in \mathfrak{X}$ ,

$$\zeta_s(X + Z, Y + Z) \leq \zeta_s(X, Y),$$

and

$$\zeta_s(cX, cY) = |c|^s \zeta_s(X, Y),$$

where  $Z$  is independent of  $X$  and  $Y$ .

- (2) If  $\zeta_s(X_n, X_0) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $X_n \xrightarrow{D} X_0$  as  $n \rightarrow \infty$ .

The following lemma states a property of Zolotarev's probability metric that will be useful in next section.

**Lemma 2.2.** *Let  $\{X_j, j \geq 1\}$  and  $\{Y_j, j \geq 1\}$  be two independent sequences of i.i.d. random variables. Then, for  $n \in \mathbb{N}$ ,*

$$(1) \quad \zeta_s \left( \sum_{j=1}^n X_j, \sum_{j=1}^n Y_j \right) \leq n \zeta_s(X_1, Y_1).$$

*Proof.* The lemma will be proved by mathematical inductive method as follows.

- For  $n = 1$ , inequality (1) holds.

- For  $n = 2$ , from Remark 2.1 it follows that

$$\begin{aligned}\zeta_s(X_1 + X_2, Y_1 + Y_2) &\leq \zeta_s(X_1 + X_2, Y_2 + X_1) + \zeta_s(Y_2 + X_1, Y_1 + Y_2) \\ &\leq \zeta_s(X_2, Y_2) + \zeta_s(X_1, Y_1) = 2\zeta_s(X_1, Y_1).\end{aligned}$$

- Assume that inequality (1) will be valid up to  $n = m \in \mathbb{N}$ , i.e.,

$$\zeta_s\left(\sum_{j=1}^m X_j, \sum_{j=1}^m Y_j\right) \leq m\zeta_s(X_1, Y_1).$$

Inequality (1) will be proved for  $n = m + 1$ . We first observe that

$$\begin{aligned}\zeta_s\left(\sum_{j=1}^{m+1} X_j, \sum_{j=1}^{m+1} Y_j\right) &= \zeta_s\left(\sum_{j=1}^m X_j + X_{m+1}, \sum_{j=1}^m Y_j + Y_{m+1}\right) \\ &\leq \zeta_s\left(\sum_{j=1}^m X_j + X_{m+1}, Y_{m+1} + \sum_{j=1}^m X_j\right) \\ &\quad + \zeta_s\left(Y_{m+1} + \sum_{j=1}^m X_j, \sum_{j=1}^m Y_j + Y_{m+1}\right) \\ &\leq \zeta_s(X_{m+1}, Y_{m+1}) + \zeta_s\left(\sum_{j=1}^m X_j, \sum_{j=1}^m Y_j\right) \\ &\leq \zeta_s(X_{m+1}, Y_{m+1}) + m\zeta_s(X_1, Y_1) \\ &= (m+1)\zeta_s(X_1, Y_1).\end{aligned}$$

The proof is complete.  $\square$

In the sequel, we recall several necessary probability distributions with their characterizations (see [17] for more details).

A random variable  $\mathcal{L}$  is said to have *symmetric Laplace distribution with parameters zero* and  $\sigma > 0$ , denoted by  $\mathcal{L} \sim \text{Laplace}(0, \sigma)$ , if its characteristic function is given in form

$$\varphi_{\mathcal{L}}(t) = \frac{1}{1 + \frac{\sigma^2}{2}t^2}, \quad t \in \mathbb{R}.$$

Furthermore, if  $\mathcal{L} \sim \text{Laplace}(0, \sigma)$ , then

$$\mathbb{E}(\mathcal{L}) = 0; \quad \mathbb{E}(\mathcal{L}^2) = \sigma^2 \quad \text{and} \quad \mathbb{E}|\mathcal{L}|^3 = \frac{3\sigma^3}{\sqrt{2}}.$$

A random variable  $\xi$  is said to have *symmetric Linnik distribution with two parameters*  $\alpha \in (0, 2]$  and  $\sigma > 0$ , denoted by  $\xi \sim \text{Linnik}(\alpha, \sigma)$ , if its characteristic function is defined by

$$\varphi_{\xi}(t) = \frac{1}{1 + \sigma^{\alpha}|t|^{\alpha}}, \quad t \in \mathbb{R}.$$

It is easily seen that, for  $\alpha = 2$ , a symmetric Linnik distribution reduces to a symmetric Laplace distribution. Thus, it is also known as  $\alpha$ -Laplace distribution.

**Definition** ([12, Definition 2, p. 793]). A random variable  $Y$  is said to be *geometrically strictly stable* (GSS) if for any  $p \in (0, 1)$ , there is a constant  $c(p) > 0$  such that

$$Y \stackrel{D}{=} c(p) \sum_{j=1}^{\nu_p} Y_j,$$

where  $Y, Y_1, Y_2, \dots$  are i.i.d. random variables, and  $\nu_p \sim \text{Geo}(p)$ , independent of  $Y_j$  for all  $j \geq 1$ .

It is clear that random variables  $Z \sim \text{Exp}(\lambda)$ ,  $\mathcal{L} \sim \text{Laplace}(0, \sigma)$  and  $\xi \sim \text{Linnik}(\alpha, \sigma)$  are GSS random variables. This will be confirmed by Proposition 3.1 with Eq. (2), Proposition 3.2 with Eq. (3) and Proposition 3.3 with Eq. (4) in the next section.

### 3. Main results

We shall begin with showing following propositions.

**Proposition 3.1.** *Let  $Z \sim \text{Exp}(\lambda)$  be an exponential distributed random variable with parameter  $\lambda > 0$ . Then*

$$(2) \quad Z \stackrel{D}{=} p \sum_{j=1}^{\nu_p} Z_j,$$

where  $\nu_p \sim \text{Geo}(p)$ , and  $Z_j$ 's are i.i.d. copies of  $Z$ , independent of  $\nu_p$  for all  $j \geq 1$ .

*Proof.* Let us denote by  $\psi_{\nu_p}(w) = \frac{pw}{1-(1-p)w}$  the generating function of  $\nu_p$  and by  $\varphi_{Z_1}(t) = \frac{\lambda}{\lambda-it}$  the characteristic function of  $Z_1$ , respectively. Then, according to [7], the characteristic function of normalized geometric sum  $p \sum_{j=1}^{\nu_p} Z_j$  is defined as follows

$$\varphi_{p \sum_{j=1}^{\nu_p} Z_j}(t) = \varphi_{\sum_{j=1}^{\nu_p} Z_j}(pt) = \psi_{\nu_p}[\varphi_{Z_1}(pt)] = \frac{p\lambda}{p\lambda - ipt} = \frac{\lambda}{\lambda - it} = \varphi_Z(t), \quad t \in \mathbb{R}.$$

The proof is complete. □

**Proposition 3.2.** *Let  $\mathcal{L} \sim \text{Laplace}(0, \sigma)$  be a symmetric Laplace distributed random variable with parameters zero and  $\sigma > 0$ . Then*

$$(3) \quad \mathcal{L} \stackrel{D}{=} p^{1/2} \sum_{j=1}^{\nu_p} \mathcal{L}_j,$$

where  $\nu_p \sim \text{Geo}(p)$ , and  $\mathcal{L}_j$ 's are i.i.d. copies of  $\mathcal{L}$ , independent of  $\nu_p$  for all  $j \geq 1$ .

*Proof.* According to [7], since  $\nu_p \sim \text{Geo}(p)$  and  $\mathcal{L}_1 \sim \text{Laplace}(0, \sigma)$ , the characteristic function of normalized geometric sum  $p^{1/2} \sum_{j=1}^{\nu_p} \mathcal{L}_j$  is defined by

$$\varphi_{p^{1/2} \sum_{j=1}^{\nu_p} \mathcal{L}_j}(t) = \varphi_{\sum_{j=1}^{\nu_p} \mathcal{L}_j}(p^{1/2}t) = \psi_{\nu_p}[\varphi_{\mathcal{L}_1}(p^{1/2}t)] = \frac{1}{1 + \frac{\sigma^2}{2}t^2} = \varphi_{\mathcal{L}}(t), \quad t \in \mathbb{R}.$$

This completes the proof. □

**Proposition 3.3.** *Let  $\xi \sim \text{Linnik}(\alpha, \sigma)$  be a symmetric Linnik distributed random variable with two parameters  $\alpha \in (0, 2]$  and  $\sigma > 0$ . Then*

$$(4) \quad \xi \stackrel{D}{=} p^{1/\alpha} \sum_{j=1}^{\nu_p} \xi_j,$$

where  $\nu_p \sim \text{Geo}(p)$ , and  $\xi_j$ 's are i.i.d. copies of  $\xi$ , independent of  $\nu_p$  for all  $j \geq 1$ .

*Proof.* On account of [7], since  $\xi_1 \sim \text{Linnik}(\alpha, \sigma)$  and  $\nu_p \sim \text{Geo}(p)$ , the characteristic function of normalized geometric sum  $p^{1/\alpha} \sum_{j=1}^{\nu_p} \xi_j$  will be given by

$$\varphi_{p^{1/\alpha} \sum_{j=1}^{\nu_p} \xi_j}(t) = \varphi_{\sum_{j=1}^{\nu_p} \xi_j}(p^{1/\alpha}t) = \psi_{\nu_p}[\varphi_{\xi_1}(p^{1/\alpha}t)] = \frac{1}{1 + \sigma^\alpha |t|^\alpha} = \varphi_\xi(t), \quad t \in \mathbb{R}.$$

The proof is complete. □

From now on, unless otherwise specified, we shall consider  $m \in \mathbb{N}$  and  $m < s \leq m + 1$ . Furthermore, for any  $p \in (0, 1)$ , let  $c(p)$  be a positive constant depending on  $p \in (0, 1)$ , such that  $c(p) \downarrow 0$  as  $p \downarrow 0$ . Next theorem will be fundamental in this paper.

**Theorem 3.4.** *Let  $\{X_j, j \geq 1\}$  and  $\{Y_j, j \geq 1\}$  be two independent sequences of i.i.d. random variables with  $\mathbb{E}|X_1|^s < +\infty$  and  $\mathbb{E}|Y_1|^s < +\infty$  for  $s \geq 1$ . Let  $\nu_p \sim \text{Geo}(p)$  be a geometric random variable with parameter  $p \in (0, 1)$ , independent of all  $X_j$  and  $Y_j$  for  $j \geq 1$ . Assume that*

$$(5) \quad \mathbb{E}(X_1^k) = \mathbb{E}(Y_1^k) \quad \text{for } 1 \leq k \leq m.$$

Moreover, let  $c(p)$  satisfy  $\frac{[c(p)]^s}{p} = o(1)$  as  $p \rightarrow 0$  for  $s \geq 1$ . Then

$$(6) \quad \zeta_s \left( c(p) \sum_{j=1}^{\nu_p} X_j, c(p) \sum_{j=1}^{\nu_p} Y_j \right) \leq \frac{[c(p)]^s}{p \cdot m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s).$$

*Proof.* Since Zolotarev’s probability metric is ideality of order  $s$ , according to Lemma 2.2, it follows that

$$\begin{aligned} \zeta_s \left( c(p) \sum_{j=1}^{\nu_p} X_j, c(p) \sum_{j=1}^{\nu_p} Y_j \right) &= [c(p)]^s \zeta_s \left( \sum_{j=1}^{\nu_p} X_j, \sum_{j=1}^{\nu_p} Y_j \right) \\ &= [c(p)]^s \sum_{n=1}^{\infty} \left\{ \mathbb{P}(\nu_p = n) \zeta_s \left( \sum_{j=1}^n X_j, \sum_{j=1}^n Y_j \right) \right\} \\ &\leq [c(p)]^s \sum_{n=1}^{\infty} \left\{ \mathbb{P}(\nu_p = n) n \zeta_s(X_1, Y_1) \right\} \\ &= [c(p)]^s \mathbb{E}(\nu_p) \zeta_s(X_1, Y_1) = \frac{[c(p)]^s}{p} \zeta_s(X_1, Y_1). \end{aligned}$$

For any  $x \in \mathbb{R}$  and  $0 < \theta < 1$ , by Taylor series expansion for a function  $f \in \mathfrak{D}_s$  with Lagrange remainder, we obtain

$$\begin{aligned} f(x) &= f(0) + \sum_{k=1}^{m-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(m)}(\theta x)}{m!} x^m \\ &= f(0) + \sum_{k=1}^m \frac{f^{(k)}(0)}{k!} x^k + \frac{x^m}{m!} \left[ f^{(m)}(\theta x) - f^{(m)}(0) \right]. \end{aligned}$$

Hence, for any  $x, y \in \mathbb{R}$  and  $f \in \mathfrak{D}_s$ , we obtain

$$f(x) - f(y) \leq \sum_{k=1}^m \frac{f^{(k)}(0)}{k!} (x^k - y^k) + \frac{1}{m!} (|x|^s + |y|^s).$$

By definition of Zolotarev’s probability metric and using assumptions of this theorem, one has

$$\begin{aligned} \zeta_s(X_1, Y_1) &\leq \sup_{f \in \mathfrak{D}_s} \left\{ \sum_{k=1}^m \frac{|f^{(k)}(0)|}{k!} |\mathbb{E}(X_1^k) - \mathbb{E}(Y_1^k)| + \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s) \right\} \\ &= \frac{1}{m!} (\mathbb{E}|X_1|^s + \mathbb{E}|Y_1|^s). \end{aligned}$$

Therefore, from (5), estimation (6) is valid. This completes the proof. □

*Remark 3.5.* Under the above assumptions of Theorem 3.4, then

$$c(p) \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} c(p) \sum_{j=1}^{\nu_p} Y_j \quad \text{as } p \rightarrow 0.$$

Rate of convergence in Rényi-type limit theorem for geometric sums (see for instance [10]) will be established by following corollary.

**Corollary 3.6.** *Let  $\{X_j, j \geq 1\}$  be a sequence of positive i.i.d. random variables with  $\mathbb{E}(X_1) = \mu_1 \in (0, +\infty)$  and  $\mathbb{E}(X_1^2) = \mu_2 \in (0, +\infty)$ . Let  $\nu_p \sim \text{Geo}(p)$  be a*

geometric random variable with parameter  $p \in (0, 1)$ , independent of  $X_j$  for all  $j \geq 1$ . Then

$$(7) \quad \zeta_2 \left( p \sum_{j=1}^{\nu_p} X_j, Z \right) \leq p(\mu_2 + 2\mu_1^2),$$

where  $Z \sim \text{Exp}(\mu_1^{-1})$ .

*Proof.* Since  $Z \sim \text{Exp}(\mu_1^{-1})$ , by Proposition 3.1, we have

$$Z \stackrel{D}{=} p \sum_{j=1}^{\nu_p} Z_j,$$

where  $Z_j$ 's are i.i.d. copies of  $Z$ , independent of  $\nu_p$  for all  $j \geq 1$ . Moreover,

$$\mathbb{E}(Z_1) = \mu_1 = \mathbb{E}(X_1) \quad \text{and} \quad \mathbb{E}(Z_1^2) = 2\mu_1^2.$$

According to Theorem 3.4 with  $m = 1$ ,  $s = 2$  and  $c(p) = p$ , then estimation (7) will be concluded. The proof is straight-forward.  $\square$

*Remark 3.7.* From (7) the Rényi's theorem in [10] will be valid as follows:

$$p \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} Z \quad \text{as} \quad p \rightarrow 0.$$

*Remark 3.8.* The upper bound in (7) was also studied by Hung (2013) via a probability metric based on Trotter's operator (see [9] for more details).

**Corollary 3.9.** *Let  $\{X_j, j \geq 1\}$  be a sequence of i.i.d. random variables with  $\mathbb{E}(X_1) = 0$ ,  $\mathbb{E}(X_1^2) = \sigma^2 \in (0, +\infty)$  and  $\mathbb{E}|X_1|^3 = \rho \in (0, +\infty)$ . Let  $\nu_p \sim \text{Geo}(p)$  be a geometric random variable with parameter  $p \in (0, 1)$ , independent of  $X_j$  for all  $j \geq 1$ . Then*

$$(8) \quad \zeta_3 \left( p^{1/2} \sum_{j=1}^{\nu_p} X_j, \mathcal{L} \right) \leq p^{1/2} \left( \frac{\rho}{2} + \frac{3\sigma^3}{2\sqrt{2}} \right),$$

where  $\mathcal{L} \sim \text{Laplace}(0, \sigma/\sqrt{2})$  with  $\sigma > 0$ .

*Proof.* Since  $\mathcal{L} \sim \text{Laplace}(0, \sigma/\sqrt{2})$ , by Proposition 3.2, it follows that

$$\mathcal{L} \stackrel{D}{=} p^{1/2} \sum_{j=1}^{\nu_p} \mathcal{L}_j,$$

where  $\mathcal{L}_j$ 's are i.i.d. copies of  $\mathcal{L}$ , independent of  $\nu_p$  for all  $j \geq 1$ . Furthermore,

$$\mathbb{E}(\mathcal{L}_1) = 0 = \mathbb{E}(X_1); \quad \mathbb{E}(\mathcal{L}_1^2) = \sigma^2 = \mathbb{E}(X_1^2) \quad \text{and} \quad \mathbb{E}|\mathcal{L}_1|^3 = \frac{3\sigma^3}{\sqrt{2}}.$$

According to Theorem 3.4 with  $m = 2$ ,  $s = 3$  and  $c(p) = p^{1/2}$ , then (8) will be established. This concludes the proof.  $\square$



*Remark 3.10.* According to Corollary 3.9, weak limit theorem for normalized geometric sum of i.i.d. random variables will be stated as follows:

$$p^{1/2} \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} \mathcal{L} \sim \text{Laplace}(0, \sigma/\sqrt{2}) \quad \text{as } p \downarrow 0.$$

This result may be found in [17, p. 30].

It is worth pointing out that, the class of symmetric Linnik distributions coincides with class of distributional limits of sum  $p^{1/\alpha} S_{\nu_p}$  as  $p \downarrow 0$ , where  $\alpha \in (0, 2]$  (see Proposition 4.3.4 in [17, p. 202]). Therefore, normalized geometric sum  $p^{1/\alpha} S_{\nu_p}$  could not be applied directly from Theorem 3.4. The following theorem will establish rate of convergence for normalized geometric sum  $p^{1/\alpha} S_{\nu_p}$  with  $1 < \alpha < 2$ .

**Theorem 3.11.** *Let  $\{X_j, j \geq 1\}$  be a sequence of i.i.d. random variables with  $\mathbb{E}(X_1) = 0$  and  $\mathbb{E}|X_1| = \varrho \in (0, +\infty)$ . Let  $\nu_p \sim \text{Geo}(p)$  be a geometric random variable with parameter  $p \in (0, 1)$ , independent of  $X_j$  for all  $j \geq 1$ . Then*

$$(9) \quad \zeta_2 \left( p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j, \xi \right) \leq 2p^{\frac{2-\alpha}{\alpha}} \sup_{f \in \mathfrak{D}_2} \|f'\| \left( \varrho + \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} \right),$$

where  $\xi \sim \text{Linnik}(\alpha, \sigma)$  with  $1 < \alpha < 2$ ,  $\sigma > 0$  and  $\|f'\| = \sup_{w \in \mathbb{R}} |f'(w)|$ .

*Proof.* Let  $\{\xi_j, j \geq 1\}$  be a sequence of i.i.d. copies of  $\xi$ , independent of  $\nu_p$ . By Proposition 3.3, we have

$$\xi \stackrel{D}{=} p^{1/\alpha} \sum_{j=1}^{\nu_p} \xi_j.$$

Consider  $m = 1$ ,  $s = 2$ , according to Remark 2.1 and Lemma 2.2, it follows that

$$\begin{aligned} \zeta_2 \left( p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j, \xi \right) &= \zeta_2 \left( p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j, p^{1/\alpha} \sum_{j=1}^{\nu_p} \xi_j \right) \\ &= p^{2/\alpha} \zeta_2 \left( \sum_{j=1}^{\nu_p} X_j, \sum_{j=1}^{\nu_p} \xi_j \right) \\ &= p^{2/\alpha} \sum_{n=1}^{\infty} \left\{ \mathbb{P}(\nu_p = n) \zeta_2 \left( \sum_{j=1}^n X_j, \sum_{j=1}^n \xi_j \right) \right\} \\ &\leq p^{2/\alpha} \sum_{n=1}^{\infty} \left\{ \mathbb{P}(\nu_p = n) n \zeta_2(X_1, \xi_1) \right\} = p^{\frac{2-\alpha}{\alpha}} \zeta_2(X_1, \xi_1). \end{aligned}$$

For all  $x, y \in \mathbb{R}$  and  $f \in \mathfrak{D}_2$ , on account of Mean Value Theorem (see [21, p. 107]), we have

$$f(x) - f(y) = (x - y)f'(z) = (x - y)f'(0) + (x - y)[f'(z) - f'(0)],$$

where  $z$  is between  $x$  and  $y$ .

Moreover, since  $f \in \mathfrak{D}_2$ , one has

$$|f'(z) - f'(0)| \leq \sup_{z \in \mathbb{R}} |f'(z)| + \sup_{w \in \mathbb{R}} |f'(w)| = 2\|f'\|.$$

Thus, we infer that

$$f(x) - f(y) \leq (x - y)f'(0) + 2\|f'\|(|x| + |y|).$$

On the other hand, since  $\xi_1 \sim \text{Linnik}(\alpha, \sigma)$  with  $\alpha \in (1, 2)$  and  $\sigma > 0$ , according to [17, p. 212], it follows that

$$\mathbb{E}(\xi_1) = 0 \quad \text{and} \quad \mathbb{E}|\xi_1| = \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} < +\infty.$$

Since  $\mathbb{E}(X_1) = 0$  and  $\mathbb{E}|X_1| = \varrho \in (0, +\infty)$ , and by definition of Zolotarev's probability metric, we conclude that

$$\begin{aligned} \zeta_2(X_1, \xi_1) &= \sup_{f \in \mathfrak{D}_2} \left| \mathbb{E}[f(X_1) - f(\xi_1)] \right| \\ &\leq \sup_{f \in \mathfrak{D}_2} \left| \mathbb{E}[(X_1 - \xi_1)f'(0) + 2\|f'\|(|X_1| + |\xi_1|)] \right| \\ &= 2 \sup_{f \in \mathfrak{D}_2} \|f'\| \left( \varrho + \frac{2\sigma}{\alpha \sin \frac{\pi}{\alpha}} \right). \end{aligned}$$

This finishes the proof.  $\square$

*Remark 3.12.* Under the assumptions of Theorem 3.11, weak limit theorem for normalized geometric sum will be stated as follows

$$p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j \xrightarrow{D} \xi \sim \text{Linnik}(\alpha, \sigma) \quad \text{as } p \downarrow 0.$$

This result may be found in [17, p. 202].

## References

- [1] S. Asmussen, *Ruin Probabilities*, Advanced Series on Statistical Science & Applied Probability, **2**, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. <https://doi.org/10.1142/9789812779311>
- [2] ———, *Applied Probability and Queues*, second edition, Applications of Mathematics (New York), **51**, Springer-Verlag, New York, 2003.
- [3] S. G. Bobkov, *Proximity of probability distributions in terms of Fourier-Stieltjes transforms*, Russian Math. Surveys **71** (2016), no. 6, 1021–1079; translated from Uspekhi Mat. Nauk **71** (2016), no. 6(432), 37–98. <https://doi.org/10.4213/rm9749>
- [4] J. L. Bon, *Geometric sums in reliability evaluation of regenerative systems*, Information Processes **2** (2002), no. 2, 161–163.
- [5] B. V. Gnedenko and V. Yu. Korolev, *Random Summation*, CRC Press, Boca Raton, FL, 1996.
- [6] J. Grandell, *Risk theory and geometric sums*, Information Processes **2** (2002), no. 2, 180–181.
- [7] A. Gut, *Probability: a graduate course*, second edition, Springer Texts in Statistics, Springer, New York, 2013. <https://doi.org/10.1007/978-1-4614-4708-5>

- [8] T. L. Hung, *On a probability metric based on Trotter operator*, Vietnam J. Math. **35** (2007), no. 1, 21–32.
- [9] ———, *On the rate of convergence in limit theorems for geometric sums*, Southeast-Asian J. of Sciences **2** (2013), no. 2, 117–130.
- [10] V. Kalashnikov, *Geometric sums: bounds for rare events with applications*, Mathematics and its Applications, **413**, Kluwer Academic Publishers Group, Dordrecht, 1997. <https://doi.org/10.1007/978-94-017-1693-2>
- [11] L. B. Klebanov, *Heavy Tailed Distributions*, Research Gate, 2003.
- [12] L. B. Klebanov, G. M. Maniya, and I. A. Melamed, *A problem of V. M. Zolotarev and analogues of infinitely divisible and stable distributions in a scheme for summation of a random number of random variables*, Teor. Veroyatnost. i Primenen. **29** (1984), no. 4, 757–760.
- [13] V. Yu. Korolev and A. V. Dorofeeva, *Bounds for the concentration functions of random sums under relaxed moment conditions*, Theory Probab. Appl. **62** (2018), no. 1, 84–97; translated from Teor. Veroyatn. Primen. **62** (2017), no. 1, 104–121. <https://doi.org/10.1137/S0040585X97T988502>
- [14] ———, *Bounds of the accuracy of the normal approximation to the distributions of random sums under relaxed moment conditions*, Lith. Math. J. **57** (2017), no. 1, 38–58. <https://doi.org/10.1007/s10986-017-9342-7>
- [15] V. Yu. Korolev and A. I. Zeifman, *Convergence of statistics constructed from samples with random sizes to the Linnik and Mittag-Leffler distributions and their generalizations*, J. Korean Statist. Soc. **46** (2017), no. 2, 161–181. <https://doi.org/10.1016/j.jkss.2016.07.001>
- [16] V. Yu. Korolev, A. I. Zeifman, and A. Y. Korchagin, *Asymmetric Linnik distributions as limit laws for random sums of independent random variables with finite variances*, Inform. Primen. **10** (2016), no. 4, 21–33.
- [17] S. Kotz, T. J. Kozubowski, and K. Podgórski, *The Laplace Distribution and Generalization*, Springer Science + Business Media, LLC, 2001.
- [18] T. J. Kozubowski, *Geometric infinite divisibility, stability, and self-similarity: an overview*, in Stability in probability, 39–65, Banach Center Publ., **90**, Polish Acad. Sci. Inst. Math., Warsaw, 2010. <https://doi.org/10.4064/bc90-0-3>
- [19] T. J. Kozubowski and K. Podgórski, *Asymmetric Laplace distributions*, Math. Sci. **25** (2000), no. 1, 37–46.
- [20] V. M. Kruglov and V. Yu. Korolev, *Limit theorems for random sums*, Moskov. Gos. Univ., Moscow, 1990.
- [21] W. Rudin, *Principles of Mathematical Analysis*, third edition, McGraw-Hill Book Co., New York, 1976.
- [22] E. Sandhya and R. N. Pillai, *On geometric infinite divisibility*, J. Kerala Stat. Assoc. **10** (1999), 1–7.
- [23] H. F. Trotter, *An elementary proof of the central limit theorem*, Arch. Math. **10** (1959), 226–234. <https://doi.org/10.1007/BF01240790>
- [24] V. M. Zolotarev, *Metric distances in spaces of random variables and of their distributions*, Mat. Sb. (N.S.) **101(143)** (1976), no. 3, 416–454, 456.
- [25] ———, *Approximation of the distributions of sums of independent random variables with values in infinite-dimensional spaces*, Teor. Veroyatnost. i Primenen. **21** (1976), no. 4, 741–758.
- [26] ———, *Probability metrics*, Teor. Veroyatnost. i Primenen. **28** (1983), no. 2, 264–287.

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