Bull. Korean Math. Soc. **57** (2020), No. 5, pp. 1083–1094
https://doi.org/10.4134/BKMS.b190483
pISSN: 1015-8634 / eISSN: 2234-3016

ON PARTIAL VALUE SHARING RESULTS OF MEROMORPHIC FUNCTIONS WITH THEIR SHIFTS AND ITS APPLICATIONS

Vangty Noulorvang and Duc Thoan Pham

Abstract. In this paper, we give some uniqueness theorems of nonconstant meromorphic functions of hyper-order less than one sharing partially three or four small periodic functions with their shifts. As an application, some sufficient conditions for periodicity of meromorphic functions are given. Our results improve and extend previous results of W. Lin, X. Lin and A. Wu [11].

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory [7, 14, 16] and in particular with the most usual of symbol $m(r,f), N(r,f)$ and $T(r,f)$ with a meromorphic function f on $\mathbb{C}$.

Consider a meromorphic function f on $\mathbb{C}$, the order $\rho(f)$ and the hyper-order $\gamma(f)$ of f are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},\ \gamma(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

We denote by $S(r,f)$ a quantity equal to $o(T(r,f))$ for all $r \in (1,\infty)$ outside a finite Borel measure set. In particular, we denote by $S_1(r,f)$ any quantity satisfying $S_1(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure.

We denote $S(f)$ as the family of all meromorphic functions α such that $T(r,\alpha) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure and let $\hat{S}(f) = S(f) \cup \{\infty\}$. For each $a \in \hat{S}(f)$, we say that two meromorphic functions f and g share a IM if $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM.

Received May 12, 2019; Revised April 22, 2020; Accepted July 8, 2020.
2010 *Mathematics Subject Classification.* Primary 32A22, 32H30; Secondary 30D35.
Key words and phrases. Meromorphic functions, sharing partially values, uniqueness theorems.

The next, for positive integers m and k (maybe $k, m = +\infty$), we denote by $E_{k)}^{[m]}(a, f)$ the set of zeros of $f - a$ with multiplicity $l \leq k$, where a zero with multiplicity l is counted l times if $l \leq m$, otherwise the zero is counted m times in the set. The counting function corresponding to $E_{k)}^{[m]}(a, f)$ is denoted by $N_{k)}^{[m]}\left(r, \frac{1}{f-a}\right)$. Similarly, we also denote by $N_{(k}^{[m]}\left(r, \frac{1}{f-a}\right)$ the counting function of those a-points of f whose multiplicities are not less than k in counting the a-points of f.

In case $m = 1$, the symbol $E_{k)}^{[1]}(a, f)$ is replaced by symbol $\overline{E}_{k)}(a, f)$. It means that $\overline{E}_{k)}(a, f)$ is the set of zeros of $f - a$ with multiplicity $l \leq k$, where a zero with multiplicity l is counted only once in the set. The reduced counting functions are denoted by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$. If $m = +\infty$ (respectively $k = +\infty$), we omit character ^{m} (respectively $_{k)}$).

Obviously, if $\overline{E}(a, f) = \overline{E}(a, f)$, then f and g share a IM and if $E(a, f) = E(a, f)$, then f and g share a CM.

The deficiency and reduced deficiency of a with respect to f are defined respectively as follows:

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}, \ \Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

In recent years, uniqueness problem of meromorphic functions sharing values with their shifts is investigated intensively by many authors (see in [2, 5, 8–13, 17, 18]). For instance, in 2009 J. Heittokangas et al. [9] considered the problem of value sharing for shift of a meromorphic function of finite order with three values CM. After that, the result was improved for the case of two shared values CM and one shared value IM also by these authors.

In early 2016, K. S Charak, R. J. Korhonen and G. Kumar [2] gave an example to show that the case one shared values CM and two shared value IM (and hence three shared values IM) does not hold in general.

The notion of partial value sharing of a meromorphic function of hyper-order less than one and its shift was also introduced by K. S. Charak, R. J. Korhonen and G. Kumar in [2]. Then, they obtained an uniqueness theorem of a nonconstant meromorphic function partially sharing four values with its shifts under an appropriate deficiency assumption as follows.

Theorem A ([2])**.** *Let f be a nonconstant meromorphic function of hyper-order $\gamma(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3, a_4 \in \hat{S}(f)$ be four distinct periodic functions with period c. If $\delta(a, f) > 0$ for some $a \in \hat{S}(f)$ and*

$$\overline{E}(a_j, f(z)) \subseteq \overline{E}(a_j, f(z + c)), \ j = 1, 2, 3, 4,$$

then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.

In 2018, W. Lin, X. Lin and A. Wu [11] obtained a counterexample which showed that Theorem A does not hold when the condition “partially shared values $\overline{E}(a_j, f(z)) \subseteq \overline{E}(a_j, f(z+c))$, $j = 1, 2$" is replaced by the condition "truncated partially shared values $\overline{E}_{k)}(a_j, f(c)) \subseteq \overline{E}_{k)}(a_j, f(z+c))$, $j = 1, 2$" with a positive integer k, even if $f(z)$ and $f(z+c)$ share a_3, a_4 CM. Then, they introduced the following results under a reduced deficiency assumption $\Theta(0, f) + \Theta(\infty, f) > \frac{2}{k+1}$. An example was also given to show that this condition is necessary and sharp.

Theorem B ([11])**.** *Let f be a nonconstant meromorphic function of hyper-order $\gamma(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let k_1, k_2 be two positive integers, and let $a_1, a_2 \in S(f) \setminus \{0\}$, $a_3, a_4 \in \hat{S}(f)$ be four distinct periodic functions with period c such that $f(z)$ and $f(z+c)$ share a_3, a_4 CM and*

$$\overline{E}_{k_j)}(a_j, f(z)) \subseteq \overline{E}_{k_j)}(a_j, f(z+c)), \ j = 1, 2.$$

If $\Theta(0, f) + \Theta(\infty, f) > \frac{2}{k+1}$, where $k := \min\{k_1, k_2\}$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Theorem C ([11])**.** *Let f be a nonconstant meromorphic function of hyper-order $\gamma(f) < 1, \Theta(\infty, f) = 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in S(f)$ be three distinct periodic functions with period c such that $f(z)$ and $f(z+c)$ share a_3 CM and*

$$\overline{E}_{k)}(a_j, f(z)) \subseteq \overline{E}_{k)}(a_j, f(z+c)), \ j = 1, 2.$$

If $k \geq 2$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

The periodic functions and elliptic functions have found a wide utilization in many fields [1, 3, 4, 19]. Therefore, it is interesting and important to study the periodicity of meromorphic functions. As an application of Theorems B and C, the above authors gave the sufficient conditions for periodicity of meromorphic functions as follows.

Theorem D ([11])**.** *Assume that f and g are two nonconstant meromorphic functions with $\Theta(\infty, f) = \Theta(\infty, g) = 1$, where f has a nonzero periodic $c \in \mathbb{C} \setminus \{0\}$ with hyper-order $\gamma(f) < 1$. Let k_1, k_2 be two positive integers, $a_1, a_2, a_3 \in S(f)$ be three distinct periodic functions with period c such that f and g share a_3 CM and*

$$\overline{E}_{k)}(a_j, f) \subseteq \overline{E}_{k)}(a_j, g), \ j = 1, 2.$$

Then g is a function with periodic T, where $T \in \{c, 2c\}$, that is $g(z) = g(z+T)$ for all $z \in \mathbb{C}$.

In this article, the first aim is to generalize and improve Theorems B and C by reducing the number of shared values. The second aim is to give some uniqueness theorems in this direction as well as some of their applications. Namely, we will prove the following results.

Theorem 1.1. *Let f be a meromorphic function of hyper-order $\gamma(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in \hat{S}(f)$ be three distinct periodic functions with* *period c and let k be a positive integer. Assume that $f(z)$ and $f(z+c)$ share partially a_1, a_2 CM, i.e.,*

$$E(a_1, f(z)) \subseteq E(a_1, f(z+c)),\ E(a_2, f(z)) \subseteq E(a_2, f(z+c))$$

and

$$\overline{E}_{k)}(a_3, f(z)) \subseteq \overline{E}_{k)}(a_3, f(z+c)).$$

If $\Theta(a,f) > \frac{2}{k+1}$ for some $a \in \hat{S}(f) \setminus \{a_3, \frac{a_3(a_1+a_2)-2a_1a_2}{2a_3-(a_1+a_2)}\}$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Corollary 1.2. *Let f be a nonconstant meromorphic function of hyper-order $\gamma(f) < 1, \Theta(\infty, f) = 1$ and $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2 \in S(f)$ be two distinct periodic functions with period c such that $f(z)$ and $f(z+c)$ share partially a_1 CM, i.e.,*

$$E(a_1, f(z)) \subseteq E(a_1, f(z+c))$$

and

$$\overline{E}_{k)}(a_2, f(z)) \subseteq \overline{E}_{k)}(a_2, f(z+c)).$$

If $k \geq 2$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Theorem 1.1 and Corollary 1.2 improve strongly Theorems B and C respectively.

Obviously, Theorem 1.1 is sharp. Indeed, we recall the example in [11]. Let $f(z) = \sin z$ and $c = \pi$. It is easy to see that $f(z)$ and $f(z+c)$ share 0 and ∞ CM, i.e., $E(0, f(z)) = E(0, f(z+c))$ and $E(\infty, f(z)) = E(\infty, f(z+c))$, and $\overline{E}_{1)}(1, f(z)) = \overline{E}_{1)}(1, f(z+c)) = \emptyset$, but $f(z+c) = -f(z)$ for all $z \in \mathbb{C}$. Here, the condition $\Theta(a,f) > \frac{2}{1+k} = 1$ with some $a \neq \pm 1$ is not satisfied.

In the case $k = \infty$, we have the following theorem.

Theorem 1.3. *Let f be a meromorphic function of hyper-order $\gamma(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. Let $a_1, a_2, a_3 \in \hat{S}(f)$ be three distinct periodic functions with period c. Assume that $f(z)$ and $f(z+c)$ share partially a_1, a_2 CM and share partially a_3 IM, i.e.,*

$$E(a_1, f(z)) \subseteq E(a_1, f(z+c)),\ E(a_2, f(z)) \subseteq E(a_2, f(z+c))$$

and

$$\overline{E}(a_3, f(z)) \subseteq \overline{E}(a_3, f(z+c)).$$

If $\Theta(a,f) > 0$ for some $a \in \hat{S}(f) \setminus \{a_3\}$, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Omitting the deficiency assumption, we will have the following results.

Theorem 1.4. *Let f be a meromorphic function of hyper-order $\gamma(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. Let k, l be two positive integers and let $a_1, a_2, a_3, a_4 \in \hat{S}(f)$ be four distinct periodic functions with period c. Assume that $f(z)$ and $f(z+c)$ share partially a_1, a_2 CM and*

$$\overline{E}_{k)}(a_3, f(z)) \subseteq \overline{E}_{k)}(a_3, f(z+c)),\ \overline{E}_{l)}(a_4, f(z)) \subseteq \overline{E}_{l)}(a_4, f(z+c)).$$

Then the following statements hold:

(i) *If* $kl > \min\{k, l\} + 2$, *then* $f(z) = f(z + c)$ *or* $\frac{f(z)-a_1}{f(z)-a_2} = -\frac{f(z+c)-a_1}{f(z+c)-a_2}$ *for all* $z \in \mathbb{C}$. *Moreover, the latter occurs only when* $\frac{a_4-a_1}{a_4-a_2} = -\frac{a_3-a_1}{a_3-a_2}$.

(ii) *If* $\max\{k, l\} = \infty$, *then* $f(z) = f(z + c)$ *for all* $z \in \mathbb{C}$.

Using the idea in the proof of Theorem D, we get a similar result which is considered an application of Theorem 1.1 and Corollary 1.2.

Theorem 1.5. *Assume that* f *and* g *are two nonconstant meromorphic functions with* $\Theta(\infty, f) = \Theta(\infty, g) = 1$, *where* f *has a nonzero periodic* $c \in \mathbb{C} \setminus \{0\}$ *with hyper-order* $\gamma(f) < 1$. *Let* k *be a positive integer and let* $a_1, a_2 \in S(f)$ *be two distinct periodic functions with period* c *such that* f *and* g *share partially* a_1 *CM and*

$$\overline{E}_{k)}(a_2, f) \subseteq \overline{E}_{k)}(a_2, g).$$

If $k \geq 2$, *then* g *is a function with periodic* c, *that is* $g(z) = g(z + c)$ *for all* $z \in \mathbb{C}$.

By the same argument as in the proof of Theorem 1.5, we also get a result in this form from applying Theorem 1.4.

Theorem 1.6. *Assume that* f *and* g *are two nonconstant meromorphic functions, where* f *has a nonzero periodic* $c \in \mathbb{C}\setminus\{0\}$ *with hyper-order* $\gamma(f) < 1$. *Let* k, l *be two positive integers and let* $a_1, a_2 \in S(f) \setminus \{0\}$ *be two distinct periodic functions with period* c *such that*

$$E(0, f) \subseteq E(0, g), \ E(\infty, f) \subseteq E(\infty, g)$$

and

$$\overline{E}_{k)}(a_1, f) \subseteq \overline{E}_{k)}(a_1, g), \ \overline{E}_{l)}(a_2, f) \subseteq \overline{E}_{l)}(a_2, g).$$

Then the following statements hold:

(i) *If* $kl > \min\{k, l\} + 2$, *then* g *is a function with periodic* T, *where* $T \in \{c, 2c\}$, *that is* $g(z) = g(z + T)$ *for all* $z \in \mathbb{C}$.

(ii) *If* $\max\{k, l\} = \infty$, *then* g *is a function with periodic* c, *that is* $g(z) = g(z + c)$ *for all* $z \in \mathbb{C}$.

2. Some lemmas

Lemma 2.1 ([15, Corollary 1])**.** *Let* f *be a nonconstant meromorphic function on* $\mathbb{C}$. *Let* $a_1, a_2, \ldots, a_q$ $(q \geq 3)$ *be* q *distinct small meromorphic functions of* f *on* $\mathbb{C}$. *Then the following holds*

$$(q-2)T(r, f) \leq \sum_{i=1}^{q} \overline{N}\big(r, \frac{1}{f - a_i}\big) + S(r, f).$$

Lemma 2.2 ([6])**.** *Let* f *be a nonconstant meromorphic function and* $c \in \mathbb{C}$. *If* f *is of finite order, then*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O\left(\frac{\log r}{r} T(r, f)\right)$$

for all r outside of a subset E zero logarithmic density. If the hyper-order $\gamma(f)$ of f is less than one, then for each $\epsilon > 0$, we have

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = o\left(\frac{T(r,f)}{r^{1-\gamma(f)-\epsilon}}\right)$$

for all r outside of a subset finite logarithmic measure.

Lemma 2.3 ([6, Lemma 8.3])**.** *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$ such that hyper-order of T is strictly less than one, i.e.,*

$$\gamma = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r)}{\log r} < 1,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{1-\gamma-\epsilon}}\right),$$

where $\epsilon > 0$ and $r \to \infty$ outside a subset of finite logarithmic measure.

For each meromorphic function f, we denote $f_c(z) = f(z+c)$ and $\Delta_c f := f_c - f$.

Lemma 2.4 ([6, Theorem 2.1])**.** *Let $c \in \mathbb{C}$, and let f be a meromorphic function of hyper-order < 1 such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$ and $a_1(z), \dots, a_q(z)$ be distinct meromorphic periodic small functions of f with period c. Then*

$$m(r,f) + \sum_{k=1}^{q} m\big(r, \frac{1}{f - a_k}\big) \leq 2T(r,f) - N_{pair}(r,f) + S_1(r,f),$$

where

$$N_{pair}(r,f) = 2N(r,f) - N(r, \Delta_c f) + N\big(r, \frac{1}{\Delta_c f}\big).$$

Lemma 2.5 ([16])**.** *Let f be a nonconstant meromorphic function. If $g = \frac{af+b}{cf+d}$, where $a,b,c,d \in S(f)$ and $ad - bc \not\equiv 0$, then*

$$T(r,g) = T(r,f) + S(r,f).$$

3. Proof of Theorem 1.1

Suppose that $f \not\equiv f_c$. Without loss of generality, we can assume that $a_1, a_2, a_3 \in S(f)$. We put

$$g = \frac{f - a_1}{f - a_2} \cdot \frac{a_3 - a_2}{a_3 - a_1} \quad \text{and } h = \frac{g_c}{g}.$$

Then, by Lemma 2.5, we have $T(r,g) = T(r,f) + S(r,f)$ and hence $\gamma(g) = \gamma(f) < 1$. Applying Lemma 2.2, we get

$$m(r,g) = S_1(r,g) = S_1(r,f).$$

We claim that $E(b_i, g) \subseteq E(b_i, g_c) \cup Pole(a_1 a_2) \cup Zero(a_1 - a_2)$ $(i = 1,2)$ and $\overline{E}_{k)}(b_3, g) \subseteq \overline{E}_{k)}(b_3, g_c) \cup Pole(a_1 a_2 a_3) \cup Zero(a_1 - a_2)$, where $b_1 := 0, b_2 := \infty$,

$b_3 := 1$ and $Pole(a_1a_2a_3)$ and $Zero(a_1 - a_2)$ denote the sets of all poles of a_1, a_2, a_3 and all zeros of $a_1 - a_2$ respectively.

Indeed, let $z_0 \in E(0, g)$ with multiplicity $k > 0$. Assume that

$$z_0 \notin Pole(a_1a_2) \cup Zero(a_1 - a_2).$$

It follows that z_0 is a zero or a pole of functions $f - a_1$ or $f - a_2$.

Assume that $z_0 \in E(a_1, f)$ with multiplicity m. By the assumption, $z_0 \in E(a_1, f_c)$ with multiplicity m. Then $f(z_0) = a_1(z_0) = a \neq \infty$ and also $f_c(z_0) = a_1(z_0) = a$. If $a_2(z_0) = a$, then $z_0 \in Zero(a_1 - a_2)$. Otherwise, $f(z_0) - a_2(z_0) \neq 0, \infty$ and $f_c(z_0) - a_2(z_0) \neq 0, \infty$. This means that $m = k$, and hence $z_0 \in E(0, g_c)$ with multiplicity k. It implies that $E(0, g) \subseteq E(0, g_c) \cup Pole(a_1a_2) \cup Zero(a_1 - a_2)$.

By the same arguments for $b_2 = \infty$ and $b_3 = 1$, we get $E(\infty, g) \subseteq E(\infty, g_c) \cup Pole(a_1a_2) \cup Zero(a_1 - a_2)$ and $\overline{E}_{k)}(1, g) \subseteq \overline{E}_{k)}(1, g_c) \cup Pole(a_1a_2a_3) \cup Zero(a_1 - a_2)$. Therefore, we get the claim.

By the Claim and since a_i $(i = 1, 2, 3)$ are small functions of f, it is easy to see that

$$\begin{aligned} &N\big(r, h\big) + N\big(r, \frac{1}{h}\big) \\ = \ &\big(N\big(r, g_c\big) - N\big(r, g\big)\big) + \Big(N\big(r, \frac{1}{g_c}\big) - N\big(r, \frac{1}{g}\big)\Big) + S_1(r, f). \end{aligned} \tag{3.1}$$

According to Lemma 2.3, for any $a \in \hat{S}(f)$ we have

$$\begin{aligned} \overline{N}\big(r, \frac{1}{g_c - a}\big) &\le \overline{N}\big(r + |c|, \frac{1}{g - a}\big) \\ &= \overline{N}\big(r, \frac{1}{g - a}\big) + o\big(\overline{N}\big(r, \frac{1}{g - a}\big)\big) \\ &\le \overline{N}\big(r, \frac{1}{g - a}\big) + S_1(r, g). \end{aligned} \tag{3.2}$$

Applying (3.2) to $a = 0, \infty$ and together this with (3.1) we obtain

$$N\big(r, h\big) + N\big(r, \frac{1}{h}\big) \le S_1(r, f).$$

Therefore, we obtain

$$T(r, h) = S_1(r, f). \tag{3.3}$$

The assumption $f \not\equiv f_c$ implies $g \not\equiv g_c$, so $h \not\equiv 1$. Then, we have

$$\overline{N}_{k)}\big(r, \frac{1}{f - a_3}\big) = \overline{N}_{k)}\big(r, \frac{1}{g - b_3}\big) = S_1(r, g). \tag{3.4}$$

Otherwise, take $z_0 \in \overline{E}_{k)}(b_3, g)$, then $z_0 \in \overline{E}_{k)}(b_3, g_c)$ and hence $h(z_0) = 1$. It follows from (3.3) that $h \equiv 1$, which contradicts our assumption.

We set

$$p = \frac{g_c - b_3}{g - b_3} = \frac{hg - b_3}{g - b_3} \not\equiv 0.$$

By the Claim again, we can show that

$$\begin{aligned}&\overline{N}\big(r,p\big)+\overline{N}\big(r,\frac{1}{p}\big)\\ \le\ &\overline{N}\big(r,g_c\big)-\overline{N}\big(r,g\big)+\overline{N}\big(r,\frac{1}{g-b_3}\big)+\overline{N}\big(r,\frac{1}{g_c-b_3}\big)+S_1(r,f).\end{aligned}$$

Also applying (3.2) again to $a=\infty$ and $a=b_3$, and using the equality (3.4), it is easy to see that

$$\begin{aligned}\overline{N}\big(r,p\big)+\overline{N}\big(r,\frac{1}{p}\big)&\le 2\overline{N}\big(r,\frac{1}{g-b_3}\big)+S_1(r,f)\\ &=2\overline{N}_{(k+1}\big(r,\frac{1}{g-b_3}\big)+S_1(r,f).\end{aligned}$$

We now apply Lemmas 2.1 and 2.5 for each $a\neq 0,\infty$, we obtain

$$\begin{aligned}T(r,p)&\le \overline{N}(r,p)+\overline{N}\big(r,\frac{1}{p}\big)+\overline{N}\big(r,\frac{1}{p-a}\big)+S_1(r,g)\\ &\le 2\overline{N}_{(k+1}\big(r,\frac{1}{g-b_3}\big)+\overline{N}\big(r,\frac{1}{p-a}\big)+S_1(r,g)\\ &\le \frac{2}{k+1}N\big(r,\frac{1}{g-b_3}\big)+\overline{N}\big(r,\frac{1}{p-a}\big)+S_1(r,g)\\ &\le \frac{2}{k+1}T(r,p)+\overline{N}\big(r,\frac{1}{p-a}\big)+S_1(r,p).\end{aligned}$$

It follows from the definition of the deficiency that

$$\Theta(a,p)\le\frac{2}{k+1},\ \forall a\neq 0,\infty,\ i.e.,\ \Theta(a,g)\le\frac{2}{k+1},\forall a\neq b_3,\frac{b_3}{h}.$$

Rewrite p as follows

$$p=\frac{g_c-b_3}{g-b_3}=\frac{g_c-b_3}{\frac{g_c}{h}-b_3}.$$

Repeating the above argument, we have

$$\Theta(a,g_c)\le\frac{2}{k+1},\forall a\neq b_3,b_3h.$$

If $h\not\equiv -1$, then $\frac{b_3}{h}\neq b_3h$. Since $\Theta(a,g)=\Theta(a,g_c)$, it is easy to see that

$$\Theta(a,g)\le\frac{2}{k+1},\forall a\neq b_3. \tag{3.5}$$

If $h\equiv -1$, then $\frac{b_3}{h}=b_3h$. It is easy to see that

$$\Theta(a,f)\le\frac{2}{k+1},\forall a\neq \pm b_3. \tag{3.6}$$

From (3.5)-(3.6) and by simple calculation, we get

$$\Theta(a,f)\le\frac{2}{k+1},\forall a\neq a_3,\frac{a_3(a_1+a_2)-2a_1a_2}{2a_3-(a_1+a_2)}.$$

The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.3

Suppose that $f \not\equiv f_c$. Without loss of generality, we consider the functions g and h as in the proof of Theorem 1.1. We split into two cases.
Case 1: $h \not\equiv -1$. By (3.5), we have $\Theta(a, g) \leq \frac{2}{k+1}, \forall a \neq b_3$. Letting $k \to \infty$, we get

$$\Theta(a, g) \leq 0, \forall a \neq b_3.$$

It follows from Lemma 2.5 that

$$\Theta(a, f) = 0, \ \forall a \neq a_3.$$

Case 2: $h \equiv -1$. Since $g_c = -g$, obviously 1 is a Picard value of g. So

$$N\big(r, \frac{1}{g-1}\big) \leq S(r, g).$$

Applying Lemma 2.4, we obtain

$$\begin{aligned}
& m(r, g) + m\big(r, \frac{1}{g}\big) + m\big(r, \frac{1}{g-1}\big) \\
\leq\ & 2T(r, g) - 2N(r, g) + N(\Delta_c g) - N\big(r, \frac{1}{\Delta_c g}\big) + S_1(r, g) \\
\leq\ & 2T(r, g) - N(r, g) - N\big(r, \frac{1}{g}\big) + S_1(r, g).
\end{aligned}$$

It implies from the definition of characteristic function that

$$T(r, g) \leq N\big(r, \frac{1}{g-1}\big) + S_1(r, g) \leq S_1(r, g)$$

which is impossible. Since the above cases, we arrive at the desired conclusion of Theorem 1.3.

5. Proof of Theorem 1.4

Suppose that $f \not\equiv f_c$.

(i) Assume that $k \geq l$. It is easy to see from the assumption $kl > \min\{k, l\}+2$ that $\frac{l}{l+1} > \frac{2}{k+1}$. By (3.4), we have

$$\begin{aligned}
\overline{N}\big(r, \frac{1}{f-a_4}\big) &= \overline{N}_{l)}\big(r, \frac{1}{f-a_4}\big) + \overline{N}_{(l+1}\big(r, \frac{1}{f-a_4}\big) + S_1(r, f) \\
&\leq \frac{1}{l+1} N\big(r, \frac{1}{f-a_3}\big) + S_1(r, g) \\
&\leq \frac{1}{l+1} T(r, f) + S_1(r, f).
\end{aligned}$$

It follows that

$$\Theta(a_4, f) \geq \frac{l}{l+1} > \frac{2}{k+1}, \ i.e., \ \Theta(b_4, g) \geq \frac{l}{l+1} > \frac{2}{k+1}, \tag{5.1}$$

where $b_4 := \frac{a_4 - a_1}{a_4 - a_2} \cdot \frac{a_3 - a_2}{a_3 - a_1}$.

Suppose that $\frac{a_4-a_1}{a_4-a_2} \neq -\frac{a_3-a_1}{a_3-a_2}$. It implies that $a_4 \neq \frac{a_3(a_1+a_2)-2a_1a_2}{2a_3-(a_1+a_2)}$. Applying Theorem 1.1, we obtain $g = g_c$ and therefore, $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Suppose that $\frac{a_4-a_1}{a_4-a_2} = -\frac{a_3-a_1}{a_3-a_2}$. It implies that $b_4 = -b_3$. Similar to the proof of Theorem 1.3, we again consider the functions g and h as in the proof of Theorem 1.1.

If $h \not\equiv -1$, then by (3.5), we have $\Theta(a,g) \leq \frac{2}{k+1}, \forall a \neq b_3$. This contradicts (5.1) and hence this case can not happen. Therefore, we obtain $h \equiv -1$, i.e., $g = -g_c$. It follows that $\frac{f(z)-a_1}{f(z)-a_2} = -\frac{f(z+c)-a_1}{f(z+c)-a_2}$ for all $c \in \mathbb{C}$.

(ii) Assume that $k = \infty$. By (5.1), we have

$$\Theta(a_4, f) \geq \frac{l}{l+1} \geq \frac{1}{2} > 0.$$

Applying Theorem 1.3, we obtain $f(z) = f(z+c)$ for all $z \in \mathbb{C}$. The proof of Theorem 1.4 is completed.

6. Proof of Theorem 1.5

Applying Lemma 2.1, we have

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,f) + \overline{N}\big(r, \frac{1}{f-a_1}\big) + \overline{N}\big(r, \frac{1}{f-a_2}\big) + S(r,f)\\
&\leq \overline{N}\big(r, \frac{1}{f-a_1}\big) + \overline{N}_{k)}\big(r, \frac{1}{f-a_2}\big) + \overline{N}_{(k+1}\big(r, \frac{1}{f-a_2}\big) + S(r,f)\\
&\leq \overline{N}\big(r, \frac{1}{g-a_1}\big) + \overline{N}_{k)}\big(r, \frac{1}{g-a_2}\big) + \frac{1}{k+1}N\big(r, \frac{1}{f-a_2}\big) + S(r,f)\\
&\leq 2T(r,g) + \frac{1}{3}T(r,f) + S(r,f).
\end{aligned}$$

It implies that

$$T(r,f) \leq 3T(r,g) + S(r,f).$$

Similarly, we also have

$$T(r,g) \leq 3T(r,f) + S(r,g).$$

It follows that $S(r,f) = S(r,g)$ which implies $\gamma(g) = \gamma(f) < 1$. Since the assumption $f(z) = f(z+c)$ for all $z \in \mathbb{C}$ and f, g share partially a_1 CM, it is easy to see that $g(z)$ and $g(z+c)$ also share partially a_1 CM. In addition,

$$\overline{E}_{k)}(a_2, g(z)) = \overline{E}_{k)}(a_2, f(z)) = \overline{E}_{k)}(a_2, f(z+c)) = \overline{E}_{k)}(a_2, g(z+c)).$$

It implies that g satisfies the conditions of Corollary 1.2. Hence, applying this corollary, we obtain $g(z) = g(z+c)$ for all $z \in \mathbb{C}$. The proof of Theorem 1.5 is completed.

Acknowledgements. The authors wish to express their thanks to the referee for his/her valuable suggestions and comments which help us improve the results, especially main result.

References

[1] G. Brosch, *Eindeutigkeitssätze für meromorphic Funktionen*, Thesis, Technical University of Aachen, 1989.

[2] K. S. Charak, R. J. Korhonen, and G. Kumar, *A note on partial sharing of values of meromorphic functions with their shifts*, J. Math. Anal. Appl. **435** (2016), no. 2, 1241–1248. https://doi.org/10.1016/j.jmaa.2015.10.069

[3] S. Chen and W. Lin, *Periodicity and uniqueness of meromorphic functions concerning sharing values*, Houston J. Math. **43** (2017), no. 3, 763–781.

[4] S. Chen and A. Xu, *Periodicity and unicity of meromorphic functions with three shared values*, J. Math. Anal. Appl. **385** (2012), no. 1, 485–490. https://doi.org/10.1016/j.jmaa.2011.06.072

[5] G. G. Gundersen, *Meromorphic functions that share three values IM and a fourth value CM*, Complex Variables Theory Appl. **20** (1992), no. 1-4, 99–106. https://doi.org/10.1080/17476939208814590

[6] R. Halburd, R. Korhonen, and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4267–4298. https://doi.org/10.1090/S0002-9947-2014-05949-7

[7] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[8] J. Heittokangas, R. Korhonen, I. Laine, and J. Rieppo, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56** (2011), no. 1-4, 81–92. https://doi.org/10.1080/17476930903394770

[9] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. Zhang, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. **355** (2009), no. 1, 352–363. https://doi.org/10.1016/j.jmaa.2009.01.053

[10] X.-M. Li and H.-X. Yi, *Meromorphic functions sharing four values with their difference operators or shifts*, Bull. Korean Math. Soc. **53** (2016), no. 4, 1213–1235. https://doi.org/10.4134/BKMS.b150609

[11] W. Lin, X. Lin, and A. Wu, *Meromorphic functions partially shared values with their shifts*, Bull. Korean Math. Soc. **55** (2018), no. 2, 469–478. https://doi.org/10.4134/BKMS.b170072

[12] X. Qi, K. Liu, and L. Yang, *Value sharing results of a meromorphic function $f(z)$ and $f(qz)$*, Bull. Korean Math. Soc. **48** (2011), no. 6, 1235–1243. https://doi.org/10.4134/BKMS.2011.48.6.1235

[13] X. Qi and L. Yang, *Sharing sets of q-difference of meromorphic functions*, Math. Slovaca **64** (2014), no. 1, 51–60. https://doi.org/10.2478/s12175-013-0186-2

[14] M. Ru, *Nevanlinna Theory and Its Relation to Diophantine Approximation*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. https://doi.org/10.1142/9789812810519

[15] K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math. **192** (2004), no. 2, 225–294. https://doi.org/10.1007/BF02392741

[16] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, **557**, Kluwer Academic Publishers Group, Dordrecht, 2003. https://doi.org/10.1007/978-94-017-3626-8

[17] J. Zhang, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. **367** (2010), no. 2, 401–408. https://doi.org/10.1016/j.jmaa.2010.01.038

[18] J. Zhang and R. Korhonen, *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. **369** (2010), no. 2, 537–544. https://doi.org/10.1016/j.jmaa.2010.03.038

[19] H. J. Zheng, *Unicity theorem for period meromorphic functions that share three values*, Chi. Sci. Bull. **37** (1992), no. 1, 12–15.

Vangty Noulorvang
Department of Mathematics
National University of Education
136-Xuan Thuy Str., Hanoi, Vietnam
Email address: vangtynoulorvang@gmail.com

Duc Thoan Pham
Department of Mathematics
National University of Civil Engineering
55 Giai Phong Str., Hanoi, Vietnam
Email address: thoanpd@nuce.edu.vn