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A CHARACTERIZATION OF SEMIGROUPS THROUGH THEIR FUZZY GENERALIZED m-BI-IDEALS

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Abstract. In this article, we initially present the concept of the fuzzy generalized m-bi-ideals in semigroups, then making use of their important types like prime, semiprime and strongly fuzzy generalized m-bi-ideals, we give the important characterizations of the semigroups. We also characterize the m-regular and m-intraregular semigroups using the properties of the irreducible and strongly irreducible fuzyy generalized m-bi-ideals.

1. Introduction and Essential Concepts

A semigroup is a non-empty set S together with an associative binary operation $\cdot$. Semigroups are primarily a generalization of groups originated in the early twentieth century. Semigroups are the simplest algebraic structures, and have uses in all branches of science, mathematics and technologies. Ideals are the most important sub-structures in a semigroup for any kind of study. The generalization of ideals is another topic of importance to explore the intrinsic properties of semigroups. Initially, ideals were generalized to one-sided ideals; one-sided

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ideals were generalized to quasi ideals and quasi ideals were generalized to bi-ideals.

Generalization of ideals in different algebraic structures in the context of the fuzzy set theory started after the introduction of the fuzzy set by Zadeh in his landmark paper [16]. The fuzzy set theory is unique in the sense that it handles the numerical data as well as the linguistic knowledge [13]. It models the complex systems of decision making, managerial tasks and huge networks. It can handle uncertainties, non-linear and distributive processes. In the field of fuzzy set theory, Shabir et. al. gave the concept of fuzzy prime bi-ideals for semigroups [14]. *Munir et. al.,* generalized the bi-ideals in the semiring through a positive integer and called them m-bi-ideals [9]. The author again gave the concept of the m-bi-ideals in the semigroups [10].

In the non-associative structure, generalization of bi-ideals is also one of the important topics to characterize these structures. Kausar et. al [15], discussed the non-associative ordered semigroups by their fuzzy bi-ideals. Kausar [3], established the ordered AG-groupoids by the properties of fuzzy ideals with thresholds $(\alpha, \beta]$. Salahuddin et. al [4], studied the anti fuzzy interior ideals on ordered AG-groupoids. Munir et. al [5], explored the intuitionistic fuzzy bi-ideals in ordered AG-groupoids and also characterized the ordered AG-groupoids in terms of intuitionistic fuzzy bi-ideals.

Getting motivation from the generalization of bi-ideals in semigroups [10], semirings [11], and fuzzy bi-ideals in semigroups [12] through positive integer m, and their applications, we, here, present the idea of the *fuzzy generalized m-bi-ideals*, and the important classes of the prime, strongly prime and semiprime generalized m-bi-ideals in Section 2. The characterization of the m-regular and m-intraregular semigroups is given in Section 3. Section 4 concludes the article.

In the following paragraphs we define the essential concepts which will be used later. Throughout the article work, the letter S is used to denote a semigroup.

Any function $\mu : S \longrightarrow [0,1]$ is known as *fuzzy subset* of S [16]. Consider the fuzzy subsets λ and μ of a semigroup S, the statement $\lambda \leq \mu$ implies that for any $x \in S$, $\lambda(x) \leq \mu(x)$ [7]. The conjunction (disjunction) of μ and ν is defined by $\mu \vee \nu(x) = \max[\mu(x), \nu(x)](\mu \wedge \nu(x) = \min[\mu(x), \nu(x)])$.

Their composition, denoted by $\mu \circ \nu$, is defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy,\, x,y\in S} \{\min[f(x), g(y)]\} \text{ if z is representable as z=xy,} \\ 0 \text{ otherwise,} \end{cases}$$

for all $z \in S$. $\mu \vee \nu$, $\mu \wedge \nu$ and $\mu \circ \nu$ are fuzzy sets of S.

The characteristic function of $A \subseteq S$ defined by

$$\chi_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \notin A. \end{cases}$$

is a fuzzy subset of S. μ is called a *fuzzy subsemigroup* of S if $\mu \circ \mu \leq \mu$. The fuzzy subsemigroup μ is called a fuzzy left respectively fuzzy right ideal of S if $\chi_S \circ \mu \leq \mu$ respectively $\mu \leq \chi_S \circ \mu$. μ is a fuzzy ideal of S if $\chi_S \circ \mu \circ \chi_S \leq \mu$. μ is a fuzzy bi-ideal of S if $\mu \circ \chi_S \circ \mu \leq \mu$. μ is known as a fuzzy quasi-ideal of S if $\mu \circ \chi_S \cap \chi_S \circ \mu \leq \mu$ [1]. A fuzzy subsemigroup μ of a semigroup S is called *fuzzy m-bi ideal* (See [12]) if $\mu \circ \chi_{S^m} \circ \mu \leq \mu$, where

$$\chi_{S^m}(t) = \begin{cases} 1 & \text{if} \quad t \in S^m, \\ 0 & \text{if} \quad t \notin S^m, \end{cases}$$

and m is a positive integer called the *bipotency* of the fuzzy m -bi ideal of S. A fuzzy subsemigroup μ of a semigroup S is called *fuzzy m-quasi ideal* if $\mu \circ \chi_{S^m} \wedge \chi_{S^m} \circ \mu \leq \mu$, m is a positive integer. Every subset of S^m is called the *m-subset* of S. Every subsemigroup of S^m is called *m -subsemigroup* of S.

Definition 1.1. Let $a \in S$ and $t \in (0, 1\}$, then the fuzzy subset of S, defined by

$$a_t(x) = \begin{cases} t & \text{if} \quad x = a \quad \text{for any} \quad x \in S^m, \\ 0 & \text{otherwise,} \end{cases}$$

is called a fuzzy m-point of S.

Definition 1.2. Let $A \subseteq S$ and $t \in (0, 1]$, then the fuzzy subset of S, defined by

$$A_t(x) = \begin{cases} t & \text{if} \quad x \in A \quad \text{for any} \quad x \in S^m, \\ 0 & \text{otherwise,} \end{cases}$$

is called a fuzzy m-subset of S.

The fuzzy m-subset A of S is the union of the fuzzy m-points a of S. In the accompanying sections, we characterize the prime fuzzy generalized m-bi and their related classes. We investigate their significant properties, and portray the m-regular and m-intraregular semigroups by their properties.

2. Fuzzy Generalized m-Bi ideals

Shabir et. al., presented the concepts of the prime fuzzy bi-ideals along with other associated fuzzy bi-ideals in the semigroups in [14], we present this ideas for the case of generalized fuzzy m-bi-ideals and demonstrate their important properties in this section.

Definition 2.1. A fuzzy subset μ of a semigroup S is called *fuzzy generalized m-bi ideal* if $\mu \circ \chi_{S^m} \circ \mu \leq \mu$, where

$$\chi_{S^m}(t) = \begin{cases} 1 & \text{if} \quad t \in S^m, \\ 0 & \text{if} \quad t \notin S^m, \end{cases}$$

and m is a positive integer called the *bipotency* of the fuzzy generalized m -bi ideal of S.

Remark 2.2. A fuzzy m-bi ideal needs to be a subsemigroup of the semigroup S, whereas the fuzzy generalized m-bi-ideals needs not to be a subsemigroup of S.

Example 2.3. Let $S = \{0, 1, 2, 3\}$, be the semigroup with the binary operation $\cdot$ defined on its elements as given in the following table [2]:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	0	0	1	1

If we take $m = 2$, we get, $S^2 = \{0, 1\}$. The 2-bi-ideals of S are $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$ and $\{0, 1, 2, 3\}$, where the generalized 2-bi-ideals of S are $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, and $\{0, 1, 2, 3\}$. This is to be noted that $\{0, 2\}$, $\{0, 3\}$, and $\{0, 2, 3\}$ are not 2-bi-ideals because they are not subsemigroups of S.

DEFINITION 2.4. A fuzzy generalized m-bi ideal, μ, of S is termed as its prime(strongly prime) fuzzy generalized m-bi ideal if for any fuzzy generalized m-bi-ideals λ, ν of S, the statement $\lambda\circ\nu \leq \mu(\lambda\circ\nu\wedge\nu\circ\lambda \leq \mu)$ implies $\lambda \leq \mu$ or $\nu \leq \mu$.

DEFINITION 2.5. A fuzzy generalized m-bi μ of S is known as its semiprime fuzzy generalized m-bi ideal if for any fuzzy generalized m-bi ideal λ of S, $\lambda \circ \lambda \leq \mu$ implies $\lambda \leq \mu$.

Strongly prime fuzzy generalized m-bi ideal $\Rightarrow$ Prime fuzzy generalized m-bi ideal $\Rightarrow$ Semiprime fuzzy generalized m-bi ideal; however, the converses do not follow.

EXAMPLE 2.6. The characteristic functions χ_S and χ_{S^m}, where m is a positive integer, are prime fuzzy generalized m-bi ideal of S. These two are also semiprime and strongly prime fuzzy generalized m-bi-ideals.

DEFINITION 2.7. A fuzzy generalized m-bi ideal μ of S is said to be an irreducible (strongly irreducible) fuzzy generalized m-bi ideal if for any two fuzzy generalized m-bi-ideals λ and ν of S, the statement $\lambda \wedge \nu = \mu(\lambda \wedge \nu \leq \mu)$ implies either $\lambda = \mu$ or $\nu = \mu(\lambda \leq \mu$ or $\nu \leq \mu)$.

DEFINITION 2.8. If the fuzzy generalized m-bi ideal λ satisfies the property $\lambda \circ \lambda = \lambda$, then λ is known as idempotent.

Strongly irreducible generalized m-bi ideal $\Rightarrow$ irreducible generalized m-bi ideal, but not the converse. See the following example.

DEFINITION 2.9. Let μ be a fuzzy subset of a semigroup S. Let $B(\mu)$ denote the intersection of all fuzzy generalized m-bi-ideals of S which contain μ. Then $B(\mu)$ is a fuzzy generalized m-bi ideal of S, called the fuzzy generalized m-bi ideal generated by μ.

THEOREM 2.10. *The characteristic function χ_I on a non-empty subset I of S is a prime fuzzy generalized m-bi ideal iff I is prime generalized m-bi ideal of S.*

Proof. Let I be prime generalized m-bi ideal of S. We want to show that χ_I is a prime fuzzy generalized m-bi ideal of S. For it, let μ and λ be fuzzy generalized m-bi-ideals of S with the property that $\mu \circ \lambda \leq \chi_I$.

If $\mu \nleq \chi_I$, then $\exists$ a fuzzy m-point $x_t \leq \mu (t > 0)$ such that $x_t \nleq \chi_I$. For any $y_r \leq \lambda (r \neq 0)$, since [14]

$$B(x_t) \circ B(y_r) \leq \mu \circ \lambda \leq \chi_I, \tag{1}$$

for all $z \in S^m$, we get

$$B(x_t) \circ B(y_r)(z) = \begin{cases} t \wedge r > 0 \ \text{ if } \ z \in B(x) \cdot B(y), \\ 0 \ \text{ otherwise.} \end{cases} \tag{2}$$

We get from (1) and (1) above, $B(x) \cdot B(y) \subseteq I$. Since I is prime, $B(x) \subseteq I$ or $B(y) \subseteq I$. Also, since $x_t \notin \chi_I$, we have $\lambda = \vee_{y_r \leq \lambda} y_r \leq \chi_I$.

For the converse, let $AB \subseteq I$ for the generalized m-bi-ideals A, B in S. As χ_A, χ_B are fuzzy generalized m-bi-ideals of S and $\chi_A \circ \chi_B = \chi_{AB} \leq \chi_I$ [14]. So by our hypothesis, we have $\chi_A \leq \chi_I$ or $\chi_B \leq \chi_I$. Hence, $A \subseteq I$ or $B \subseteq I$ making I a fuzzy prime generalized m-bi ideal of S. □

Theorem 2.11. *A nonempty subset I of a semigroup S is strongly prime generalized m-bi ideal of S if and only if the characteristic function χ_I is a strongly prime fuzzy generalized m-bi ideal.*

Proof. Suppose that I is a strongly prime generalized m-bi ideal of S, then χ_I is a fuzzy generalized m-bi ideal of S. To prove that $\chi_I$is strongly prime, we consider $\mu \circ \lambda \wedge \lambda \circ \mu \leq \chi_I$ for fuzzy generalized m-bi-ideals μ and λ. If $\mu \nleq C_I$, $\exists$ a fuzzy m-point $x_t \leq \mu (t > 0)$ such that $x_t \nleq \chi_I$. Now for any $y_r \leq \lambda (r \neq 0)$, we get

$$B(x_t) \circ B(y_r) \wedge B(y_r) \circ B(x_t) \leq \mu \circ \lambda \wedge \lambda \circ \mu \leq \chi_I. \tag{3}$$

Moreover, for all $z \in S^m$, we have

$$(B(x_t) \circ B(y_r) \wedge B(y_r) \circ B(x_t))(z) = \begin{cases} t \wedge r \quad \text{if} \quad z \in B(x)B(y) \cap B(y)B(x) \\ 0 \quad \text{otherwise.} \end{cases} \tag{4}$$

From (3) and (4), $B(x)B(y) \cap B(y)B(x) \subseteq I$. Since I is *strongly prime*, so $B(x) \subseteq I$ or $B(y) \subseteq I$. Also, since $x_t \nleq \chi_I$, $B(x) \nsubseteq I$. Eventually, $B(y) \subseteq I$ bringing $y_r \leq \chi_I$. Thus, $\lambda = \bigvee_{y_r \leq g} y_r \leq \chi_I$.

Conversely, if A and B are fuzzy generalized m-bi-ideals of S such $AB \cap BA \subseteq I$, then χ_A, χ_B are fuzzy generalized m-bi-ideals of S and $\chi_A \circ \chi_B \wedge \chi_B \circ \chi_A = \chi_{AB \cap BA} \leq \chi_I$. From the hypothesis, we get $\chi_A \leq \chi_I$ or $\chi_B \leq C_I$. Eventually, either $A \subseteq I$ or $B \subseteq I$. □

THEOREM 2.12. *A non-empty subset A of S is a semiprime generalized m-bi ideal of S iff χ_A is a semiprime fuzzy generalized m-bi ideal of S.*

Proof. Suppose that χ_A is a semiprime fuzzy generalized m-bi ideal of S, then A is a generalized m-bi ideal of S. Let B be a generalized m-bi ideal in S with the property that $B^2 \subseteq A$. That is, $\chi_B \circ \chi_B = \chi_B \leq \chi_A$. Since χ_A is semiprime, $\chi_B \leq \chi_A$, that is $B \subseteq I$.

Conversely, suppose that A is a semiprime generalized m-bi ideal of S, and λ is a fuzzy generalized m-bi ideal of S such that $\lambda \circ \lambda \leq \chi_A$. If $\lambda \nleq \chi_A$, then $\exists$ an m-point $x_t \leq \lambda(t > 0)$ such that $x_t \nleq \chi_A$. Now since, $B(x_t) \circ B(x_t) \leq \lambda \circ \lambda \leq \chi_I$, therefore, for all, $z \in S$, we have

$$B(x_t) \circ B(x_t)(z) = \begin{cases} t & \text{for} \quad z \in B(x)B(x) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $B(x)B(x) \subseteq A$. This implies that $B(x) \subseteq A$; because A is semiprime. Hence, $B(x_t) \leq \chi_A$. But $x_t \leq B(x_t)$ so $x_t \leq \chi_A$; which is a contradiction. Hence $\lambda \leq \chi_I$.

□

LEMMA 2.13. *If $\{\mu_i : i \in I\}$ is a family of prime fuzzy generalized m-bi-ideals of S, then their intersection $\mathcal{A} = \bigwedge_{i \in I} \mu_i$ is a semiprime fuzzy generalized m-bi ideal of S.*

Proof. $\mathcal{A}$ is clearly a fuzzy generalized m-bi ideal of S. Let δ be a fuzzy generalized m-bi ideal with the property that $\delta \circ \delta \leq \mathcal{A}$. So, $\delta \circ \delta \leq \bigwedge_{i \in I} \mu_i$. Then, $\delta \circ \delta \leq \mu_i$ for all $i \in I$. Since each μ_i is a fuzzy prime generalized m-bi ideal of S, so $\delta \leq \mu_i$ for all $i \in I$. Hence, $\delta \leq \bigwedge_{i \in I} \mu_i = \mathcal{A}$. Thus, $\mathcal{A}$ is a fuzzy semiprime generalized m-bi ideal of S. □

COROLLARY 2.14. *The intersection of the fuzzy prime(strongly prime) generalized m-bi-ideals of S is a fuzzy prime (strongly prime) generalized m-bi ideal of S.*

PROPOSITION 2.15. *Strongly irreducible semiprime fuzzy generalized m-bi coincides with the strongly prime fuzzy generalized m-bi ideal in a semigroup S.*

Proof. Let μ is a semiprime fuzzy generalized m-bi ideal of S which is also strongly irreducible, and λ and ν are fuzzy generalized m-bi-ideals of S such that $\lambda \circ \nu \wedge \nu \circ \lambda \leq \mu$. As $(\lambda \wedge \nu)^2 \leq \lambda \circ \nu$ and $(\lambda \wedge \nu)^2 \leq \nu \circ \lambda$. Thus $(\lambda \wedge \nu)^2 \leq \lambda \circ \nu \wedge \nu \circ \lambda$ and so $(\lambda \wedge \nu)^2 \leq \mu$. By the hypothesis, $\lambda \wedge \nu \leq \mu$; μ being semiprime. Again by our hypothesis, either $\lambda \leq \mu$ or $\nu \leq \mu$; μ being strongly irreducible. Thus μ is strongly prime fuzzy generalized -m-bi ideal. □

3. m-Regular and m-Intraregular Semigroups

Every subset of a regular semigroup is generalized bi ideal. In this way, every subset of m-regular and m-intraregular semigroups [10] is a generalized m-bi ideal. In this section, we shall explore when the generalized m-bi-ideals are prime, semiprime and strongly prime generalized m-bi-ideals, and when they are irreducible and strongly irreducible.

PROPOSITION 3.1. *There exists an irreducible fuzzy generalized m-bi ideal λ of S such that $\mu \leq \lambda$ and $\lambda(a) = \alpha$ for every fuzzy generalized m-bi ideal μ of S satisfying $\mu(a) = \alpha$, for $a \in S^m$ and $\alpha \in (0, 1]$,*

Proof. Let $\mathcal{C}$ be the collection of all fuzzy generalized m-bi ideal ν of S such that $\nu(a) = \alpha$ and $\mu \leq \nu$. $\mathcal{C} \neq \emptyset$ because $\mu \in X$. $\mathcal{C}$ is a partially ordered set with respect to *inclusion* of sets. If $\mathcal{D} = \{h_i : i \in I\}$ is a totally ordered subset of $\mathcal{C}$, then $\underset{i \in I}{\vee} h_i$ is a fuzzy generalized m-bi ideal of S containing μ. Because, if $a, b \in S$ and $x \in S^m$ then, $(\underset{i \in I}{\vee} h_i)(axb) = \underset{i \in I}{\vee}(h_i(axb)) \geq \underset{i \in I}{\vee}(h_i(a)) \wedge h_i(b)) = (\underset{i \in I}{\vee} h_i(a)) \wedge (\underset{i \in I}{\vee}(h_i(b)) = (\underset{i \in I}{\vee} h_i)(a) \wedge (\underset{i \in I}{\vee} h_i)(b)$. Hence, $\underset{i \in I}{\vee} h_i$ is a fuzzy generalized m-bi ideal of S. As $\mu \leq h_i$ for each $i \in I$, so $\mu \leq \underset{i \in I}{\vee} h_i$. Also, $(\underset{i \in I}{\vee} h_i)(a) = \underset{i \in I}{\vee} h_i(a) = \alpha$. Thus, $\underset{i \in I}{\vee} h_i$ is the supremum of $\mathcal{D}$. By Zorn's Lemma, there exists a maximal fuzzy generalized m-bi ideal λ of S such that $\mu \leq \lambda$ and $\lambda(a) = \alpha$. It remains to show that λ is an irreducible fuzzy generalized m-bi ideal of S. Let $\lambda = \lambda_1 \wedge \lambda_2$ where λ_1 and λ_2 are fuzzy generalized m-bi-ideals of S. It shall be shown that $\lambda = \lambda_1$ or $\lambda = \lambda_2$. Suppose on contradiction that, $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Since $\lambda(a) = \alpha$, it follows that $\lambda_1(a) \neq \alpha$ and $\lambda_2(a) \neq \alpha$. Therefore, $\alpha = \lambda(a) = (\lambda_1 \wedge \lambda_2)(a) \neq \alpha$; a contradiction. Hence, our supposition is wrong, so either $\lambda = \lambda_1$ or $\lambda = \lambda_2$. Thus, λ is an irreducible fuzzy generalized m-bi ideal of S. □

In the following theorems, we describe the semigroups in which every fuzzy generalized m-bi ideal is a semiprime generalized m-bi ideal.

THEOREM 3.2. *The following conditions are equivalent for a semigroup S [6]:*

1. *S is both m-regular and m-intraregular.*
2. *$\mu \circ \mu = \mu$ for each fuzzy generalized m-bi ideal μ of S such that $\mu \circ \mu \leq \mu$.*
3. *$\lambda \wedge \nu = \lambda \circ \nu \wedge \nu \circ \lambda$ for all fuzzy generalized m-bi-ideals λ and ν of S.*
4. *Each fuzzy generalized m-bi ideal of S is semiprime.*
5. *Each proper fuzzy generalized m-bi ideal of S is the intersection of irreducible semiprime fuzzy generalized m-bi-ideals of S which contain it.*

Proof. (1) $\Rightarrow$ (2): Given that $\mu \circ \mu \leq \mu$. For $a \in S$, since S is both m-regular and m-intra-regular, so there exist $u, v, w \in S^m$ so that $a = aua$ and $a = va^2w$. Since we can write $a = (auva)(awua)$. Hence, $\mu \circ \mu(a) = \sup\limits_{a=bc}\{\min\{\mu(b), \mu(c)\}\} \geq \sup\limits_{a=bc}\{\min\{\mu(auva), \mu(awua)\}\}{=}\min\limits_{a=bc}\{\mu(b), \mu(c)\}\}$ $=\mu \wedge \mu(a) = \mu(a)$. That is, $\mu \circ \mu(a) \geq \mu(a)$. Thus, $\mu \circ \mu = \mu$.

(2) $\Rightarrow$ (3): By the hypothesis, $\lambda \wedge \nu = \lambda \wedge \nu \circ \lambda \wedge \nu = (\lambda \wedge \nu \circ \lambda) \wedge (\lambda \wedge \nu \circ \nu) = \lambda \circ \nu \wedge \nu \circ \lambda$, as required.

(3) $\Rightarrow$ (4) Suppose that μ, λ be fuzzy generalized m-bi-ideals of S such that $\mu \leq \lambda$. By the hypothesis, $\mu = \mu \wedge \mu = \mu \circ \mu \wedge \mu \circ \mu = \mu \circ \mu = \mu^2$. Thus $\mu \leq \lambda$. Hence every fuzzy generalized m-bi ideal of S is semiprime.

(4) $\Rightarrow$ (5): Let $\mathcal{C} = \{\mu_i : i \in I\}$ be the collection of all irreducible fuzzy generalized m-bi-ideals of S which contain a proper fuzzy generalized m-bi ideal μ of S. $\mathcal{C} \neq \emptyset$ because $\mu \in \mathcal{C}$. Moreover, $\mu \leq \bigwedge\limits_{i \in I} \mu_i$. For, $a \in S$, by Proposition 3.1, $\exists$ an irreducible fuzzy generalized m-bi ideal μ of S such that $\mu \leq \mu$ and $\mu(a) = \mu(a)$. Therefore, $\mu \in \{\mu_i : i \in I\}$. Hence $\bigwedge\limits_{i \in I} \mu_i \leq \mu$. So, $\bigwedge\limits_{i \in I} \mu_i(a) \leq \mu_\alpha(a) = \mu(a)$. By hypothesis, each fuzzy generalized m-bi ideal of S is semiprime. So each fuzzy generalized m-bi ideal of S is the intersection of all irreducible semiprime fuzzy generalized m-bi-ideals of S which contain it.

(5) $\Rightarrow$ (2): Let μ be a fuzzy generalized m-bi ideal of S, this gives that μ^2 is a fuzzy generalized m-bi ideal of S as well. μ contains μ^2. By our hypothesis $\mu^2 = \bigwedge\limits_{i \in I} \mu_i$ where each μ_i is irreducible semiprime fuzzy generalized m-bi ideal of S. So $\mu^2 \leq \mu_i$ for all $i \in I$. Thus $\mu \leq \mu_i$ for all $i \in I$, since μ_i are semiprime. Consequently, $\mu \leq \bigwedge\limits_{i \in I} \mu_i = \mu^2$, so that $\mu^2 = \mu$. $\square$

Proposition 3.3. *Suppose S is a m-regular and m-intraregular semigroup. Then the following assertions for a fuzzy generalized m-bi ideal μ of S are equivalent [14].*

1. *μ is strongly irreducible m-bi ideal.*
2. *μ is strongly prime m-bi ideal.*

Proof. (1) $\Rightarrow$ (2): Let μ is strongly irreducible m-bi ideal of S with the property that $\lambda \circ \nu \wedge \nu \circ \lambda \leq \mu$, for any two fuzzy generalized m-bi-ideals λ and ν of S. Theorem 3.2 (3) gives $\lambda \wedge \nu \leq \mu$. By our hypothesis, $\lambda \leq \mu$ or $\nu \leq \mu$ making μ a strongly prime m-bi ideal of S.

(2) $\Rightarrow$ (1): Similarly.

□

Theorem 3.4. *The following statements for a semigroup S are equivalent [14]:*

1. *S is both m-regular and m-intraregular and the collection of fuzzy generalized m-bi-ideals of S is totally ordered by set inclusion.*
2. *Every fuzzy generalized m-bi ideal of S is strongly prime.*

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. We show that an arbitrary fuzzy generalized m-bi-ideal μ of S is strongly prime. Let λ and ν are two fuzzy generalized m-bi-ideals of S satisfying the condition that $\lambda \circ \nu \wedge \nu \circ \lambda \leq \mu$. By the use of Theorem 3.2, we have $\lambda \circ \nu \wedge \nu \circ \lambda = \lambda \wedge \nu$. This gives that $\lambda \wedge \nu \leq \mu$. But the collection of fuzzy generalized m-bi-ideals of S is totally ordered, so either $\lambda \leq \nu$ or $\nu \leq \lambda$. Consequently, either $\lambda \leq \mu$ or $\nu \leq \mu$ making μ strongly prime.

(2) $\Rightarrow$ (1): Suppose the proposition (2) holds. Since a strongly prime fuzzy generalized m-bi-ideal is semiprime, so every fuzzy generalized m-bi ideal of S is semiprime. Using Theorem 3.2, S turns out to be both a m-regular and m-intraregular. We need to show that the set of fuzzy generalized m-bi-ideals of S totally ordered. For this purpose, let λ and ν be arbitrary two fuzzy generalized m-bi-ideals of S. Theorem 3.2 implies that $\lambda \wedge \nu = (\lambda \circ \nu) \wedge (\nu \circ \lambda)$. Since every fuzzy generalized m-bi-ideal is strongly prime, so $\lambda \wedge \nu$ is strongly prime. This implies either $\lambda \leq \lambda \wedge \nu$ or $\nu \leq \lambda \wedge \nu$. If $\lambda \leq \lambda \wedge \nu$, then $\lambda \leq \nu$ and if $\nu \leq \lambda \wedge \nu$, then $\nu \leq \lambda$. This completes the proof.

□

Theorem 3.5. *For a semigroup S, such that the collection of its fuzzy generalized m-bi-ideals is totally ordered under set inclusion, the following statements are equivalent [14] :*

1. *A fuzzy generalized m-bi-ideal of S is prime.*
2. *S is m-regular and m-intraregular.*

Proof. (1) $\Rightarrow$ (2): Suppose that every fuzzy generalized m-bi-ideal of S is prime; and so is semiprime clearly. By Theorem 3.2, S is both m-regular and m-intraregular.

(2) $\Rightarrow$ (1): If μ is any fuzzy generalized m-bi-ideal of an m-regular and m-intraregular semigroup S, such that $\lambda \circ \nu \leq \mu$ for generalized m-bi ideal λ and ν of S. Since the fuzzy generalized m-bi ideals of S produce a totally ordered set, so $\lambda \leq \nu$ or $\nu \leq \lambda$. Consider $\lambda \leq \nu$, then $\lambda \circ \lambda \leq \lambda \circ \nu \leq \mu$. By Theorem 3.2, μ is semiprime, so $\lambda \leq \mu$. Thus, μ becomes a prime fuzzy generalized m-bi-ideal of S. □

THEOREM 3.6. *For a semigroup S, the following statements are equivalent [14]:*

1. *The set of fuzzy generalized m-bi-ideals of S is totally ordered under inclusion.*
2. *Every fuzzy generalized m-bi-ideal of S is strongly irreducible.*
3. *Every fuzzy generalized m-bi-ideal of S is irreducible.*

Proof. (1) $\Rightarrow$ (2): Suppose μ is an arbitrary fuzzy generalized m-bi-ideal of S and λ, ν are two fuzzy generalized m-bi-ideals of S such that $\lambda \wedge \nu \leq \mu$. Since the fuzzy generalized m-bi-ideals of S produce a totally ordered set, so either $\lambda \leq \nu$ or $\nu \leq \lambda$. This gives $\lambda \wedge \nu = \nu$ or $\lambda \wedge \nu = \lambda$. Hence $\lambda \wedge \nu \leq \mu$ results in either $\nu \leq \mu$ or $\lambda \leq \mu$. Eventually, μ is strongly irreducible.
(2) $\Rightarrow$ (3): trivial.
(3) $\Rightarrow$ (1): Let us suppose that generalized m-bi-ideals of S are irreducible. Then for a given generalized m-bi ideal, δ, of S, there exist two generalized m-bi-ideals, ν and μ, of S, such that

$$\nu \wedge \mu = \delta \tag{5}$$

implies either $\nu = \delta$ or $\mu = \delta$(By hypothesis). But from Equation 5, it also follows that $\delta \leq \nu$ and $\delta \leq \mu$. So, by combining these four results, we get either $\nu \leq \mu$ or $\mu \leq \nu$. This makes the set of fuzzy generalized m-bi-ideals of S a totally ordered. □

In the above theorems, we showed that every m-bi-ideal of S is semiprime iff every fuzzy generalized m-bi-ideal of S is semiprime. However, the following examples show that if every generalized m-bi-ideal of S is prime (strongly prime), then it is not necessarily true that every fuzzy generalized m-bi ideals of S is prime (strongly prime).

EXAMPLE 3.7. We have semigroup $S = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ given in the following table.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	1	1
2	0	2	1	2	2
3	0	1	2	1	1
4	0	1	2	1	1

Taking $m = 2$, we get, $S^2 = \{0, 1, 2\}$.

PROPOSITION 3.8. 1. *All generalized* 2*-bi-ideals of* S *are semiprime generalized* 2*-bi-ideals of* S.

Proof. The generalized 2-bi-ideals in S are $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$. As S is 2-regular and 2-intraregular, every fuzzy generalized 2-bi-ideal is semiprime3.2. Therefore, all generalized 2-bi ideals are also semiprime generalized 2-bi-ideals of S. □

2. *All prime generalized* 2*-bi-ideals of* S *except the trivial ideal* $\{0\}$ *are also strongly prime.*

Proof. The prime generalized 2-bi-ideals of S are $\{0\}$, $\{0, 1\}$, $\{0, 2\}$ and $\{0, 1, 2, 3\}$. These are also strongly prime except the prime generalized 2-bi ideal, $\{0\}$, which is not strongly prime generalized 2-bi ideal because $\{0, 1\}\{0, 2\} \cap \{0, 2\}\{0, 1\} = \{0, 1\} \cap \{0, 2\} = \{0\}$, but none of $\{0, 1\}$ and $\{0, 2\}$ is contained in $\{0\}$. □

3. *Every fuzzy* 2*-subset of* S *is a fuzzy* 2*-subsemigroup of* S.

Proof. Obvious from the table. □

4. *A fuzzy* 2*-subset* μ *of* S *is a fuzzy generalized* 2*-bi ideal of* S *iff* $\mu(0) \geq \mu(x)$ *for all* $x \in S^m$.

Proof. Since $\mu(0) = \mu(x0x) \geq \mu(x) \wedge \mu(x) = \mu(x)$, for all, $x \in S^m$. This gives that $\mu(0) \geq \mu(x)$, for all, $x \in S^m$.
Conversely, suppose that $\mu(0) \geq \mu(x)$, for all $x \in S^m$. Then we get $xyz = x$ for $x, y, z \in \{1, 2\}$. $xyz = 0$ if either of x, y or z is 0.

$\mu(xyz) \geq \mu(x) \wedge \mu(z)$. That is, μ is fuzzy generalized 2-bi ideal of S. □

5. *Every fuzzy generalized 2-bi-ideal of S is not prime.*

Proof. We take the fuzzy generalized 2-bi-ideals μ, λ, ν of S given as $\mu(0) = 0.7$, $\mu(1) = 0.6$, $\mu(2) = 0.4$, $\lambda(0) = 1$, $\lambda(1) = 0.5$, $\lambda(2) = 0.4$, $\nu(0) = 0.7$, $\nu(1) = 0.65$, $\nu(2) = 0.3$, and $\lambda \circ \nu(0) = 0.7$, $\lambda \circ \nu(1) = 0.5$, $\lambda \circ \nu(2) = 0.4$. Thus $\lambda \circ \nu \leq \mu$ but neither $\lambda \leq \mu$ nor $\nu \leq \mu$. Hence, μ is not a prime fuzzy generalized 2-bi ideal of S. □

Example 3.9. We have the semigroup $S = \{0, 1, 2, 3, 4\}$ with the *Caley* multiplication table given below:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	2	1
3	0	1	2	3	1
4	0	1	1	1	1

If we take $m = 2$, we get $S^2 = \{0, 1, 2, 3\}$. Here, S is neither regular and nor intraregular. However, S is both 2-regular and 2- intraregular. The generalized 2-bi-ideals are $\{0\}$, $\{0, 2\}$, $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$ and S. All these generalized 2-bi-ideals are strongly prime generalized 2-bi-ideals.

Proposition 3.10. 1. *Each fuzzy m-subset of S is its fuzzy m-subsemigroup.*

Proof. Obvious from the multiplication table. □

2. *A fuzzy m-subset μ of S is its fuzzy m-bi ideal iff $\mu(0) \geq \mu(1) \geq \mu(2)$ for all $x \in S^m$.*

Proof. Since $\mu(0) = \mu(x0x) \geq \mu(x) \wedge \mu(x) = \mu(x)$, for all, $x \in S^m$. This gives that $\mu(0) \geq \mu(x)$, for all, $x \in S^m$. Also, $\mu(1) = \mu(212) \geq \mu(2) \wedge \mu(2) = \mu(2)$, i.e.,$\mu(1) \geq \mu(2)$. Consequently, $\mu(0) \geq \mu(1) \geq \mu(2)$.
Conversely, suppose that $\mu(0) \geq \mu(1) \geq \mu(2)$. Obviously, μ is a fuzzy m-subsemigroup of S. Also $uvw = 2$ if u, v, $w \in \{2, 1\}$ and

one of these is 2 and $uvw = 1$ if $u = v = w = 1$. So $\mu(uvw) \geq \mu(u) \wedge \mu(w)$. Eventually, μ is a fuzzy generalized m-bi ideal of S. □

3. *Every fuzzy generalized m-bi-ideal of S is not strongly prime, for $m = 2$.*

   Proof. We consider the fuzzy generalized m-bi-ideals μ, λ and ν of S defined by $\mu(0) = 0.7$, $\mu(1) = 0.6$, $\mu(2) = 0.5$, $\lambda(0) = 1$, $\lambda(1) = 0.5$, $\lambda(2) = 0.4$, $\nu(0) = 0.7$, $\nu(1) = 0.65$, $\nu(2) = 0.3$. Then $\lambda \circ \nu(0) = \nu \circ \lambda(\alpha) = 0.7$, $\lambda \circ \nu(1) = \nu \circ \lambda(1) = 0.5$, $\lambda \circ \nu(2) = \nu \circ \lambda(2) = 0.3$. Thus $\lambda \circ \nu \wedge \nu \circ \lambda \leq \mu$ but neither $\lambda \leq \mu$ nor $\nu \leq \mu$. Hence μ is not strongly prime fuzzy generalized m-bi-ideal of S. □

The following example demonstrates when the m-bi-ideals and the subsets of a semigroup are also prime and semiprime generalized m-bi-ideals.

Example 3.11. For a left zero semigroup S with the cardinality $|S| > 1$, we have $xy = x$ for all $x, y \in S$. So, for an arbitrary $x \in S$, $xx = x$, i.e., x is idempotent, $S^2 = S$. That is, every subset of S is a generalized m-bi ideal of S. Then S is both m-regular and m-intraregular. Every fuzzy m-subset of S is a fuzzy generalized m-bi ideal of S. If μ is a fuzzy generalized m-subset of S , $\mu(xy) = \mu(x) \leq \mu(x) \wedge \mu(y)$, $\forall x, y \in S^m$. Moreover, $\mu(xyz) = \mu(x) \leq \mu(x) \wedge \mu(z)$, for all $x, y, z \in S$ [14]. This implies that μ is a fuzzy generalized m-bi ideal of S. Since S is both m-regular and m-intraregular, every fuzzy m-subset of S is a fuzzy semiprime generalized m-bi ideal of S.

4. Conclusions

We presented the concepts of the prime, strongly prime and semiprime generalized m-bi ideals, along with their basic properties. These concepts can be interpreted to semiring theory.

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