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ON THE GENERALIZED BOUNDARY AND THICKNESS

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Abstract. We introduced the concepts of the generalized accumulation points and the generalized density of a subset of the Euclidean space in [1] and [2]. Using those concepts, we introduce the concepts of the generalized closure, the generalized interior, the generalized exterior and the generalized boundary of a subset and investigate some properties of these sets. The generalized boundary of a subset is closely related to the classical boundary. Finally, we also introduce and study a concept of the thickness of a subset.

1. Introduction

In this section, we introduce a concept of the generalized closure of a set and study some properties of the generalized dense subset which we need later. Throughout this paper, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number. We denote the open ball, the closed ball and the sphere with radius ϵ and center at α in the space R^m by $B(\alpha, \epsilon) = \{x \in R^m : \|x - \alpha\| < \epsilon\}$, $\overline{B}(\alpha, \epsilon) = \{x \in R^m : \|x - \alpha\| \leq \epsilon\}$ and $S(\alpha, \epsilon) = \{x \in R^m : \|x - \alpha\| = \epsilon\}$, respectively.

Definition 1.1. Let S be a subset of R^m. A point $a \in R^m$ is an ϵ_0-accumulation point of the subset S if and only if $B(a, \epsilon) \cap (S - \{a\}) \neq \emptyset$ for all $\epsilon > \epsilon_0$. And a point $a \in S$ is an ϵ_0-isolated point of S if and only if $B(a, \epsilon_1) \cap (S - \{a\}) = \emptyset$ for some positive number $\epsilon_1 > \epsilon_0$.

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Definition 1.2. For a subset S of R^m, we define the ϵ_0-derived set of S as the set of all the ϵ_0-accumulation points of S and denote it by $S'_{(\epsilon_0)}$.

Definition 1.3. Let S be a subset of R^m. The ϵ_0-closure of S is defined by $\overline{S}_{(\epsilon_0)} = Cl_{\epsilon_0}(S) = S'_{(\epsilon_0)} \cup S$.

Definition 1.4. Let E be any non-empty and open subset of R^m and $\epsilon_0 \geq 0$. And let a subset D of E be given. We define that D is an ϵ_0-dense subset of E in E if and only if $E \subseteq \overline{D}_{(\epsilon_0)}$. In this case, we say that D is ϵ_0-dense in E.

Definition 1.5. Let E be an open non-empty subset of R^m. And let D be an ϵ_0-dense subset of E in E. An element $a \in D$ is called a point of the ϵ_0-dense ace of D in E if and only if $D - \{a\}$ is not ϵ_0-dense in E.

Lemma 1.6. *Let E be an open subset of R^m and D be a non-empty subset of E. Suppose that $E \subseteq \bigcup_{b\in D} \overline{B}(b, \epsilon_0)$. Then D is ϵ_0-dense in E.*

Proof. See the proof of the lemma 2.10 in [1]. □

Lemma 1.7. *Let D be a non-empty subset of an open subset E of R^m and $\overline{D} = D'_{(0)} \cup D$. Then D is ϵ_0-dense in E if and only if $E \subseteq \bigcup_{b\in \overline{D}} \overline{B}(b, \epsilon_0)$.*

Proof. See the proof of the theorem 2.11 in [1]. □

2. The generalized interior and boundary

In this section, we investigate about the concepts of the ϵ_0-interior, the ϵ_0-exterior and the ϵ_0-bouundary of subsets in R^m and research the shapes of these sets. Throughout this section, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number unless otherwise stated.

Definition 2.1. Let S be a subset of R^m. A point x is called the ϵ_0-interior point of S if and only if there is a positive real number $\epsilon_1 > \epsilon_0$ such that $x \in B(x, \epsilon_1) \subseteq S$. Let's denote the set of all the ϵ_0-interior points of S in R^m by $Int_{\epsilon_0}(S)$ or $S^o_{(\epsilon_0)}$.

Definition 2.2. Let S be a subset of R^m. A point x is called the ϵ_0-boundary point of S if and only if $B(x,\epsilon_1)\cap S \neq \emptyset$ and $B(x,\epsilon_1)\cap S^C \neq \emptyset$ for each positive real number $\epsilon_1 > \epsilon_0$. Let's denote the set of all the ϵ_0-boundary points of S in R^m by $Bd_{\epsilon_0}(S)$ or $\partial_{\epsilon_0} S$.

Definition 2.3. Let S be a subset of R^m. A point x is called the ϵ_0-exterior point of S if and only if x is an ϵ_0-interior point of $S^C = R^m - S$. Let's denote the set of all the ϵ_0-exterior points of S in R^m by $Ext_{\epsilon_0}(S)$.

Remark 2.4. The union $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the mutually disjoint one, $S^o_{(0)} = S^o$ and $Int_{\epsilon_0}(S) \subseteq Int_0(S) = S^o$ for all $\epsilon_0 \geq 0$.

Lemma 2.5. *Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then $Int_{\epsilon_0}(S)$ and $Ext_{\epsilon_0}(S)$ are open subsets of R^m. Hence $Bd_{\epsilon_0}(S)$ is closed in R^m.*

Proof. Let any element $x \in Int_{\epsilon_0}(S)$ be given. Then there is a positive real number $\epsilon_1 > \epsilon_0$ such that $x \in B(x,\epsilon_1) \subseteq S$. Consider the set $B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0))$. For any point $y \in B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0))$, we have, for any point $z \in B(y, \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0))$,

$$\begin{aligned} \|x - z\| &\leq \|x - y\| + \|y - z\| \\ &< \frac{1}{3}(\epsilon_1 - \epsilon_0) + \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0) \\ &< \epsilon_0 + \epsilon_1 - \epsilon_0 = \epsilon_1. \end{aligned}$$

Hence we have $B(y, \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0)) \subseteq B(x,\epsilon_1) \subseteq S$. Thus we have $y \in Int_{\epsilon_0}(S)$ since $\epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0) > \epsilon_0$. Therefore, we have

$$x \in B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0)) \subseteq Int_{\epsilon_0}(S).$$

This implies that $Int_{\epsilon_0}(S)$ is open. And $Ext_{\epsilon_0}(S)$ is also open since it is the ϵ_0-interior of S^C. Since $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the disjoint union, $Bd_{\epsilon_0}(S) = R^m - \{Int_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)\}$ is closed in R^m. □

Lemma 2.6. *Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then we have $S'_{(\epsilon_0)} \subseteq Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$.*

Proof. Let any element $x \in S'_{(\epsilon_0)}$ be given. Since $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is a disjoint union, we need only to show that $x \notin$

$Ext_{\epsilon_0}(S)$. To the contrary, assume that $x \in Ext_{\epsilon_0}(S)$. Then there is $\epsilon_1 > \epsilon_0$ such that $x \in B(x, \epsilon_1) \subseteq S^C$. Hence $B(x, \epsilon_1) \cap S = \emptyset$. This is a contradiction since $x \in S'_{(\epsilon_0)}$. □

THEOREM 2.7. *Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then $\overline{S}_{(\epsilon_0)} = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$.*

Proof. Since $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the disjoint union and S is disjoint from $Ext_{\epsilon_0}(S)$, we have $S \subseteq Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$. Hence, by lemma 2.6, we have

$$\overline{S}_{(\epsilon_0)} = S \cup S'_{(\epsilon_0)} \subseteq Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S).$$

In order to prove the equality, let any element $x \in Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$ be given. If $x \in S$ then we are done. Suppose that $x \notin S$. Then $x \notin Int_{\epsilon_0}(S)$. Thus we have $x \in Bd_{\epsilon_0}(S)$. Hence we have

$$\forall \epsilon_1 > \epsilon_0, B(x, \epsilon_1) \cap S \neq \emptyset \text{ and } B(x, \epsilon_1) \cap S^C \neq \emptyset.$$

Thus we have $\exists y_{\epsilon_1} \in S$ s.t. $y_{\epsilon_1} \in B(x, \epsilon_1)$. Since $y_{\epsilon_1} \neq x$, we have

$$\forall \epsilon_1 > \epsilon_0,\ y_{\epsilon_1} \in B(x, \epsilon_1) \cap (S - \{x\}) \neq \emptyset.$$

This implies that $x \in S'_{(\epsilon_0)}$ which completes the proof. □

COROLLARY 2.8. *Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then*

$$\overline{S}_{(\epsilon_0)} = [\{S^C\}^o_{(\epsilon_0)}]^C.$$

Proof. Since $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the disjoint union and $\overline{S}_{(\epsilon_0)} = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$, the union of the equation $R^m = \overline{S}_{(\epsilon_0)} \cup Ext_{\epsilon_0}(S)$ is the disjoint one. Hence we have $\{\overline{S}_{(\epsilon_0)}\}^C = Ext_{\epsilon_0}(S) = \{S^C\}^o_{(\epsilon_0)}$. Thus we have $\overline{S}_{(\epsilon_0)} = [\{S^C\}^o_{(\epsilon_0)}]^C$. □

THEOREM 2.9. *Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then $R^m - \overline{(R^m - S)}_{(\epsilon_0)} = Int_{\epsilon_0}(S)$, i.e., $\overline{[S^C]}^C_{(\epsilon_0)} = Int_{\epsilon_0}(S)$.*

Proof. By the definition of the ϵ_0−closure of the set S^C, we have $\overline{[S^C]}_{(\epsilon_0)} = [S^C]'_{(\epsilon_0)} \cup S^C$. Hence we have $R^m - \overline{[S^C]}_{(\epsilon_0)} = \{[S^C]'_{(\epsilon_0)}\}^C \cap S$. Thus we need only to show that $Int_{\epsilon_0}(S) = \{[S^C]'_{(\epsilon_0)}\}^C \cap S$. Let any element $x \in Int_{\epsilon_0}(S)$ be given. Then we have

$$\begin{aligned} & \exists \epsilon_1 > \epsilon_0 \text{ s.t. } x \in B(x, \epsilon_1) \subseteq S \\ \Rightarrow\ & B(x, \epsilon_1) \cap S^C = \emptyset \text{ and } x \in S \\ \Rightarrow\ & x \notin [S^C]'_{(\epsilon_0)} \text{ and } x \in S \\ \Rightarrow\ & x \in \{[S^C]'_{(\epsilon_0)}\}^C \cap S. \end{aligned}$$

Conversely, let any element $x \in \{[S^C]'_{(\epsilon_0)}\}^C \cap S$ be given. Since $x \in S$ is not a member of $[S^C]'_{(\epsilon_0)}$, we have

$$\exists \epsilon_1 > \epsilon_0 \text{ s.t. } B(x, \epsilon_1) \cap (S^C - \{x\}) = \emptyset.$$

Since $x \in S$ and $S^C - \{x\} = S^C$, we also have $B(x, \epsilon_1) \cap S^C = \emptyset$. Thus we have $x \in B(x, \epsilon_1) \subseteq S$. Therefore, we have $x \in Int_{\epsilon_0}(S)$ which completes the proof. □

THEOREM 2.10. *(Representation) Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then we have*

$$Bd_{\epsilon_0}(S) = \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0).$$

Moreover, if $\epsilon_0 > 0$ then ∂S is an ϵ_0-dense subset of the interior of the subset $Bd_{\epsilon_0}(S)$.

Proof. Let $x \in \partial S$ and any element $y \in \overline{B}(x, \epsilon_0)$ be given. For each positive real number $\epsilon > \epsilon_0$, we have $x \in B(y, \epsilon)$. Hence $x \in B(x, \epsilon - \epsilon_0) \subseteq B(y, \epsilon)$. Since $x \in \partial S$,

$$B(x, \epsilon - \epsilon_0) \cap S \neq \emptyset \text{ and } B(x, \epsilon - \epsilon_0) \cap S^C \neq \emptyset.$$

Thus we have

$$B(y, \epsilon) \cap S \neq \emptyset \text{ and } B(y, \epsilon) \cap S^C \neq \emptyset.$$

Hence we have $y \in Bd_{\epsilon_0}(S)$. Thus we have $\overline{B}(x, \epsilon_0) \subseteq Bd_{\epsilon_0}(S)$ for all elements $x \in \partial S$. Therefore, we have $\bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0) \subseteq Bd_{\epsilon_0}(S)$. Conversely, let any element $y \in Bd_{\epsilon_0}(S)$ be given. For each natural number n, we have

$$B(y, \epsilon_0 + \frac{1}{n}) \cap S \neq \emptyset \text{ and } B(y, \epsilon_0 + \frac{1}{n}) \cap S^C \neq \emptyset.$$

Hence there are two sequences $\{w_n\}, \{z_n\}$ in R^m such that $\{w_n\} \subseteq S$, $\{z_n\} \subseteq S^C$ and $w_n, z_n \in B(y, \epsilon_0 + \frac{1}{n})$ for each natural number n. Since they are bounded, we may assume by using their subsequences that $\lim_{n \to \infty} w_n = w_0$ and $\lim_{n \to \infty} z_n = z_0$ for some elements $w_0 \in \overline{S}$ and $z_0 \in \overline{S^C}$.

Note that $\partial S = \partial S^C$. If $w_0 \in \partial S$ or $z_0 \in \partial S$ then we are done since $y \in \overline{B}(w_0, \epsilon_0)$ with $w_0 \in \partial S$ or $y \in \overline{B}(z_0, \epsilon_0)$ with $z_0 \in \partial S$. Now suppose that $w_0 \notin \partial S$ and $z_0 \notin \partial S$. Then we must have $w_0 \in Int(S)$ and $z_0 \in Ext(S)$. Now consider the line segment $\overline{w_0 z_0}$ joining the points w_0 and z_0. We have $\overline{w_0 z_0} \cap \partial S \neq \emptyset$ since $\overline{w_0 z_0}$ is connected. Choosing an element $x_0 \in \overline{w_0 z_0} \cap \partial S$, we have $x_0 = t_0 w_0 + (1 - t_0) z_0$ for some real number $0 < t_0 < 1$. Thus we have

$$\begin{aligned} \|y - x_0\| &= \|t_0 y + (1-t_0)y - \{t_0 w_0 + (1-t_0) z_0\}\| \\ &\leq t_0\|y - w_0\| + (1-t_0)\|y - z_0\| \\ &\leq t_0\epsilon_0 + (1-t_0)\epsilon_0 = \epsilon_0. \end{aligned}$$

Hence $y \in \overline{B}(x_0, \epsilon_0) \subseteq \underset{x \in \partial S}{\cup} \overline{B}(x, \epsilon_0)$. Moreover, if $\epsilon_0 > 0$ then ∂S is a subset of the interior of $Bd_{\epsilon_0}(S)$. Thus ∂S is an ϵ_0-dense subset of the interior of the subset $Bd_{\epsilon_0}(S)$ by the lemma 1.6. □

THEOREM 2.11. *(Core) Let S be a subset of R^m and suppose that $\epsilon_0 \geq 0$. Then*

$$Int_{\epsilon_0}(S) = S - \underset{x \in \partial S}{\cup} \overline{B}(x, \epsilon_0).$$

Proof. By the theorem just above, we need only to show that $Int_{\epsilon_0}(S)$ $= S - Bd_{\epsilon_0}(S)$. Let any element $x \in S - Bd_{\epsilon_0}(S)$ be given. Then $x \in S$ and $x \notin Bd_{\epsilon_0}(S)$. Since $R^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the disjoint union, we must have $x \in Int_{\epsilon_0}(S)$. Conversely, let any element $x \in Int_{\epsilon_0}(S)$ be given. Then we clearly have $x \in S$, $x \notin Ext_{\epsilon_0}(S)$ and $x \notin Bd_{\epsilon_0}(S)$. Thus we have $x \in S - Bd_{\epsilon_0}(S)$. □

LEMMA 2.12. *A subset F of R^m is the boundary of some open subset in R^m if and only if F is closed and nowhere dense.*

Proof. First, suppose that F is the boundary of some open subset S in R^m. Then it is clear that F is closed. Since the interior S of the set S is disjoint from the boundary F of S, we have $S \cap F = \emptyset$. If some point $x \in F$ is an interior point of F then there is a positive real number $\epsilon_1 > 0$ such that $x \in B(x, \epsilon_1) \subseteq F$. Since $S \cap F = \emptyset$, this implies that $B(x, \epsilon_1) \cap S = \emptyset$. Thus we have $x \in B(x, \epsilon_1) \subseteq S^C$. This implies that $x \in Ext(S)$. This is a contradiction since the boundary is disjoint from the exterior. This contradiction implies that F is nowhere dense. Now suppose that F is closed and nowhere dense. Take $S = F^C$. Then S is an open subset of R^m. We need only to prove that $F = \partial F^C$. First, we have $\partial F^C \cap F^C = \emptyset$ since F^C is open. Hence we have $\partial F^C \subseteq F$. Next, let any element $x \in F$ be given. Then $B(x,\epsilon) \cap F \neq \emptyset$ for all positive real number $\epsilon > 0$ since this intersection contains the element x. Moreover, the open ball $B(x,\epsilon)$ cannot be a subset of F for all positive real number $\epsilon > 0$ since F is nowhere dense. Thus we also have $B(x,\epsilon) \cap F^C \neq \emptyset$ for all positive real number $\epsilon > 0$. Hence we have $x \in \partial F^C = \partial S$. Thus $F = \partial S$. □

Corollary 2.13. *Let S be any subset of R^m. Then $\{\partial \overline{S}\}^o = \emptyset$.*

Proof. Let S be any subset of R^m. Since $\overline{S}^C$ is open, $\partial \overline{S}^C$ is nowhere dense by the lemma just above. But we have $\partial \overline{S} = \partial \overline{S}^C$. Hence we have $\{\partial \overline{S}\}^o = \{\partial \overline{S}^C\}^o = \emptyset$. □

Theorem 2.14. *Let F be a non-empty subset of R^m and $\epsilon_0 \geq 0$. Then F is the ϵ_0-boundary of some open subset of R^m if and only if $F = \underset{x\in S}{\cup} \overline{B}(x,\epsilon_0)$ for some closed and nowhere dense subset S of R^m.*

Proof. First, suppose that F is the ϵ_0-boundary of some open subset G of R^m. Then the boundary $S = \partial G$ of G is closed and nowhere dense subset of R^m by the lemma just above. Moreover, we have

$$F = Bd_{\epsilon_0}(G) = \underset{x\in \partial G}{\cup} \overline{B}(x,\epsilon_0)$$

by the theorem 2.10. Hence we have $F = \underset{x\in S}{\cup} \overline{B}(x,\epsilon_0)$. Conversely, suppose that $F = \underset{x\in S}{\cup} \overline{B}(x,\epsilon_0)$ for some closed and nowhere dense subset S of R^m. Then S is the boundary ∂G of some open subset G of R^m by the lemma just above. The ϵ_0-boundary of this open subset G is given by $Bd_{\epsilon_0}(G) = \underset{x\in \partial G}{\cup} \overline{B}(x,\epsilon_0)$ by the theorem 2.10. Thus F is the ϵ_0-boundary of the open subset G. □

Lemma 2.15. *Let S, T be any subsets of R^m and suppose that $\epsilon_0 \geq 0$. Then*

(1) $Int_{\epsilon_0}(S \cap T) = Int_{\epsilon_0}(S) \cap Int_{\epsilon_0}(T)$.

(2) $Ext_{\epsilon_0}(S \cup T) = Ext_{\epsilon_0}(S) \cap Ext_{\epsilon_0}(T)$.

Proof. (1) Since $S \cap T$ is a subset of S and T, we have $Int_{\epsilon_0}(S \cap T) \subseteq Int_{\epsilon_0}(S) \cap Int_{\epsilon_0}(T)$. Conversely, if $x \in Int_{\epsilon_0}(S) \cap Int_{\epsilon_0}(T)$ is any element then we have

$$\exists \epsilon_1 > \epsilon_0 \text{ s.t. } \ x \in B(x,\epsilon_1) \subseteq S$$

and

$$\exists \epsilon_2 > \epsilon_0 \text{ s.t. } \ x \in B(x,\epsilon_2) \subseteq T.$$

Hence we have the statement

$$\exists\epsilon_3 = min\{\epsilon_1, \epsilon_2\} > \epsilon_0 \text{ s.t. } \ x \in B(x, \epsilon_3) \subseteq S \cap T$$

which implies that $x \in Int_{\epsilon_0}(S \cap T)$. (2) By (1), we have

$$Int_{\epsilon_0}(S^C \cap T^C) = Int_{\epsilon_0}(S^C) \cap Int_{\epsilon_0}(T^C).$$

Since $Int_{\epsilon_0}(S^C) = Ext_{\epsilon_0}(S)$, we have the desired result $Ext_{\epsilon_0}(S \cup T) = Ext_{\epsilon_0}(S) \cap Ext_{\epsilon_0}(T)$. □

Note that $Int_{\epsilon_0}(S) \cup Int_{\epsilon_0}(T) \subseteq Int_{\epsilon_0}(S \cup T)$ in general.

THEOREM 2.16. *Let S, T be any subsets of R^m and suppose that $\epsilon_0 \geq 0$. Then $Cl_{\epsilon_0}(S \cup T) = Cl_{\epsilon_0}(S) \cup Cl_{\epsilon_0}(T)$.*

Proof. By the corollary 2.8 and the lemma 2.15, we have

$$\begin{aligned} \overline{(S \cup T)}_{(\epsilon_0)} &= [\{(S \cup T)^C\}^o_{(\epsilon_0)}]^C \\ &= [\{(S^C \cap T^C)\}^o_{(\epsilon_0)}]^C \\ &= \{(S^C)^o_{(\epsilon_0)} \cap (T^C)^o_{(\epsilon_0)}\}^C \\ &= \{(S^C)^o_{(\epsilon_0)}\}^C \cup \{(T^C)^o_{(\epsilon_0)}\}^C \\ &= \overline{S}_{(\epsilon_0)} \cup \overline{T}_{(\epsilon_0)} \end{aligned}$$

which completes the proof. □

3. Thickness

By the corollary 2.13, we have $\{\partial\overline{S}\}^{\text{o}} = \emptyset$ for all subsets of R^m. But the similar relation $\{\partial_{\epsilon_0}\overline{S}\}^{\text{o}}_{(\epsilon_0)} = \emptyset$ is not true in general if $\epsilon_0 \neq 0$. For if $S = \{A, B, C\}$ is the vertices of the equilateral triangle in R^2, then we have $\frac{A+B+C}{3} \in \{\partial_{\epsilon_0}\overline{S}\}^{\text{o}}_{(\epsilon_0)}$ with $\epsilon_0 = \|A - B\|$. This leads us to the following concept of the thickness.

DEFINITION 3.1. Let S be a non-empty subset of R^m and $\epsilon_0 \geq 0$. Then S is said to be ϵ_0-thick at a point $p \in S$ if and only if $p \in Int_{\epsilon_0}(S)$. In this case, we call that p is an ϵ_0-thick point or spot of S.

Note that $Int_{\epsilon_0}(S)$ is the set of all the ϵ_0-thick points of S. We call the closure $\overline{Int_{\epsilon_0}(S)}$ the ϵ_0-core of S. In according to the theorem 2.11, the ϵ_0-core of S is the closure of the set $Int_{\epsilon_0}(S) = S - \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0)$.

Note also that if S is ϵ_0-thick at a point $p \in S$ then S is $\epsilon-$thick at a point $p \in S$ for all $0 < \epsilon < \epsilon_1$ for some ϵ_1 with $\epsilon_0 < \epsilon_1$.

DEFINITION 3.2. Let S be a non-empty subset of R^m and $\epsilon_0 \geq 0$. Then S is said to be not ϵ_0-thick anywhere or nowhere ϵ_0-thick if and only if $Int_{\epsilon_0}(S) = \emptyset$.

THEOREM 3.3. *Let S be any subsets of R^m and suppose that $\epsilon_0 \geq 0$. If $Bd_{\epsilon_0}(S)$ is nowhere ϵ_0-thick then $Bd(S)$ is closed and nowhere dense, but not conversely.*

Proof. The boundary $Bd(S)$ is clearly closed in R^m. Suppose that $Bd(S)$ is not nowhere dense. Then $Int(Bd(S)) \neq \emptyset$. Hence there is a point $x_0 \in Bd(S)$ such that $B(x_0, \epsilon_1) \subseteq Bd(S)$ for some positive real number $\epsilon_1 > 0$. Then we have

$$x_0 \in B(x_0, \epsilon_0 + \frac{\epsilon_1}{2}) \subseteq \bigcup_{x \in Bd(S)} \overline{B}(x, \epsilon_0) = Bd_{\epsilon_0}(S).$$

Thus we have $x_0 \in Int_{\epsilon_0}(Bd_{\epsilon_0}(S))$. Hence $Bd_{\epsilon_0}(S)$ is ϵ_0-thick at x_0. In order to show that the converse is not true in general, choose the open set $S = B(0, \epsilon_0)$ with $\epsilon_0 > 0$. Then we have $Bd(S) = \{x \in R^m | \|x - 0\| = \epsilon_0\} = S(0, \epsilon_0)$. The sphere $S(0, \epsilon_0)$ is closed and nowhere dense. But we have

$$0 \in B(0, \frac{3}{2}\epsilon_0) \subseteq \bigcup_{x \in Bd(S)} \overline{B}(x, \epsilon_0) = Bd_{\epsilon_0}(S).$$

Hence $Bd_{\epsilon_0}(S)$ is ϵ_0-thick at the origin 0. □

Let u be any non-zero vector in R^m. Let's denote the orthogonal space by $u^{\perp} = \{z \in R^m : z \cdot u = 0\}$. Recall that the projection of a vector $x \in R^m$ along the vector u is given by $proj_u(x) = \frac{u \cdot x}{u \cdot u} u$. Let's denote the parallel projection from R^m to $u^{\perp}$ by $\Pi_{(u^{\perp})}(x) = x - proj_u(x)$.

THEOREM 3.4. *Let S be any subsets of R^m and suppose that $\epsilon_0 \geq 0$. If S is ϵ_0-thick at a point $p \in S$ in R^m then for any non-zero vector $u \in R^m$ the set $\Pi_{(u^{\perp})}(S) = \{\Pi_{(u^{\perp})}(x) : x \in S\}$ is ϵ_0-thick at the point $\Pi_{(u^{\perp})}(p)$ in the $m-1$ dimensional space $\Pi_{(u^{\perp})}(R^m)$, but not conversely.*

Proof. Suppose that S is ϵ_0-thick at a point $p \in S$ in R^m and let u be any non-zero vector in R^m. Then there is a positive real number $\epsilon_1 > \epsilon_0$ such that $p \in B(p, \epsilon_1) \subseteq S$. Hence we have

$$\Pi_{(u^{\perp})}(p) \in \Pi_{(u^{\perp})}(B(p, \epsilon_1)) \subseteq \Pi_{(u^{\perp})}(S).$$

This completes the proof of the first part since $\Pi_{(u^{\perp})}(B(p, \epsilon_1))$ is an open ball in $\Pi_{(u^{\perp})}(R^m)$ with the same radius ϵ_1. Now let $\{A, B, C\}$ be the vertices of the equilateral triangle in R^2 with $\|A - B\| = 2\epsilon_0$. Then the set $S = B(A,\epsilon_0) \cup B(B,\epsilon_0) \cup B(C,\epsilon_0)$ is not ϵ_0-thick at any point. But the set $\Pi_{(u^\perp)}(S)$ is obviously ϵ_0-thick at some point for any direction u in R^m. □

Lemma 3.5. *Let $\epsilon_0 > 0$ be given. If $P, Q \in R^2$ are distinct points with $\|P - Q\| < 2\epsilon_0$, then there are two points $U, V \in R^2$ such that $\|U - P\| = \|U - Q\| = \epsilon_0 = \|V - P\| = \|V - Q\|$.*

Proof. We clearly have $S(P,\epsilon_0) \cap S(Q,\epsilon_0) = \{U, V\}$. □

Remark 3.6. It is obvious that $Int_{\epsilon_0}\left[\overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0)\right] = \emptyset$ for any two points P, Q in R^2.

Theorem 3.7. *Let $P, Q, U, V \in R^2$ be the four points in the above lemma with P on the left, Q on the right, U at the top and V at the bottom. If a point $T \in R^2$ is an element of the intersection $\overline{B}(U,\epsilon_0) \cap \overline{B}(V,\epsilon_0)$ then we have*

$$Int_{\epsilon_0}\left[\overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0) \cup \overline{B}(T,\epsilon_0)\right] = \emptyset.$$

Proof. Put $Z = \overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0) \cup \overline{B}(T,\epsilon_0)$. If T is a boundary point of the intersection $\overline{B}(U,\epsilon_0) \cap \overline{B}(V,\epsilon_0)$ then the three spheres $S(T,\epsilon_0), S(P,\epsilon_0)$ and $S(Q,\epsilon_0)$ meet at the point U or V. Suppose that they meet at the point V. Then for any point $x \in \overline{B}(V,\epsilon_0)$ we have $\|x - V\| \leq \epsilon_0$. Since V is a boundary point of the union Z, this implies that any point x in the set $\overline{B}(V,\epsilon_0) \cap Z$ is not an ϵ_0-interior point of Z. Since the sphere $S(V,\epsilon_0)$ passes through the center points P, Q, T of the three spheres $S(P,\epsilon_0)$, $S(Q,\epsilon_0)$ and $S(T,\epsilon_0)$, we also have $dist(x, \partial(Z - \overline{B}(V,\epsilon_0))) \leq \epsilon_0$ for all the points $x \in Z - \overline{B}(V,\epsilon_0)$. Thus we have $Int_{\epsilon_0}(Z) = \emptyset$. The proof of the case where they meet at the point U is similarly handled. On the other hand, suppose that the point T is in the interior of the intersection $\overline{B}(U,\epsilon_0) \cap \overline{B}(V,\epsilon_0)$. Then the center points U, V are in the open ball $B(T,\epsilon_0)$ and the sphere $S(T,\epsilon_0)$ meets the boundary of the union $\overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0)$ at the four points, say A, B, C and D. Let's call the point on the upper left A, the point on the lower left B, the point on the upper right C and the point on the lower right D. Then, for any point x of the union of the rhombi $\Diamond APBT$ and $\Diamond CTDQ$, we have $dist(x, \partial(Z)) \leq \epsilon_0$ since the points A, B, C and D are in the boundary of Z. And, for any point x in the union of the four circular sectors $\circlearrowleft APB$, $\circlearrowleft ATC$, $\circlearrowleft BTD$ and $\circlearrowleft CQD$, we also have $dist(x, \partial(Z)) \leq \epsilon_0$ since all of the circular arcs of these four circular sectors are parts of the boundary of Z. Therefore, we have $dist(x, \partial(Z)) \leq \epsilon_0$ for all the points $x \in Z$. Consequently, we have $Int_{\epsilon_0}(Z) = \emptyset$. □

Corollary 3.8. *Let P_1, P_2, P_3 be three points in R^2. Suppose that*

$$Int_{\epsilon_0}\left[\overline{B}(P_1,\epsilon_0) \cup \overline{B}(P_2,\epsilon_0) \cup \overline{B}(P_3,\epsilon_0)\right] \neq \emptyset.$$

Then we have

(1) $S(P_1,\epsilon_0) \cap S(P_2,\epsilon_0) = \{U_1, V_2\}$ *and* $P_3 \notin \overline{B}(U_1,\epsilon_0) \cap \overline{B}(V_2,\epsilon_0)$

(2) $S(P_2,\epsilon_0) \cap S(P_3,\epsilon_0) = \{U_2, V_3\}$ *and* $P_1 \notin \overline{B}(U_2,\epsilon_0) \cap \overline{B}(V_3,\epsilon_0)$

(3) $S(P_3,\epsilon_0) \cap S(P_1,\epsilon_0) = \{U_3, V_1\}$ *and* $P_2 \notin \overline{B}(U_3,\epsilon_0) \cap \overline{B}(V_1,\epsilon_0)$.

Proof. (1) From the theorem just above, if $S(P_1,\epsilon_0) \cap S(P_2,\epsilon_0) = \{U_1, V_2\}$ and $P_3 \in \overline{B}(U_1,\epsilon_0) \cap \overline{B}(V_2,\epsilon_0)$ then

$$Int_{\epsilon_0}\left[\overline{B}(P_1,\epsilon_0) \cup \overline{B}(P_2,\epsilon_0) \cup \overline{B}(P_3,\epsilon_0)\right] = \emptyset.$$

The proofs of (2) and (3) are quite similar to the proof of (1) and we omit them. □

Theorem 3.9. *Let P, Q, U, V be the four mutually distinct points in R^2 such that $S(P,\epsilon_0) \cap S(Q,\epsilon_0) = \{U, V\}$ with P on the left, Q on the right, U at the top and V at the bottom. If a point $T \in R^2$ is an element of the union*

$$\left[B(U,\epsilon_0) - \overline{B}(V,\epsilon_0)\right] \cup \left[B(V,\epsilon_0) - \overline{B}(U,\epsilon_0)\right]$$

then $Z = \overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0) \cup \overline{B}(T,\epsilon_0)$ is ϵ_0-thick at some point.

Proof. We need only to prove the case where $T \in \left[B(U,\epsilon_0) - \overline{B}(V,\epsilon_0)\right]$ since the another case is similarly handled. Then we have $U \in B(T,\epsilon_0)$ and $V \notin \overline{B}(T,\epsilon_0)$. And the sphere $S(T,\epsilon_0)$ meets the boundary of the set $\overline{B}(P,\epsilon_0) \cup \overline{B}(Q,\epsilon_0)$ at two points, say L on the left, R on the right. Consider the triangle $\triangle LVR$. Let's denote by V' the point at which the line segment connecting the midpoint $\frac{L+R}{2}$ and the vertex V intersects the sphere $S(T,\epsilon_0)$. Now if $\angle LV'R \leq \frac{\pi}{2}$ then the radius of the circumscribed circle of the triangles $\triangle LV'R$ is ϵ_0 and $0 < \angle LVR < \angle LV'R \leq \frac{\pi}{2}$. Hence if r is the radius of the circumscribed circle of the triangle $\triangle LVR$ then we have

$$2\epsilon_0 = \frac{\overline{LR}}{sin(\angle LV'R)} < \frac{\overline{LR}}{sin(\angle LVR)} = 2r, \text{ i.e., } \epsilon_0 < r.$$

On the other hand, if $\angle LV'R > \frac{\pi}{2}$ then the point T is positioned higher than the line segment $\overline{LR}$. In this case, let C be the image of the reflection of the circle $S(T, \epsilon_0)$ with respect to the line segment $\overline{LR}$. Let's denote by V'' the point at which the line segment connecting the midpoint $\frac{L+R}{2}$ and the vertex V intersects this circle C. Then the point V'' lies inside the triangle $\triangle LVR$ and we have $\angle LV''R \leq \frac{\pi}{2}$. Hence the radius r of the circumscribed circle of the triangle $\triangle LVR$ still satisfies the relation $\epsilon_0 < r$ since the radius of the circumscribed circle of the triangles $\triangle LV''R$ is ϵ_0 and $0 < \angle LVR < \angle LV''R \leq \frac{\pi}{2}$. Since the three sides $\overline{LV}$, $\overline{RV}$ and $\overline{LR}$ of the triangle $\triangle LVR$ are parts of the closed balls $\overline{B}(P, \epsilon_0)$, $\overline{B}(Q, \epsilon_0)$ and $\overline{B}(T, \epsilon_0)$, respectively, the circumscribed circle and its interior of the triangle $\triangle LVR$ is a subset of the union Z. Thus Z contains an open ball with radius $\frac{\epsilon_0+r}{2}$ which implies that $Int_{\epsilon_0}(Z) \neq \emptyset$. □

THEOREM 3.10. *(Three points thickness) Let P, Q be the two distinct points in R^2 with $\|P - Q\| < 2\epsilon_0$ such that $S(P, \epsilon_0) \cap S(Q, \epsilon_0) = \{U, V\}$ with P on the left, Q on the right, U at the top and V at the bottom. For a point $T \in R^2$, the union $Z = \overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0) \cup \overline{B}(T, \epsilon_0)$ is ϵ_0-thick at some point of Z if and only if*

$$T \in \left\{B(U, \epsilon_0) - \overline{B}(V, \epsilon_0)\right\} \cup \left\{B(V, \epsilon_0) - \overline{B}(U, \epsilon_0)\right\}.$$

Proof. By means of the theorems 3.7 and 3.9, we need only to prove that if $T \notin B(U, \epsilon_0) \cup B(V, \epsilon_0)$ then Z is nowhere ϵ_0-thick. Suppose that $T \notin B(U, \epsilon_0) \cup B(V, \epsilon_0)$. Then we have $U, V \notin B(T, \epsilon_0)$. Now there are three cases depending on the relative position of the two points U, V with respect to the sphere $S(T, \epsilon_0)$.

Case I. $U, V \notin S(T, \epsilon_0)$. In this case, the intersection of the sphere $S(T, \epsilon_0)$ and the boundary of the union $\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0)$ is a subset A of R^2 consisting of no point, one point, two points, three points or four points. But all the points of the union $A \cup \{U, V\}$ are the boundary point of the union Z. Hence we have $Int_{\epsilon_0}(Z) = \emptyset$.

Case II. U or $V \in S(T, \epsilon_0)$ and $S(T, \epsilon_0) \cap \partial\left[\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0)\right]$ is consisting of the two elements. In this case, we may assume that this intersection contains the point V since the case where it contains U is similarly handled. Then we have $\|x - V\| \leq 2\epsilon_0$ for all the points $x \in Z$. Since V is a boundary point of Z, this implies that $Int_{\epsilon_0}(Z) = \emptyset$.

Case III. U or $V \in S(T, \epsilon_0)$ and $S(T, \epsilon_0) \cap \partial\left[\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0)\right]$ is consisting of the three elements. In this case, we may also assume that the set of the last intersection is $\{E, V, F\}$ with $E \in S(T, \epsilon_0) \cap S(P, \epsilon_0)$. Since the quadrilaterals $\square PETV$ and $\square QVTF$ are the rhombi, we have $\overline{PQ} = \overline{EF}$. Similarly, we have $\overline{EU} = \overline{TQ}$ and $\overline{PT} = \overline{UF}$ by using the appropriate rhombi. Thus the triangles $\triangle UEF$ and $\triangle PQT$ are congruent. Since $\overline{PV} = \overline{TV} = \overline{QV} = \epsilon_0$, the point V is the circumcenter of the triangle $\triangle PQT$. Hence the radius of the circumscribed circle of $\triangle UEF$ is ϵ_0. Since all the three points U, E, F are the boundary points of Z, this implies that $Int_{\epsilon_0}(Z) = \emptyset$. □

References

[1] Buhyeon Kang, *An Introduction to ϵ_0-Density and ϵ_0-Dense Ace,* JCMS. **32** (1) (2019).

[2] Buhyeon Kang, *The sequential attainability and attainable ace,* Korean J. Math. **26** (4)(2018).

[3] Yong Chan Kim and Jung Mi Ko, *The properties of rough approximations,* Korean J. Math. **19** (2)(2011).

[4] Yong Chan Kim and Jung Mi Ko, *L-fuzzy bi-closure systems and L-fuzzy bi-closure operators,* Korean J. Math. **27** (2) (2019).

[5] Shyamapada Modak and Ahmad Abdullah Al-omari, *Generalized Closed Sets in Binary Ideal Topological Spaces,* JCMS. **31** (1) (2018).

[6] M.Bala Prabhakar, S.Kalesha Vali, and M. Sambasiva Rao, *Closed and Dense Elements of BE-algebras,* JCMS. **32** (1) (2019).

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