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CHARACTERIZATIONS FOR TOTALLY GEODESIC SUBMANIFOLDS OF (κ, μ)-PARACONTACT METRIC MANIFOLDS

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Abstract. The aim of the present paper is to study pseudoparallel invariant submanifold of a (κ, μ)-paracontact metric manifold. We consider pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudo parallel invariant submanifolds of a (κ, μ)-paracontact metric manifold and we obtain new results contribute to geometry.

1. Introduction

Invariant submanifolds are used to discuss properties of non-linear antronomous systems. Also totally geodesic submanifolds play an important role in the relativity theory for the geodesic of the ambient manifolds remain geodesic in the submanifolds.

Pseudoparallel submanifolds have been studied intensively by many geometers(for the more detail, see references).

The present paper, we are deal with pseoduparallel submanifolds of (κ, μ)-paracontact metric manifold which have not been attempted so far. Also, we obtain some necessary and sufficient conditions that an invariant submanifold to be pseudoparallel, generalized Ricci-pseudoparallel,

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2-pseudoparallel and 2-Ricci-generalized pseudoparallel under the some conditions.

Many geometers studied paracontact metric manifolds and researched some important properties of these manifolds. The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [8], Authors introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (κ,μ)-nullity condition for some real constants κ and μ. Such manifolds are known as (κ,μ)-paracontact metric manifolds.

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}^{2n+1}$ has an almost paracontact structure (φ,ξ,η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\varphi^2 = I - \eta\otimes\xi, \quad \eta(\xi)=1, \quad \varphi\xi = 0. \tag{1}$$

A semi-Riemannian metric g is said to be an associated metric if tensor field φ is related

$$d\eta(X,Y) = g(X,\varphi Y), \quad \eta(X) = g(X,\xi),. \tag{2}$$

In this case, the structure (φ,ξ,η,g) on $\widetilde{M}^{2n+1}$ is called a paracontact metric structure. A manifold has such structure is also called paracontact metric manifold [9].

In a paracontact metric manifold, the following relations hold:

$$g(\varphi X,\varphi Y) = \eta(X)\eta(Y) - g(X,Y), g(\varphi X,Y) + g(X,\varphi Y) = 0, \tag{3}$$

for all vector fields X,Y on $\widetilde{M}^{2n+1}$.

Now, let $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$ be a paracontact metric manifold and we define a $(1,1)$-type tensor field h by $2h = \ell_\xi\varphi$, where ℓ denotes the Lie-derivative and it defined by

$$2hX = (\ell_\xi\varphi)X = \ell_\xi\varphi X - \varphi\ell_\xi X = [\xi,\varphi X] - \varphi[\xi,X], \tag{4}$$

for any vector field X on $\widetilde{M}^{2n+1}$. Then h is symmetric and satisfies

$$h\xi = 0, \quad Trh = Tr(\varphi h) = 0, \quad h\varphi = -\varphi h. \tag{5}$$

By $\widetilde{\nabla}$, we denote the Riemannian connection on $\widetilde{M}^{2n+1}$, then we have the following relation

$$\widetilde{\nabla}_X\xi = -\varphi X + \varphi hX, \tag{6}$$

for any vector field X on $\widetilde{M}^{2n+1}$ [10].

An important class among paracontact metric manifolds is that of the (κ, μ)-space forms which satisfy the nullity condition

$$\widetilde{R}(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{7}$$

for all vector fields X, Y on $\widetilde{M}^{2n+1}$, where $\widetilde{R}$ is the Riemannian curvature tensor of $\widetilde{M}^{2n+1}$, κ and μ are arbitrary constants [6].

The geometry behaviour of the (κ, μ)-paracontact metric manifold is different according as $\kappa < -1$, $\kappa = -1$ and $\kappa > -1$. In particular, for the case, $\kappa < -1$ and $\kappa > -1$, (κ, μ)-nullity condition (7) determines the whole curvature tensor field completely [8].

Fortunately, for both the case $\kappa < -1$ and $\kappa > -1$ same formula holds. For this reason, in this paper we consider the (κ, μ)-paracontact metric manifolds with the condition $\kappa \neq -1$ [7].

We can easily see that $\kappa = -1$ and $\mu = 0$ in a normal paracontact metric manifold.

For a (κ, μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$, the following identities hold;

$$\begin{aligned}
&(8) & (\widetilde{\nabla}_X\varphi)Y &= -g(X - hX, Y)\xi + \eta(Y)(X - hX),\\
&(9) & S(X,Y) &= [2(1-n) + n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y)\\
& & &+ [2(n-1) + n(2\kappa - \mu)]\eta(X)\eta(Y)\\
&(10) & S(X,\xi) &= 2n\kappa\eta(X)\\
&(11) & Q\varphi - \varphi Q &= 2[2(n-1)+\mu]h\varphi,\\
&(12) & h^2 &= (1+\kappa)\varphi^2,
\end{aligned}$$

for any vector fields X, Y on $\widetilde{M}^{2n+1}$, where S and Q denote the Ricci tensor and Ricci operator of M^{2n+1}, respectively.

Now, let M be an immersed submanifold of a paracontact metric manifold $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{13}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{14}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the connections on M and $\Gamma(T^{\perp}M)$ and σ and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V). \tag{15}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{16}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then submanifold is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of M, we have the following Gauss equation

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y,Z) \\ &- (\widetilde{\nabla}_Y\sigma)(X,Z),\end{aligned} \tag{17}$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$.

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned}Q(A,T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, ..., X_k)... \\ &- T(X_1, X_2, ...X_{k-1}, (X \wedge_A Y)X_k),,\end{aligned} \tag{18}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{19}$$

Definition 1.1. A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if

$$\begin{aligned}\widetilde{R}\cdot\sigma &\quad and \quad Q(g,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma &\quad and \quad Q(g,\widetilde{\nabla}\sigma) \\ \widetilde{R}\cdot\sigma &\quad and \quad Q(S,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma &\quad and \quad Q(S,\widetilde{\nabla}\sigma)\end{aligned}$$

are linearly dependent, respectively [1].

Equivalently, these can be expressed by the following equations;

$$\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma), \tag{20}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(g,\widetilde{\nabla}\sigma), \tag{21}$$

$$\widetilde{R}\cdot\sigma = L_3 Q(S,\sigma), \tag{22}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4 Q(S,\widetilde{\nabla}\sigma), \tag{23}$$

where L_1, L_2, L_3 and L_4 are, respectively, functions defined on $M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$.

Particularly, if $L_1 = 0$(resp. $L_2 = 0$), the submanifold is said to be semiparallel(resp. 2-semiparallel) [11].

2. Invariant Submanifolds of (κ,μ)-Paracontact Metric Manifold

Now, we will investigate the above cases for the invariant submanifold M of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$.

Now, let M be an immersed submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,g,\eta)$. If $\varphi(T_xM) \subseteq T_xM$, for each point at $x \in M$, then M is said to be invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold.

In the rest of this paper, we will assume that M is invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. In this case, from (5), we have

$$\varphi hX = -h\varphi X, \tag{24}$$

for all $X \in \Gamma(TM)$, that is, M is also invariant with respect to the tensor field h. In this connection, from (6), (8) and (13), we obtain

$$\sigma(X,\xi) = 0, \quad \sigma(\varphi X, Y) = \sigma(X,\varphi Y) = \varphi\sigma(X,Y), \tag{25}$$

for all $X, Y \in \Gamma(TM)$.

Theorem 2.1. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If M is a pseudoparallel submanifold, then $L_1 = \kappa \mp \mu\sqrt{1+\kappa}$. Furthermore, $L_1 \neq \kappa \mp \mu\sqrt{1+\kappa}$ if and only if M is totally geodesic.*

Proof. Let M be pseudoparallel, then from (20) we have

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V) = L_1 Q(g,\sigma)(X,Y;U,V),$$

for all $X,Y,U,V \in \Gamma(TM)$. This implies that

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V) \\ &= -L_1\{\sigma((X\wedge_g Y)U,V) + \sigma(U,(X\wedge_g Y)V)\}, \end{aligned} \tag{26}$$

for all $X,Y,U,V \in \Gamma(TM)$, where $(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$. Taking $V=\xi$ in (26) and making use of (25), we obtain

$$\begin{aligned} \sigma(R(X,Y)\xi,U) &= -L_1\{\sigma(U,\eta(Y)X-\eta(X)Y)\} \\ &= L_1\{\eta(Y)\sigma(U,X) - \eta(X)\sigma(U,Y)\} \end{aligned}$$

and from (7), we have

$$\begin{aligned} &\kappa\{\eta(Y)\sigma(X,U) - \eta(X)\sigma(Y,U)\} + \mu\{\eta(Y)\sigma(hX,U) \\ -\ &\eta(X)\sigma(hY,U)\} = L_1\{\eta(Y)\sigma(X,U) - \eta(X)\sigma(Y,U)\}, \end{aligned}$$

that is,

$$(L_1-\kappa)\{\eta(Y)\sigma(X,U) - \eta(X)\sigma(U,Y)\} = \mu\{\eta(Y)\sigma(hX,U) - \eta(X)\sigma(hY,U)\},$$

which from for $Y=\xi$,

$$(L_1-\kappa)\sigma(X,U) = \mu\sigma(hX,U). \tag{27}$$

Substituting X by hX in (27) and making use of (6), (12) and (25), we obtain

$$\begin{aligned} (L_1-\kappa)\sigma(hX,U) &= \mu\sigma(h^2X,U) \\ &= \mu(1+\kappa)\sigma(\varphi^2X,U) \\ &= \mu(1+\kappa)\sigma(X,U). \end{aligned} \tag{28}$$

From (27) and (28), we conclude that

$$\mu^2(1+\kappa)\sigma(X,U) = (L_1-\kappa)^2\sigma(X,U).$$

This completes the proof. □

From Theorem 2.1, we have following corollary.

COROLLARY 2.2. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is semiparallel if and only if M is totally geodesic.*

THEOREM 2.3. *let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If M is 2-pseudoparallel submanifold, then the function L_2 satisfies $L_2 = \kappa \mp \mu\sqrt{1+\kappa}$. Moreover, $L_2 \neq \kappa \mp \mu\sqrt{1+\kappa}$ if and only if M is totally geodesic.*

Proof. Let M be 2-pseudoparallel. Then from (21), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W) = L_2 Q(g,\widetilde{\nabla}\sigma)(U,V,W;X,Y),$$

for all $X,Y,U,V,W \in \Gamma(TM)$. This means that

$$\begin{aligned} &\widetilde{R}^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,W) - (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,W)\\ -\;&(\widetilde{\nabla}_U\sigma)(R(X,Y)V,W) - (\widetilde{\nabla}_U\sigma)(V,R(X,Y)W)\\ =\;&-L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,W) + (\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,W)\\ +\;&(\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)W)\},\end{aligned}$$

that is,

$$\begin{aligned} &R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,W) - (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,W)\\ -\;&(\widetilde{\nabla}_U\sigma)(R(X,Y)V,W) - (\widetilde{\nabla}_U\sigma)(V,R(X,Y)W)\\ =\;&-L_2\{g(Y,U)(\widetilde{\nabla}_X\sigma)(V,W) - g(X,U)(\widetilde{\nabla}_Y\sigma)(V,W)\\ +\;&(\widetilde{\nabla}_U\sigma)(g(Y,V)X - g(X,V)Y,W) + (\widetilde{\nabla}_U\sigma)(V,g(Y,W)X - g(X,W)Y)\}.\end{aligned}$$

In the last equation, taking $X = W = \xi$, we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) - (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) = -L_2\{g(Y,U)(\widetilde{\nabla}_\xi\sigma)(V,\xi)\\ &- \eta(U)(\widetilde{\nabla}_Y\sigma)(V,\xi) + (\widetilde{\nabla}_U\sigma)(g(Y,V)\xi - \eta(V)Y,\xi)\\ &+ (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi - Y)\}. \end{aligned} \tag{29}$$

Now, let's calculate each of these expressions. From (6), (16) and (25), we obtain

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(V,\xi) &= \nabla_U^{\perp}\sigma(V,\xi) - \sigma(\nabla_U V,\xi) - \sigma(V,\nabla_U\xi)\\ &= -\sigma(V,\nabla_U\xi) = -\sigma(V,-\varphi U + \varphi hU)\\ &= \sigma(V,\varphi U) - \sigma(V,\varphi hU). \end{aligned} \tag{30}$$

From (17), we can easily to see that

$$\begin{aligned} \widetilde{R}(\xi,U)Y &= \kappa\{g(Y,U)\xi - \eta(Y)U\}\\ &+ \mu\{g(Y,hU)\xi - \eta(Y)hU\}. \end{aligned}$$

Moreover, taking into account of (6) and (25), we have

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U})\sigma(V,\xi) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi) - \sigma(\nabla_{R(\xi,Y)U}V,\xi) \\
&- \sigma(V,\nabla_{R(\xi,Y)U}\xi) \\
&= -\sigma(-\varphi R(\xi,Y)U + \varphi hR(\xi,Y)U,V) \\
&= \sigma(\kappa(-\eta(U)\varphi Y) - \mu\eta(U)\varphi hY,V) \\
&- \sigma(-\kappa\eta(U)\varphi hY - \mu\eta(U)\varphi h^2Y,V)
\end{aligned}
$$

that is,

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U})\sigma(V,\xi) &= \eta(U)\{\mu(1+\kappa)-\kappa)\sigma(\varphi Y,V) \\
&+ (\kappa-\mu)\sigma(\varphi hY,V)\}.
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_U\sigma(R(\xi,Y)V,\xi) - \sigma(\nabla_U R(\xi,Y)V,\xi) \\
&- \sigma(R(\xi,Y)V,\nabla_U\xi) = -\sigma(R(\xi,Y)V,-\varphi U + \varphi hU) \\
&= \sigma(R(\xi,Y)V,\varphi U) - \sigma(R(\xi,Y)V,\varphi hU) \\
&= \kappa\sigma(g(Y,V)\xi - \eta(V)Y,\varphi U) \\
&+ \mu\sigma(g(hV,Y)\xi - \eta(V)hY,\varphi U) \\
&- \kappa\sigma(g(Y,V)\xi - \eta(V)Y,\varphi hU) \\
&+ \mu\sigma(g(hV,Y)\xi - \eta(V)hY,\varphi hU) \\
&= -\kappa\eta(V)\sigma(Y,\varphi U) - \mu\eta(V)\sigma(hY,\varphi U) \\
&+ \kappa\eta(V)\sigma(Y,\varphi hU) + \mu\eta(V)\sigma(hY,\varphi hU) \\
&= \kappa\eta(V)[\sigma(Y,\varphi hU) - \sigma(Y,\varphi U)] \\
&+ \mu\eta(V)[\sigma(hY,\varphi hU) - \sigma(hY,\varphi U)],
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,k[\eta(Y)\xi-Y]-\mu hY)\\
&= \kappa(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)-\kappa(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \mu(\widetilde{\nabla}_U\sigma)(V,hY)\\
&= \kappa\{\nabla^{\perp}_U\sigma(V,\eta(Y)\xi)-\sigma(\nabla_UV,\eta(V)\xi)\\
&- \sigma(V,\nabla_U\eta(V)\xi)\}\\
&- \kappa(\widetilde{\nabla}_U\sigma)(V,Y)-\mu(\widetilde{\nabla}_U\sigma)(V,hY)\\
&= -\kappa\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi)-\kappa(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \mu(\widetilde{\nabla}_U\sigma)(V,hY)\\
&= -\kappa\eta(Y)\sigma(V,-\varphi U+\varphi hU)-\kappa(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \mu(\widetilde{\nabla}_U\sigma)(V,hY)\\
&= \kappa\eta(Y)\{\sigma(V,\varphi U)-\sigma(V,\varphi hU)\}-\kappa(\widetilde{\nabla}_U\sigma)(V,Y)\\
&- \mu(\widetilde{\nabla}_U\sigma)(V,hY), \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_Y\sigma)(V,\xi) &= \nabla^{\perp}_Y\sigma(V,\xi)-\sigma(\nabla_YV,\xi)-\sigma(\nabla_Y\xi,V)\\
&= -\sigma(V,-\varphi Y+\varphi hY)=\sigma(\varphi Y,V)-\sigma(\varphi hY,V). \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(g(Y,V)\xi-\eta(V)Y,\xi) &= -(\widetilde{\nabla}_U\sigma)(\eta(V)Y,\xi)\\
&= -\nabla^{\perp}_U\sigma(\eta(V)Y,\xi)+\sigma(\nabla_U\eta(V)Y,\xi)\\
&+ \sigma(\eta(V)Y,\nabla_U\xi)\\
&= \eta(V)\{\sigma(Y,\varphi hU)-\sigma(Y,\varphi U)\}. \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y) &= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= \nabla^{\perp}_U\sigma(V,\eta(Y)\xi)-\sigma(\nabla_UV,\eta(Y)\xi)\\
&- \sigma(V,\nabla_U\eta(Y)\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= -\sigma(V,U\eta(Y)\xi+\eta(Y)\nabla_U\xi)-(\widetilde{\nabla}_U\sigma)(V,Y)\\
&= \eta(Y)\{\sigma(V,\varphi U)-\sigma(V,\varphi hU)\}-(\widetilde{\nabla}_U\sigma)(V,Y). \qquad (36)
\end{aligned}
$$

Consequently, from (30), (31), (32), (33), (34), (35) and (36), we reach at

$$\begin{aligned}
\sigma(\varphi U, V) &- \sigma(\varphi hU, V) - \eta(U)(\mu(1+\kappa) - \kappa)\sigma(\varphi Y, V) \\
&- (\kappa - \mu)\eta(U)\sigma(\varphi hY, V) - \kappa\eta(V)\{\sigma(\varphi hU, Y) - \sigma(\varphi U, Y)\} \\
&- \mu\eta(V)\{\sigma(\varphi hU, hY) - \sigma(\varphi U, hY)\} - \kappa\eta(Y)\{\sigma(\varphi U, V) \\
&- \sigma(\varphi hU, V)\} + \kappa(\widetilde{\nabla}_U\sigma)(Y, V) + \mu(\widetilde{\nabla}_U\sigma)(V, hY) \\
&= -L_2\{-\eta(U)[\sigma(\varphi Y, V) - \sigma(\varphi hY, V)] + \eta(V)\sigma(Y, \varphi hU) \\
&- \eta(V)\sigma(Y, \varphi U) + \eta(Y)\sigma(V, \varphi U) - \eta(Y)\sigma(V, \varphi hU) \\
&- (\widetilde{\nabla}_U\sigma)(V, Y)\}. \qquad (37)
\end{aligned}$$

Replacing $V = \xi$ at (37) and considering (25), we get

$$\begin{aligned}
-\kappa\sigma(Y, \varphi hU) &+ \kappa\sigma(Y, \varphi U) - \mu\sigma(hY, \varphi hU) + \mu\sigma(hY, \varphi U) \\
&+ \kappa(\widetilde{\nabla}_U\sigma)(\xi, Y) + \mu(\widetilde{\nabla}_U\sigma)(\xi, hY) \\
&= -L_2\{\sigma(Y, \varphi hU) - \sigma(Y, \varphi U) - (\widetilde{\nabla}_U\sigma)(\xi, Y)\}, \qquad (38)
\end{aligned}$$

where

$$\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi, Y) &= \nabla^{\perp}_U\sigma(Y, \xi) - \sigma(\nabla_U Y, \xi) - \sigma(Y, \nabla_U\xi) \\
&= -\sigma(-\varphi U + \varphi hU, Y) = \sigma(\varphi U, Y) - \sigma(\varphi hU, Y). \qquad (39)
\end{aligned}$$

From (38) and (39), we conclude that

$$\begin{aligned}
&- \kappa\sigma(Y, \varphi hU) + \kappa\sigma(Y, \varphi U) - \mu\sigma(hY, \varphi hU) + \mu\sigma(hY, \varphi U) \\
&+ \kappa\sigma(Y, \varphi U) - \kappa\sigma(\varphi hU, Y) + \mu\sigma(\varphi U, hY) - \mu\sigma(\varphi hU, hY) \\
&= -L_2\{\sigma(Y, \varphi hU) - \sigma(Y, \varphi U) + \sigma(\varphi hU, Y) - \sigma(\varphi U, Y)\},
\end{aligned}$$

that is,

$$\begin{aligned}
L_2\{\sigma(Y, \varphi U) - \sigma(Y, \varphi hU)\} &= \kappa\{\sigma(Y, \varphi U) - \sigma(Y, \varphi hU)\} \\
&+ \mu\{\sigma(hY, \varphi U) - \sigma(hY, \varphi hU)\},
\end{aligned}$$

from which

$$\begin{aligned}
(L_2 - \kappa)\{\sigma(Y, \varphi U) &- \sigma(Y, \varphi hU)\} \\
&= \mu\{\sigma(hY, \varphi U) - \sigma(hY, \varphi hU)\}. \qquad (40)
\end{aligned}$$

Here substituting hY by Y in (40), we get

$$\begin{aligned}
(L_2 - \kappa)(\sigma(hY, \varphi U) &- \sigma(hY, \varphi hU)) \\
&= \mu(1+\kappa)(\sigma(Y, \varphi U) - \sigma(Y, \varphi hU)). \qquad (41)
\end{aligned}$$

From (40) and (41), we conclude that

$$\left((L_2 - k)^2 - \mu^2(1+\kappa)\right)(\sigma(Y,\varphi U) - \sigma(Y,\varphi hU)) = 0,$$

which is proves our assertions. We note that $\sigma(Y,\varphi U) - \sigma(Y,\varphi hU) = 0$ if and only if $\kappa\sigma(Y,U) = 0$. □

From Theorem 2.3, we have the following corollary.

COROLLARY 2.4. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is 2-semiparallel if and only if M is totally geodesic.*

THEOREM 2.5. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If M is a Ricci-generalized pseudoparallel submanifold, then $L_3 = \frac{1}{2n}(1 \mp \frac{\mu}{\kappa}\sqrt{1+\kappa})$. On the other hand, $L_3 \neq \frac{1}{2n}(1 \mp \frac{\mu}{\kappa}\sqrt{1+\kappa})$ if and only if M is totally geodesic submanifold.*

Proof. If M is Ricci-generalized pseudoparallel, then from (18) and (22), we have

$$\begin{aligned}(\widetilde{R}(X,Y)\cdot\sigma)(U,V) &= L_3 Q(S,\sigma)(U,V;X,Y) \\ &= -L_3\left\{\sigma((X\wedge_S Y)U,V) + \sigma(U,(X\wedge_S Y)V)\right\},\end{aligned}$$

for all $X,Y,U,V \in \Gamma(TM)$. This implies that

$$\begin{aligned}R^\perp(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V) - \sigma(U,R(X,Y)V) \\ &= -L_3\{\sigma(S(Y,U)X - S(X,U)Y,V) \\ &+ \sigma(S(V,Y)X - S(X,V)Y,U)\}.\end{aligned}$$

Here taking $U=\xi$ and by using (10), we reach at

$$\begin{aligned}R^\perp(X,Y)\sigma(\xi,V) &- \sigma(R(X,Y)\xi,V) - \sigma(\xi,R(X,Y)V) \\ &= -L_3\{S(Y,\xi)\sigma(X,V) - S(X,\xi)\sigma(Y,V) \\ &+ S(Y,V)\sigma(X,\xi) - S(X,V)\sigma(Y,\xi)\}.\end{aligned}$$

From (7), (10) and (25), we obtain

$$\sigma(R(X,Y)\xi,V) = 2n\kappa L_3\{\eta(Y)\sigma(X,V) - \eta(X)\sigma(Y,V)\}. \tag{42}$$

Thus

$$\begin{aligned}\sigma(\kappa[\eta(Y)X &- \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],V) \\ &= 2n\kappa L_3\{\eta(Y)\sigma(X,V) - \eta(X)\sigma(Y,V)\}.\end{aligned}$$

Consequently,

$$\kappa(2nL_3-1)(\sigma(\eta(Y)X-\eta(X)Y,V)=\mu\sigma(\eta(Y)hX-\eta(X)hY,V).$$

This equality reduce for $Y=\xi$,

$$\kappa(2nL_3-1)\sigma(X,V)=\mu\sigma(hX,V). \tag{43}$$

Substituting hX by X in (43) and making use of (12) we obtain

$$\begin{aligned}\kappa(2nL_3-1)\sigma(hX,V) &= \mu\sigma(h^2X,V)\\ &= \mu(1+\kappa)\sigma(\varphi^2X,V)\\ &= \mu(1+\kappa)\sigma(X,V).\end{aligned} \tag{44}$$

From (43) and (44), we conclude that

$$\{\kappa^2(2nL_3-1)^2-\mu^2(1+\kappa)\}\sigma(X,V)=0.$$

This proves our assertion. □

THEOREM 2.6. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. If M is a 2-Ricci-generalized pseudoparallel submanifold, then $L_4=\frac{2\mu}{2n\kappa-1}\sqrt{1+\kappa}$. As the other case, $L_4\neq\frac{2\mu}{2n\kappa-1}\sqrt{1+\kappa}$ if and only if M is totally geodesic submanifold.*

Proof. Let us assume that M is 2-Ricci-generalized pseudoparallel submanifold. Then from (23), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W)=L_4Q(S,\widetilde{\nabla}\sigma)(U,V,W;X,Y),$$

for all $X,Y,U,V,W\in\Gamma(TM)$. This implies that

$$\begin{aligned}R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,W) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,W)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,W)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)W)=L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,W)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,W)\\ &+ (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)W)\}.\end{aligned}$$

Here taking $X=V=\xi$, we have

$$\begin{aligned}R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,W) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W)\\ &- (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)W)=-L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W)\\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,W)+(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W)\}.\end{aligned} \tag{45}$$

Now, let's calculate each of these expressions. Also taking into account of (6) and (25),

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,W) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(\xi,W)-\sigma(\nabla_U\xi,W) \\ &- \sigma(\xi,\nabla_U W)\} \\ &= -R^{\perp}(\xi,Y)\sigma(-\varphi U+\varphi hU,W) \\ &= R^{\perp}(\xi,Y)\{\sigma(\varphi U,W)-\sigma(\varphi hU,W)\}. \end{aligned} \tag{46}$$

On the other hand, by using (7),

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,W)-\sigma(\nabla_{R(\xi,Y)U}\xi,W) \\ &- \sigma(\xi,\nabla_{R(\xi,Y)U}W) \\ &= -\sigma(-\varphi R(\xi,Y)U+\varphi hR(\xi,Y)U,W) \\ &= \sigma(\kappa[-\eta(U)\varphi Y]+\mu[-\eta(U)\varphi hY],W) \\ &- \sigma(\kappa[-\eta(U)\varphi hY]+\mu[-\eta(U)\varphi h^2Y],W) \\ &= -\kappa\eta(U)\sigma(\varphi Y,W)-\mu\eta(U)\sigma(\varphi hY,W) \\ &+ \kappa\eta(U)\sigma(\varphi hY,W)+\mu\eta(U)\sigma(\varphi h^2Y,W) \\ &= \eta(U)\{-\kappa\sigma(\varphi Y,W)-\mu\sigma(\varphi hY,W) \\ &+ \kappa\sigma(\varphi hY,W)+\mu(1+\kappa)\sigma(\varphi^3Y,W)\} \\ &= \eta(U)\{(\mu(1+\kappa)-\kappa)\sigma(\varphi Y,W) \\ &+ (\kappa-\mu)\sigma(\varphi hY,W)\}. \end{aligned} \tag{47}$$

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W) &= (\widetilde{\nabla}_U\sigma)(\kappa(\eta(Y)\xi-Y)-\mu hY,W) \\ &= \kappa(\widetilde{\nabla}_U\sigma)(\eta(Y)\xi-Y,W)-\mu(\widetilde{\nabla}_U\sigma)(hY,W) \\ &= \kappa\{(\widetilde{\nabla}_U\sigma)(\eta(Y)\xi,W)-(\widetilde{\nabla}_U\sigma)(Y,W)\} \\ &- \mu(\widetilde{\nabla}_U\sigma)(hY,W) \\ &= \kappa\{\nabla^{\perp}_U\sigma(\eta(Y)\xi,W)-\sigma(\nabla_U\eta(Y)\xi,W) \\ &- \sigma(\eta(Y)\xi,\nabla_U W)-(\widetilde{\nabla}_U\sigma)(Y,W)\} \\ &- \mu(\widetilde{\nabla}_U\sigma)(hY,W)\} \\ &= -\kappa\{\sigma(U\eta(Y)\xi,W)+\sigma(\eta(Y)\nabla_U\xi,W) \\ &- (\widetilde{\nabla}_U\sigma)(Y,W)\}-\mu(\widetilde{\nabla}_U\sigma)(hY,W) \\ &= \kappa\eta(Y)\{\sigma(\varphi U,W)-\sigma(\varphi hU,W)\} \\ &+ \kappa(\widetilde{\nabla}_U\sigma)(Y,W)-\mu(\widetilde{\nabla}_U\sigma)(hY,W). \end{aligned} \tag{48}$$

and

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi, R(\xi,Y)W) &= \nabla^{\perp}_U\sigma(\xi, R(\xi,Y)W) - \sigma(\nabla_U\xi, R(\xi,Y)W) \\
&- \sigma(\xi, \nabla_U R(\xi,Y)W) \\
&= -\sigma(-\varphi U + \varphi hU, R(\xi,Y)W) \\
&= -\sigma(-\varphi U + \varphi hU, \kappa(g(Y,W)\xi \\
&- \eta(W)Y) + \mu(g(hW,Y)\xi - \eta(W)hY)) \\
&= \eta(W)\{-\kappa\sigma(\varphi U, Y) - \mu\sigma(\varphi U, hY) \\
&+ \kappa\sigma(\varphi hU, Y) + \mu\sigma(\varphi hU, hY)\}. \qquad (49)
\end{aligned}
$$

Now, let's calculate the left side of equality. Making use of (6), (10) and (25),

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi, W) &= \nabla^{\perp}_{(\xi\wedge_S Y)U}\sigma(\xi, W) - \sigma(\nabla_{(\xi\wedge_S Y)U}\xi, W) \\
&- \sigma(\xi, \nabla_{(\xi\wedge_S Y)U}) \\
&= -\sigma(\nabla_{S(Y,U)\xi - S(\xi,U)Y}\xi, W) \\
&= -\sigma(-\varphi(S(Y,U)\xi - S(\xi,U)Y) + \varphi h(S(Y,U)\xi \\
&- S(\xi,U)Y), W) \\
&= S(U,\xi)\{\sigma(\varphi hY, W) - \sigma(\varphi Y, W)\} \\
&= 2n\kappa\eta(U)\{\sigma(\varphi hY, W) - \sigma(\varphi Y, W)\}. \qquad (50)
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi, W) &= (\widetilde{\nabla}_U\sigma)(S(Y,\xi)\xi - Y, W) \\
&= (\widetilde{\nabla}_U\sigma)(S(Y,\xi)\xi, W) - (\widetilde{\nabla}_U\sigma)(Y, W) \\
&= \nabla^{\perp}_U\sigma(S(Y,\xi)\xi, W) - \sigma(\nabla_U S(Y,\xi)\xi, W) \\
&- \sigma(\nabla_U W, S(Y,\xi)\xi) - (\widetilde{\nabla}_U\sigma)(Y, W) \\
&= -\sigma(US(Y,\xi)\xi + S(Y,\xi)\nabla_U\xi, W) \\
&- (\widetilde{\nabla}_U\sigma)(Y, W) \\
&= -S(Y,\xi)\sigma(-\varphi U + \varphi hU, W) - (\widetilde{\nabla}_U\sigma)(Y, W) \\
&= 2n\kappa\eta(Y)\{\sigma(\varphi U, W) - \sigma(\varphi hU, W)\} \\
&- (\widetilde{\nabla}_U\sigma)(Y, W). \qquad (51)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W) &= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,W)\xi-S(\xi,W)Y)\\
&= (\widetilde{\nabla}_U\sigma)(\xi,S(Y,W)\xi)-(\widetilde{\nabla}_U\sigma)(\xi,2n\kappa\eta(W)Y)\\
&= \nabla^{\perp}_U\sigma(\xi,S(Y,W)\xi)-\sigma(\nabla_U\xi,S(Y,W)\xi)\\
&- \sigma(\xi,\nabla_U S(Y,W)\xi)-2n\kappa(\widetilde{\nabla}_U\sigma)(\xi,\eta(W)Y)\\
&= -2n\kappa\{\nabla^{\perp}_U\sigma(\xi,\eta(W)Y)-\sigma(\nabla_U\xi,\eta(W)Y)\\
&- \sigma(\xi,U\eta(W)Y)\}\\
&= 2n\kappa\sigma(-\varphi U+\varphi hU,\eta(W)Y)\\
&= 2n\kappa\eta(W)\{\sigma(\varphi hU,Y)-\sigma(\varphi U,Y)\}. \qquad (52)
\end{aligned}
$$

By substituting (46), (47), (48), (49), (50), (51) and (52) into (45) we reach at

$$
\begin{aligned}
R^{\perp}(\xi,Y)\{\sigma(\varphi U,W) &- \sigma(\varphi hU,W)\}-\eta(U)\{(\mu(1+\kappa)-\kappa)\sigma(\varphi Y,W)\\
&+ (\kappa-\mu)\sigma(\varphi hY,W)\}-\kappa(\widetilde{\nabla}_U\sigma)(Y,W)\\
&+ \mu(\widetilde{\nabla}_U\sigma)(hY,W)-\kappa\eta(Y)\{\sigma(\varphi U,W)-\sigma(\varphi hU,W)\}\\
&- \eta(W)\{\kappa\sigma(\varphi hU,Y)+\mu\sigma(\varphi hU,hY)-\kappa\sigma(\varphi U,Y)\\
&- \mu\sigma(\varphi U,hY)\}\\
&= -L_4\{2n\kappa\eta(U)(\sigma(\varphi hY,W)-\sigma(\varphi Y,W))\\
&+ 2n\kappa\eta(Y)(\sigma(\varphi U,W)-\sigma(\varphi hU,W)\\
&+ 2n\kappa\eta(W)(\sigma(\varphi hU,Y)-\sigma(\varphi U,Y))\\
&- (\widetilde{\nabla}_U\sigma)(Y,W))\}. \qquad (53)
\end{aligned}
$$

Here taking $W=\xi$, (53) reduce

$$
\begin{aligned}
L_4\{(\widetilde{\nabla}_U\sigma)(Y,\xi) &+ 2n\kappa\{\sigma(\varphi hU,Y)-\sigma(\varphi U,Y)\}\}=-\kappa(\widetilde{\nabla}_U\sigma)(Y,\xi)\\
&+ \mu(\widetilde{\nabla}_U\sigma)(hY,\xi)\\
&- \{\kappa\sigma(\varphi hU,Y)+\mu\sigma(\varphi hU,hY)-\kappa\sigma(\varphi U,Y)-\mu\sigma(\varphi U,hY)\},
\end{aligned}
$$

that is,

$$
\begin{aligned}
L_4\{\nabla^{\perp}_U\sigma(Y,\xi) &- \sigma(\nabla_U Y,\xi)-\sigma(\nabla_U\xi,Y)-\sigma(\varphi hU,Y)\\
&+ \sigma(\varphi U,Y)\}=\kappa\{\nabla^{\perp}_U\sigma(Y,\xi)-\sigma(\nabla_U Y,\xi)-\sigma(Y,\nabla_U\xi)\}\\
&+ \mu\{\nabla^{\perp}_U\sigma(hY,\xi)-\sigma(\nabla_U hY,\xi)-\sigma(\nabla_U\xi,hY)\}\\
&- \kappa\sigma(\varphi hU,Y)-\mu\sigma(\varphi hU,hY)+\kappa\sigma(\varphi U,Y)+\mu\sigma(\varphi U,hY).
\end{aligned}
$$

On the other hand, by a direct calculation, we have

$$(\widetilde{\nabla}_U\sigma)(Y,\xi)=\sigma(\varphi U,Y)-\sigma(\varphi hU,Y).$$

Thus we conclude that

$$\begin{aligned} L_4\{(2n\kappa-1)(\sigma(Y,\varphi hU) &- \sigma(Y,\varphi U))\}=2\mu\{\sigma(\varphi U,hY)\\ &- \sigma(\varphi hU,hY)\}. \end{aligned} \tag{54}$$

Substituting hY by Y into (54), we can derive

$$\begin{aligned} L_4(2n\kappa-1)\{\sigma(hY,\varphi hU) &- \sigma(hY,\varphi U)\}=2\mu\{\sigma(\varphi U,h^2Y)-\sigma(\varphi hU,h^2Y)\}\\ &= 2\mu(1+\kappa)\{\sigma(\varphi U,\varphi^2Y)-\sigma(\varphi hU,\varphi^2Y)\}\\ &= 2\mu(1+\kappa)\{\sigma(\varphi U,Y)-\sigma(\varphi hU,Y)\}. \end{aligned} \tag{55}$$

From (54) and (55), we find

$$L_4^2(2n\kappa-1)^2=4\mu^2(1+\kappa)\{\sigma(\varphi U,Y)-\sigma(\varphi hU,Y)\},$$

which proves our assertion. □

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