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THE EXTENDIBILITY OF DIOPHANTINE PAIRS WITH PROPERTY $D(-1)$

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Abstract. A set $\{a_1, a_2, \cdots, a_m\}$ of m distinct positive integers is called a $D(-1)$-m-tuple if the product of any distinct two elements in the set decreased by one is a perfect square. In this paper, we find a solution of Pellian equations which is constructed by $D(-1)$-triples and using this result, we prove the extendibility of $D(-1)$-pair with some conditions.

1. Introduction

Let n be an integer. A set $\{a_1, a_2, \cdots, a_m\}$ of m positive integers is called a Diophantine m-tuple with the property $D(n)$, if the product of any two of them increased by n is a perfect square. The set called simply $D(n)$-m-tuple. For the case $n = 1$, Fermat [3] first found the Diophantine quadruple $\{1, 3, 8, 120\}$. This set is called Fermat's set. Euler found that the number 777480/8288641 makes the Fermat's set to rational Diophantine quintuple. Actually, Diophantus found the Diophantine quadruple $\{1/16, 33/16, 17/4, 105/16\}$ which consists of distinct rational numbers. In 1969, A. Baker and H. Davenport proved that the $D(1)$-triple $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. This

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result was generalized by Y. Fujita [9], who showed that the set $\{k-1, k+1\}$ for an integer k cannot be extended to a $D(1)$-quintuple.

Let us consider about the importance of extendibility of $D(1)$-triple. We can find the answer in the relation with the elliptic curve. We have to solve the equations

$$ax+1=\Box,\ bx+1=\Box,\ cx+1=\Box$$

to extend the $D(1)$-triple $\{a, b, c\}$ to $D(1)$-quadruple. Hence, we have the equation

$$E: y^2=(ax+1)(bx+1)(cx+1),$$

which is the elliptic curve by the product of three equations. If we prove the non-extendibility of $D(1)$-triple then it suggests that there exist only finitely many integer points on E induced by some $D(1)$-triples. For example, A. Dujella [5] proved that the elliptic curve

$$E_k: y^2=((k-1)x+1)((k+1)x+1)(4kx+1)$$

has four integer points

$$(0, \pm 1),\ (16k^3-4k, \pm(128k^6-112k^4-20k^2-1))$$

under assumption that $\operatorname{rank}(E_k(\mathbb{Q}))=1$. There are various papers which contain the similar results [6, 10]. Hence, it is important not only how to prove the extendibility of $D(1)$-triple but also which cases are proved. A folklore conjecture was that there does not exist a $D(1)$-quintuple. Recently, the conjecture has been proved by B. He, A. Togbé and V. Ziegler [12].

In the case $n=-1$, the conjecture is that there does not exist $D(-1)$-quadruple. The case $n=-1$ is related to an old problem of Diophantus and Euler. Diophantus studied the problem of finding numbers such that the product of any two numbers increased by the sum of these gives a square, that is

$$xy+x+y=(x+1)(y+1)-1=\Box.$$

We see that the problem of finding integer m-tuples with the above property is equivalent to finding $D(-1)$-m-tuples.

It is known that some particular $D(-1)$-triples cannot be extended to $D(-1)$-quadruple. In 1985, E. Brown [2] proved that the set $\{1, 2, 5\}$ cannot be extended to $D(-1)$-quadruple. Also, Mohanty and Ramasamy proved the extendiblity of $D(-1)$-triple $\{1, 5, 10\}$. Furthermore, the extendiblity of the sets $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$,

$\{1, 10, 17\}$, $\{1, 26, 37\}$ to $D(-1)$-quadruple proved by Kedlaya [13]. The important result of the extendiblity of $D(-1)$-m-tuple was proved by A. Dujella and C. Fuchs [7]. They proved that there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ with $2 < a < b < c < d$. This means if $\{a, b, c, d\}$ is a $D(-1)$-quadurple then $a = 1$, $b \geq 5$ and obviously b should have the form $\alpha^2 + 1$ for an integer α.

Let F_k be the k-th Fibonacci number which defined by $F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$. The aim of this paper is to give a solution of Pellian equations which is constructed by $D(-1)$-triples

$$\{1, F_{2k+2}^2 + 1, 4F_{2k+2}^4 + 1\}$$

and

$$\{1, F_{2k+2}^2 + 1, (4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1\}.$$

According to these results, we prove the extendiblity of $D(-1)$-pair

$$\{1, F_{2k+2}^2 + 1\}$$

with some conditions.

2. Preliminaries

2.1. The properties of third elements. Let $\{1, b, c\}$ be a Diophantine triple with the property $D(-1)$ and $1 < b < c$. Then there exist positive integers r, s, t which satisfy the following equations

$$b - 1 = r^2, \quad c - 1 = s^2, \quad bc - 1 = t^2.$$

Eliminating c from these equations, we obtain the Diophantine equation

$$t^2 - bs^2 = b - 1 = r^2, \tag{1}$$

and easily find the form of solutions of equation is

$$(t + s\sqrt{b}) = (t_0 + s_0\sqrt{b})(2b - 1 + 2r\sqrt{b})^\nu.$$

From the definition of s, we find the form of third elements $c = c_\nu$ of Diophantine triple with the property $D(-1)$, that is,

$$c_\nu = \frac{1}{4b}[(t_0+s_0\sqrt{b})^2(2b-1+2\sqrt{b})^{2\nu}+(t_0-s_0\sqrt{b})^2(2b-1-2\sqrt{b})^{2\nu}+2r^2+4].$$

The equation (1) has at least three classes of solutions belonging to $(t_0, s_0) = (r, 0), (b - r, \pm(r - 1))$. We call a positive solution (t, s) of equation (1) regular if (t, s) belongs to one of these three classes. The following theorem gives us the upper bound of third elements c in the Diophantine triple with the property $D(-1)$.

THEOREM 2.1. *[8, Theorem 1] If $\{1, b, c, d\}$ with $b < c < d$ is a $D(-1)$-quadruple, then $c < 9.6b^4$.*

2.2. The fundamental solutions of the Pell equations. We have to solve the system

$$d - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2 \tag{2}$$

to extend the Diophantine triple with the property $D(-1)$ to the Diophantine quadruple with the property $D(-1)$. From the equations (2), we obtain the following system of Pell equations

$$z^2 - cx^2 = c - 1 \tag{3}$$
$$bz^2 - cy^2 = c - b. \tag{4}$$

Using the following theorem, we get the fundamental solutions of Pell equations.

LEMMA 2.2. *[6, Lemma 1] If (z, x) and (z, y) with positive integers x, y, z are solutions of (3) and (4), respectively then there exist integers z_0, x_0 and z_1, y_1, with*

1. (z_0, x_0) and (z_1, y_1) are solutions of (3) and (4), respectively.
2. the folliwng inequalities are satisfied:

$$|x_0| < s, \qquad 0 < z_0 < c, \tag{5}$$
$$|y_1| < t, \qquad 0 < z_1 < c, \tag{6}$$

and there exist integers $m, n \geq 0$ such that

$$z + x\sqrt{c} = (z_0 + x_0\sqrt{c})(s + \sqrt{c})^{2m}, \tag{7}$$
$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^{2n}. \tag{8}$$

By (7), we may write $z = v_m$, where

$$v_0 = z_0, v_1 = (2c - 1)z_0 + 2scx_0, v_{m+2} = (4c - 2)v_{m+1} - v_m, \tag{9}$$

and (8), we may writhe $z = w_n$, where

$$w_0 = z_1, w_1 = (2bc - 1)z_1 + 2tcy_1, w_{m+2} = (4bc - 2)w_{n+1} - w_n. \tag{10}$$

Our system of equations (3) and (4) is thus transformed to finitely many equations of the form $z = v_m = w_n$. Moreover, we have the following properties.

Lemma 2.3. *[6, Lemma 2] If* $v_m = w_n, n \neq 0$ *then*

1. $m \equiv n \pmod 2$,
2. $n \leq m \leq 2n$,
3. $m^2 z_0 + smx_0 \equiv bn^2 z_1 + tny_1 \pmod{4c}$.

2.3. Congruence relation between solutions of Pell equations. From (9) and (10), we have the following congruence equations by induction.

Lemma 2.4. *[7, Lemma 2]*

$$\begin{aligned} v_m &\equiv (-1)^m (z_0 - 2cm^2 z_0 - 2csmx_0) \pmod{8c^2}, \\ w_n &\equiv (-1)^m (z_1 - 2bcn^2 z_1 - 2ctny_1) \pmod{8c^2}. \end{aligned}$$

It is natually to ask when does $v_m = w_n$ have a solution and if there exists a solution then what values are possible. The following lemma gives us the answer.

Lemma 2.5. *[6, Lemma 5] Let the integers* z_0, z_1, x_0, y_1 *be as in Lemma 2.2. If* $c \leq b^9$ *then*

$$z_0 = z_1 = s, x_0 = 0, y_1 = \pm\sqrt{b-1} = \pm r.$$

By Theorem 2.1, we know that the set $\{1, b, c\}$ can be a $D(-1)$-triple when $c \leq 9.6b^4$. Therefore, the fundamental solution of Pell equation is as expressed in the Lemma 2.5.

2.4. Some theorems for applying the reduction method. From (7) and (8), and sum of their conjugates, respectively, we have

$$\begin{aligned} v_m &= \frac{1}{2}[(z_0 + x_0\sqrt{c})(s + \sqrt{c})^{2m} + (z_0 - x_0\sqrt{c})(s - \sqrt{c})^{2m}], \\ w_n &= \frac{1}{2\sqrt{b}}[(z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^{2n} + (z_1\sqrt{b} - y_1\sqrt{c})(t - \sqrt{bc})^{2n}]. \end{aligned}$$

Hence, we transform the equation $v_m = w_n$ into the following inequality.

Lemma 2.6. *[6, Lemma 11] If* $v_m = w_n, n \neq 0$ *then*

$$0 < 2n\log(t + \sqrt{bc}) - 2m\log(s + \sqrt{c}) + \log(\frac{s\sqrt{b} \pm \sqrt{c}}{2\sqrt{b}}) < (3.96bc)^{-n+1}.$$

THEOREM 2.7. *[1, p.20] For a linear form*

$$\Lambda = \beta_1 \log \alpha_1 + \cdots + \beta_l \log \alpha_l \neq 0$$

in logarithms of l algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ with rational coefficients $\beta_1, \beta_2, \ldots, \beta_l$, we have

$$\log |\Lambda| \geq -18(l+1)! l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log \beta,$$

where $\beta := \max\{|\beta_1|, \ldots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \cdots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

LEMMA 2.8. *[7, Lemma 5] Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

1. If $\epsilon > 0$ then there is no solution of the inequality

$$0 < n\kappa - m + \mu < AB^{-n} \tag{11}$$

in integers n and m with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq n \leq M.$$

2. Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$ then there is no solution of inequality (11) in integers n and m with

$$\max\left\{\frac{\log(3Aq)}{\log B}, 1\right\} < n \leq M.$$

3. The extendibility with fundamental solution $(r, 0)$

3.1. The lower bound of n. To find the extendibility of $D(-1)$-pair $\{1, F_{2k+2}^2 + 1\}$, we should find which case c makes $D(-1)$-pair to $D(-1)$-triple $\{1, F_{2k+2}^2 + 1, c\}$. The extendibility of $D(-1)$-triple $\{1, 2, c\}$ is proved by A. Dujella in [4], we may assume that $k \geq 1$, that is $F_{2k+2}^2 + 1 \geq 10$. Using the Theorem 2.1, we obtain the case of third element belonging to the $D(-1)$-triple $\{1, F_{2k+2}^2 + 1, c\}$.

Lemma 3.1. *Let* $\{1, F_{2k+2}^2+1, c\}$ *is a* $D(-1)$*-triple with the fundamental solution* $(r, 0)$*. Then the third element* $c > F_{2k+2}^2 + 1$ *is only*

$$4F_{2k+2}^4 + 1.$$

Proof. As mentioned before, the form of third element $c = c_\nu$ is

$$c_\nu = \frac{1}{4b}[(t_0+s_0\sqrt{b})^2(2b-1+2\sqrt{b})^{2\nu}+(t_0-s_0\sqrt{b})^2(2b-1-2\sqrt{b})^{2\nu}+2r^2+4].$$

If the fundamental solution of Pell equation is $(r, 0)$ then c can be expressed by

$$c = c_\nu = \frac{1}{4b}[r^2((2b-1+2\sqrt{b})^{2\nu} + (2b-1-2b\sqrt{b})^{2\nu}) + 2r^2 + 4].$$

Since if $\nu = 0$ implies $c = 1$, we may assume that $\nu \geq 1$.

1. $\nu = 1$

   In the case of $\nu = 1$, we get the form of third element $c = c_1$ is

   $$4F_{2k+2}^4 + 1,$$

   and we easily find that $4F_{2k+2}^4 + 1 < 9.6(F_{2k+2}^2 + 1)^4$.
2. $\nu \geq 2$

   In the case of $\nu \geq 2$, we get the lower bound of third elements c is

   $$16F_{2k+2}^4(2F_{2k+2}^2 + 1)^2,$$

   but this lower bound is greater than $9.6(F_{2k+2}^2 + 1)^4$. Therefore, these third elements cannot be extended to $D(-1)$-triple to $D(-1)$-quadruple. Hence, the only possible c is

   $$4F_{2k+2}^4 + 1.$$

□

The next lemma gives us the lower bound of n. This lower bound will be used in the reduction methods.

Lemma 3.2. *Let* $c = c_1 = 4F_{2k+2}^2 + 1$*. Then if the equation* $v_m = w_n$ *has a solution with* $n \geq 2$ *then*

$$n > \sqrt{2F_{2k+2}^2 - 1} - \frac{1}{2}.$$

Proof. Using the lemma 2.3, we have the congruence equation

$$2F_{2k+2}^2(m^2 - (F_{2k+2}^2 + 1)n^2) \equiv nF_{2k+2}(2F_{2k+2}^3 + F_{2k+2}) \pmod{4c}.$$

Moreover, since $\gcd(F_{2k+2}^2, c) = 1$, we have

$$2(m^2 - (F_{2k+2}^2 + 1)n^2) \equiv \pm n(2F_{2k+2}^2 + 1) \pmod{c}.$$

This means

$$2(F_{2k+2}^2+1)n^2 \pm (2F_{2k+2}^2+1)n - 2m^2 > 2(F_{2k+2}^2+1)n^2 - (2F_{2k+2}^2+1)n - 8n^2 > 0,$$

since $2 \leq n \leq m \leq 2n$. Therefore, we have

$$2(F_{2k+2}^2 + 1)n^2 + (2F_{2k+2}^2 + 1)n - 2n^2 = 2F_{2k+2}^2 n^2 + (2F_{2k+2}^2 + 1)n > c.$$

This inequality shows that

$$(2F_{2k+2}^2 + 1)(n + \frac{1}{2})^2 - \frac{2F_{2k+2}^2 + 1}{4} > c.$$

This implies n has the lower bound

$$n > \sqrt{\frac{c}{2F_{2k+2}^2 + 1}} - \frac{1}{2} > \sqrt{2F_{2k+2}^2 - 1} - \frac{1}{2}.$$

Hence, we have the desired result. □

3.2. The Theorem of Baker and Wüstholz.

LEMMA 3.3. *If* $v_m = w_n$, $n \neq 0$ *then*

$$0 < 2n \log(t + \sqrt{bc}) - 2m \log(s + \sqrt{c}) + \log \frac{s\sqrt{b} \pm r\sqrt{c}}{s\sqrt{b}} < 0.625(bc - 1)^{-n}.$$

Proof. Using the Lemma 2.5 with the solving the recurrences (7) and (8), we get

$$v_m = \frac{s}{2}[(s + \sqrt{c})^{2m} + (s - \sqrt{c})^{2m}],$$

$$w_n = \frac{s\sqrt{b} \pm r\sqrt{c}}{2\sqrt{b}}[(t + \sqrt{bc})^{2n} + (t - \sqrt{bc})^{2n}].$$

Let

$$P = s(s + \sqrt{c})^{2m}, \quad Q = \frac{s\sqrt{b} \pm r\sqrt{c}}{\sqrt{b}}(t + \sqrt{bc})^{2n}.$$

The equation $v_m = w_n$ implies

$$P + (c - 1)P^{-1} = Q + \frac{c - b}{b}Q^{-1}.$$

From this equation, we have

$$P - Q = \frac{c - b}{b}Q^{-1} - (C - 1)P^{-1} < (c - 1)(P - Q)P^{-1}Q^{-1}.$$

Since $P > c-1$ and $Q > 2c \geq 2$, we have $Q > P$. Furthermore,

$$P > Q - (c-1)P^{-1} > Q - 1$$

which implies that

$$\frac{Q-P}{Q} < Q^{-1} \leq \frac{1}{2}.$$

These means

$$0 < \log\frac{Q}{P} = -\log\frac{P}{Q} = -\log(1 - \frac{Q-P}{Q}) < \frac{Q-P}{Q} + (\frac{Q-P}{Q})^2.$$

From this inequlity, we have the following result in our situation.

$$\begin{aligned} 0 &< \log\frac{Q}{P} < \frac{1}{Q} + \frac{1}{Q^2} < \frac{2}{Q} = \frac{2\sqrt{b}}{s\sqrt{b} \pm r\sqrt{c}}(t+\sqrt{bc})^{-2n} \\ &< \frac{2\sqrt{b}}{s\sqrt{b} - r\sqrt{c}}(2\sqrt{bc-1})^{-2n} < \frac{8F_{2k+2}^2 + 8}{4(4F_{2k+2}^2 - 1)}(bc-1)^{-n} \\ &< \frac{F_{2k+2}^2 + 1}{2(F_{2k+2}^2 - 1)}(bc-1)^{-n} < \frac{1.25}{4}(bc-1)^{-n} = 0.625(bc-1)^{-n}. \end{aligned}$$

□

Now, we apply the theorem of Baker and Wüstholz. We consider the equation $v_m = w_n$. By applying the theorem, we have $l = 3, d = 4, B = 2m$, where

$$\begin{aligned} \alpha_1 &= (t+\sqrt{bc})^2 = 2bc - 1 + 2t\sqrt{bc}, \\ \alpha_2 &= (s+\sqrt{c})^2 = 2c - 1 + 2s\sqrt{c}, \\ \alpha_3 &= \frac{s\sqrt{b} \pm r\sqrt{c}}{s\sqrt{b}}. \end{aligned}$$

Then we obtain the following equations by $\alpha_1, \alpha_2, \alpha_3$.

$$\begin{aligned} \alpha_1^2 - 2(2bc-1)\alpha_1 + 1 = 0 \\ \alpha_2^2 - 2(2c-1)\alpha_2 + 1 = 0 \\ b(c-1)\alpha_3^2 - 2b(c-1)\alpha_3 + c - b = 0. \end{aligned}$$

Therefore,

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log(4bc),$$
$$h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log(4c),$$
$$h'(\alpha_3) = \frac{1}{2}\log\frac{s\sqrt{b}+r\sqrt{c}}{s\sqrt{b}} < \frac{1}{2}\log 2,$$

and

$$\log|\Lambda| \geq -18\cdot 4!3^4(32\cdot 4)^5\frac{1}{2}\log(4bc)\frac{1}{2}\log(4c)\frac{1}{2}\log 2\log 24\log 2m.$$

Since

$$\log 0.625(bc-1)^{-n} < \log(bc-1)^{-n+1} < (-n+1)\log(bc),$$

we have

$$\frac{n-1}{\log 4n} < 6.64\cdot 10^{14}\log 4c. \tag{12}$$

1. In the case $v_{2m} = w_{2n}$, we get the inequality
   $$2\sqrt{2F_{2k+2}^2-1}-2 < 2.66\cdot 10^{15}\log(2.12F_{2k+2}))\log(5.66F_{2k+2})$$
   from the inequality (12).
2. In the case $v_{2m+1} = w_{2n+1}$, similarly as above, we get the inequality
   $$2\sqrt{2F_{2k+2}^2-1}-1 < 2.66\cdot 10^{15}\log(2.12F_{2k+2})\log(5.66F_{2k+2}).$$

From these inequalities, we obtain the following results

$$F_{2k+2} < 1.76\cdot 10^{18}, \quad c < 3.84\cdot 10^{73}.$$

Since $F_k = (\alpha^k - \bar{\alpha}^k)/\sqrt{5}$, where $\alpha = (1+\sqrt{5})/2 > 1.618$, we have the following inequality from Fibonacci numbers

$$(1.618)^k < (\alpha)^k = \bar{\alpha}^k + \sqrt{5}\cdot F_k. \tag{13}$$

We can find the following upper bounds

$$n \leq 5.05\cdot 10^{18}, \quad k \leq 43,$$

using the inequalities (12) and (13), respectively.

3.3. The reduction method. Now dividing logarithmic inequalities from Lemma 3.3 by $\log \alpha_2$ leads us to the inequality

$$0 < n\kappa - m + \mu < AB^{-n},$$

where

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{2 \log \alpha_2}, \quad A = \frac{0.625}{2 \log \alpha_2}, \quad B = (bc - 1).$$

We apply Lemma 2.8 to the logarithmic inequality with $M = 1.01 \cdot 10^{19}$ and we have to examine $2 \cdot 43 = 86$ cases(the doubling comes from the signs $\pm$ in α_3). The program was developed in **PARI/GP** running with 150 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\epsilon > 0$ then we use the next convergent until we find the one that satisfies the conditions. Then we have the following results.

TABLE 1. Results from **PARI/GP** running

Sign of α_3	Use the next convergent	Upper bounds of n
$+$ sign	0 case	5
$-$ sign	0 case	5

After few steps of reduction in all cases, we get $n < 2$. Therefore, we have the following first main theorem.

THEOREM 3.4. *Let k be an integer. Then the system of Pellian equations*

$$y^2 - (F_{2k+2}^2 + 1)x^2 = F_{2k+2}^2,$$
$$z^2 - (4F_{2k+2}^4 + 1)x^2 = 4F_{2k+2}^4$$

has only the trivial solutions $(x, y, z) = (0, \pm F_{2k+2}, \pm 2F_{2k+2}^2)$, where F_k is the k-th Fibonacci number defined by $F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$.

Consequently, Theorem 3.4 gives

COROLLARY 3.5. *Let k be an integer. Then the set $\{1, F_{2k+2}^2 + 1, 4F_{2k+2}^4 + 1\}$ cannot be extended to a $D(-1)$-quadruple.*

4. The extendibility with fundamental solution $(b-r, \pm(r-1))$

4.1. The lower bound of n. In this section, we consider the extendibility of $D(-1)$-triple $\{1, F_{2k+2}^2+1, c\}$, where c is induced by the fundamental solutions $(b-r, \pm(r-1))$. The following lemma gives us the answer for the possible cases of c.

LEMMA 4.1. *Let $\{1, F_{2k+2}^2+1, c\}$ be the $D(-1)$-triple with the fundamental solutions $(b-r, \pm(r-1))$. Then the third element $c > F_{2k+2}^2+1$ is only*

$$(4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1.$$

Proof. We already find the form of third elements c is

$$c_\nu = \frac{1}{4b}[(t_0+s_0\sqrt{b})^2(2b-1+2\sqrt{b})^{2\nu}+(t_0-s_0\sqrt{b})^2(2b-1-2\sqrt{b})^{2\nu}+2r^2+4].$$

If $\nu \geq 2$ then the lower bound of c is greater than $9.6(F_{2k+2}^2+1)^4$. Hence, we may check the cases $\nu \leq 1$.

1. If $\nu = 0$ then we have $D(-1)$-triple $\{1, F_{2k+2}^2+1, (F_{2k+2}-1)^2+1\}$. This case is proved by B. He and A. Togbé in [11].
2. If $\nu = 1$ with the fundamental solution $(b-r, -(r-1))$ then $D(-1)$-triple has a third element $(F_{2k+2}+1)^2+1$. This case is also proved in [11].

Lastly, if $\nu = 1$ with the fundamental solution $(b-r, (r-1))$, then the third element c of $D(-1)$-triple $\{1, F_{2k+2}^2+1, c\}$ is

$$(4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1.$$

Therefore, we have the desired result. □

From the third element c, we have the following result

$$r = F_{2k+2}, \quad s = 4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1,$$

and

$$t = 4F_{2k+2}^4 - 4F_{2k+2}^3 + 5F_{2k+2}^2 - 3F_{2k+2} + 1.$$

Let us consider the lower bound of n. The lemma gives us the answer.

LEMMA 4.2. *Let $c = c_1 = (4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1$. Then if the equation $v_m = w_n$ has a solution with $n \geq 2$ then*

$$n \geq \frac{\sqrt[4]{c}}{F_{2k+2}}.$$

Proof. Using the lemma 2.3, we have the congruence equation

$$s((F_{2k+2}^2+1)n^2-m^2) \equiv \mp ntr \pmod{4c}. \tag{14}$$

We assume that $n \le \sqrt[4]{c}/F_{2k+2}$. Then we have

$$s((F_{2k+2}^2+1)n^2-m^2) < \sqrt{c}F_{2k+2}^2 n^2 < F_{2k+2}^2\sqrt{c}(\sqrt[4]{c}/F_{2k+2})^2 = c,$$

and since $c > F_{2k+2}\sqrt[4]{c^3}$,

$$ntr < nF_{2k+2}\sqrt{(F_{2k+2}^2-1)c} < nF_{2k+2}^2\sqrt{c} < F_{2k+2}\sqrt[4]{c^3} < c.$$

This means

$$s((F_{2k+2}^2+1)n^2-m^2) = ntr, \tag{15}$$

since $m \le 2n$. From (14), we also have

$$s^2((F_{2k+2}^2+1)n^2-m^2)^2 \equiv (ntr)^2 \pmod{4c}.$$

Since $s^2 \equiv t^2 \equiv -1 \pmod{c}$, the congruence equation becomes

$$((F_{2k+2}^2+1)n^2-m^2)^2 \equiv n^2r^2 \pmod{c}.$$

The left side of congruence equation becomes

$$((F_{2k+2}^2+1)n^2-m^2)^2 \le F_{2k+2}^4 n^4 \le F_{2k+2}^4\left(\frac{\sqrt[4]{c}}{F_{2k+2}}\right)^4 = c,$$

and the right side of congruence equation becomes

$$n^2r^2 < F_{2k+2}^2\left(\frac{\sqrt[4]{c}}{F_{2k+2}}\right)^2 < c.$$

Hence, we have

$$((F_{2k+2}^2+1)n^2-m^2)^2 = n^2r^2. \tag{16}$$

From (15) and (16), we have $t^2 = s^2$ which is a contradiction. Therefore, we have desired result. □

4.2. The Theorem of Baker and Wüstholz.

LEMMA 4.3. *If $v_m = w_n, n \neq 0$ then*

$$0 < 2n\log(t+\sqrt{bc}) - 2m\log(s+\sqrt{c}) + \log\frac{s\sqrt{b}\pm r\sqrt{c}}{s\sqrt{b}} < 0.028(bc-1)^{-n}.$$

Proof. The proof follows along the same line as that of Lemma 3.3. In this case, we have

$$\begin{aligned} 0 &< \log\frac{Q}{P} < \frac{1}{Q} + \frac{1}{Q^2} < \frac{2}{Q} = \frac{2\sqrt{b}}{s\sqrt{b} \pm r\sqrt{c}}(t+\sqrt{bc})^{-2n} \\ &< \frac{2\sqrt{b}}{s\sqrt{b} - r\sqrt{c}}(2\sqrt{bc-1})^{-2n} < \frac{F_{2k+2}^2+1}{4F_{2k+2}^3 - 4F_{2k+2}^2 + 2F_{2k+2} - 1}(bc-1)^{-n} \\ &< \frac{101}{3619}(bc-1)^{-n} < 0.028(bc-1)^{-n}. \end{aligned}$$

□

By the theorem of Baker and Wüstholz with α_1, α_2, and α_3, where we used in Chapter 3.2, we have the equation (12), since

$$0.028(bc-1)^{-n} < (-n+1)\log(bc).$$

From the inequatily (12), we have the following two inequalities.

1. In the case $v_{2m} = w_{2n}$, we get the inequality
$$2\sqrt{4r-4} - 1 < 6.64 \cdot 10^{14} \log(2(4r^3 - 4r^2 + 3r))\log(256r),$$
where $r = F_{2k+2}$.
2. In the case $v_{2m+1} = w_{2n+1}$, similarly as above, we get the inequality
$$2\sqrt{4r-4} < 6.64 \cdot 10^{14} \log(2(4r^3 - 4r^2 + 3r))\log(324r),$$
where $r = F_{2k+2}$.

From these two inequalities, we obtain the following results

$$F_{2k+2} < 1.55 \cdot 10^{37}, \quad c < 2.22 \cdot 10^{224}.$$

Using the upper bound of c and inequality (13), we have

$$n \le 1.57 \cdot 10^{19}, \quad k \le 87.$$

4.3. The reduction method. Now dividing logarithmic inequalities from Lemma 4.3 by $\log\alpha_2$ leads us to the inequality

$$0 < n\kappa - m + \mu < AB^{-n},$$

where

$$\kappa = \frac{\log\alpha_1}{\log\alpha_2}, \quad \mu = \frac{\log\alpha_3}{2\log\alpha_2}, \quad A = \frac{0.028}{2\log\alpha_2}, \quad B = (bc-1).$$

We apply Lemma 2.8 to the logarithmic inequality with $M = 3.14 \cdot 10^{19}$ and we have to examine $2 \cdot 87 = 174$ cases(the doubling comes from the signs $\pm$ in α_3). The program was developed in **PARI/GP** running with 150 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\epsilon > 0$ then we use the next convergent until we find the one that satisfies the conditions. Then we have the following results.

TABLE 2. Results from **PARI/GP** running

Sign of α_3	Use the next convergent	Upper bounds of n
$+$ sign	0 case	3
$-$ sign	0 case	4

After few steps of reduction in all cases, we get $n < 2$. Therefore, we have the following second main theorem.

THEOREM 4.4. *Let k be an integer and*

$$c = (4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1.$$

Then the system of Pellian equations

$$\begin{cases} y^2 - (F_{2k+2}^2 + 1)x^2 = F_{2k+2}^2, \\ z^2 - cx^2 = (4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 \end{cases}$$

has only the trivial solutions

$$(x, y, z) = (0, \pm F_{2k+2}, \pm 4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1),$$

where F_k is the k-th Fibonacci number defined by $F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$.

Consequently, Theorem 4.4 gives

COROLLARY 4.5. *The set*

$$\{1, F_{2k+2}^2 + 1, (4F_{2k+2}^3 - 4F_{2k+2}^2 + 3F_{2k+2} - 1)^2 + 1\}$$

cannot be extended to a $D(-1)$-quadruple.

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