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CLOSURE FILTERS AND PRIME FUZZY CLOSURE FILTERS OF $MS-$ALGEBRAS

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Abstract. The notion of fuzzy closure filters is introduced and discussed in an MS-algebra. In particular, we characterize the prime fuzzy closure filters in terms of boosters. Some relationship between the lattice of fuzzy closure filters and the fuzzy ideal lattice of boosters are explored and investigated.

1. Introduction

In the last decades, various generalization of Boolean algebras have emerged. Along this direction, the class of $MS-$ algebras were first introduced by T.S. Blyth and J.C. Varlet [5, 6] as a generalization of de Morgan algebras and Stone algebras.

In the literature, Alaba and Alemayehu first introduced the concept of closure fuzzy ideals of $MS-$algebras in [4]. As a consequence, A. Badawy and R. El-Fawal have recently introduced the new notion of boosters, that is, the closure filters in $MS-$algebras and studied some of their properties in [1]. In this paper, we introduce the notions of fuzzy closure filters in $MS-$algebras. It is proved that the lattice of fuzzy closure filters is isomorphic to the fuzzy ideals of lattice of boosters. We also prove that the class of all fuzzy closure filters forms a complete distributive lattice. We also observe that the minimal elements of the

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poset of all prime fuzzy filters of an $MS-$algebra are fuzzy closure filters, and every proper fuzzy closure filters of L is the intersection of all prime fuzzy closure filters containing it. Throughout this paper we consider an $MS-$Algebra L as a decomposable $MS-$algebra.

2. Preliminaries

We first recall some basic concepts and outcomes in this section which will be used in this paper. For ordinary crisp theory of closure filters of $MS-$algebras,

Definition 2.1. [5] An algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is said to be an $MS-$algebra if $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $x \to x^\circ$ is a unary operation satisfies the following

1. $x \leq x^\circ$
2. $(x \wedge y)^\circ = x^\circ \vee y^\circ$
3. $1^\circ = 0$, for all $x, y \in L$.

Lemma 2.2. [5] *In any $MS-$algebra L, the following statements hold:*

1. $0^\circ = 1$
2. $x \leq y \Rightarrow y^\circ \leq x^\circ$
3. $x^{\circ\circ\circ} = x^\circ$
4. $(x \vee y)^\circ = x^\circ \wedge y^\circ$
5. $(x \vee y)^{\circ\circ} = x^{\circ\circ} \vee y^{\circ\circ}$
6. $(x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ}$, *for all $x, y, z \in L$*

Definition 2.3. [1] Let L be an $MS-$algebra. Then L is said to be decomposable MS-algebra if for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{\circ\circ} \wedge d$.

For any $a \in L$, we define the booster of a as follows: $(a)^+ = \{x \in L \mid a \leq x^{\circ\circ}\}$. Note that $(0)^+ = L$ and $(1)^+ = D(L)$, where $D(L) = \{x \in L \mid x^\circ = 0\}$. Let us denote the set of all boosters of an $MS-$algebra L by $M_0(L)$. For an $MS-$algebra L, the set $M_0(L)$ of all boosters is a complete distributive lattice on its own. Note that for any boosters $(a)^+; (b)^+$ of $M_0(L)$, define the operations $\cap$ and $\sqcup$ as $(a)^+ \cap (b)^+ = (a \vee b)^+$ and $(a)^+ \sqcup (b)^+ = (a \wedge b)^+$. Note that $(a \vee b)^+$ and $(a \wedge b)^+$ are infimum and supremum for both $(a)^+$ and $(b)^+$ in $M_0(L)$ respectively.

Definition 2.4. [1]
For any filter F of L, define an operator α as $\alpha(F) = \{(a)^+ \mid a \in F\}$. For any ideal $\widetilde{I}$ of $M_0(L)$, define an operator $\overleftarrow{\alpha}$ as $\overleftarrow{\alpha}(\widetilde{I}) = \{a \in L \mid (a)^+ \in \widetilde{I}\}$.

We recall that for any nonempty set T, the characteristic function of S,

$$\chi_T(a) = \begin{cases} 1 & \text{if } a \in T \\ 0 & \text{if } a \notin T. \end{cases}$$

Definition 2.5. [12] A proper filter P of L is said to be prime if $F \cap G \subset P$ implies $F \subseteq P$ or $G \subseteq P$ for any fuzzy filters of F and G of L.

Definition 2.6. [10] Let μ be a fuzzy subset of S and $t \in [0;1]$. Then, the set $\mu_t = \{a \in L \mid t \leq \mu(a)\}$ is called a level subset of μ.

A fuzzy subset μ of L is said to be proper if it is a non constant function. A fuzzy subset μ such that $\mu(a) = 0$ for all $a \in L$ is an improper fuzzy subset.

Definition 2.7. [13] Let μ and θ be fuzzy subsets of a set L. Then, we define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of L as follows: for each $a \in L$, $(\mu \cup \theta)(a) = \mu(a) \vee \theta(a)$ and $(\mu \cap \theta)(a) = \mu(a) \wedge \theta(a)$. Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

We define the binary operations $\vee$ and $\wedge$ on all fuzzy subsets of a lattice L as $(\mu \vee \theta)(a) = \sup\{\mu(x) \wedge \theta(y) \mid x, y \in L; x \vee y = a\}$ and $(\mu \wedge \theta)(a) = \sup\{\mu(x) \wedge \theta(y) \mid x, y \in L; x \wedge y = a\}$.

If μ and θ are fuzzy ideals of L, then $\mu \wedge \theta = \mu \cap \theta$ and $\mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

Definition 2.8. [7] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy ideal of L, if for all $a, b \in L$,

1. $\mu(0) = 1$
2. $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$
3. $\mu(a \wedge b) \geq \mu(a) \vee \mu(b)$ for all $a, b \in L$.

In [7], Swamy and Raju proved that, a fuzzy subset μ of a bounded lattice L is a fuzzy ideal of L if and only if $\mu(0) = 1$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$ for all $a, b \in L$.

DEFINITION 2.9. [7] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy filter of L, if for all $a, b \in L$,

1. $\mu(1) = 1$
2. $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$
3. $\mu(a \wedge b) \geq \mu(a) \vee \mu(b)$ for all $a, b \in L$.

In [7], Swamy and Raju have shown that a fuzzy subset μ of a bounded lattice L is a fuzzy filter of L if and only if $\mu(1) = 1$ and $\mu(a \vee b) = \mu(a) \wedge \mu(b)$, for all $a, b \in L$. However,for fuzzy ideals of a bounded lattice L. we now have the following theorem.

THEOREM 2.10. [7] *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if, for any $t \in [0; 1]$, μ_t is an ideal of L.*

DEFINITION 2.11. [11]

1. A proper fuzzy ideal μ of L is called a prime fuzzy ideal if for any two fuzzy ideals η, ν, of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.
2. A proper fuzzy filter μ of L is called a prime fuzzy filter if for any two fuzzy filters η, ν of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.

We now have the following theorem.

THEOREM 2.12. [8] *For any $t \in [0; 1)$, the fuzzy subset P_t^1 of L given by*

$$P_t^1(a) = \begin{cases} 1 \text{ if } a \in P \\ t \text{ if } a \notin P \end{cases}$$

for all $a \in L$ is a prime fuzzy filter if and only if P is a prime filter of L.

Throughout this paper, L stands for a decomposable MS-algebra unless otherwise stated.

3. Fuzzy Closure filters of $MS-$Algebras

In this section, we introduce the notion of fuzzy closure filters in $MS-$algebras and study the properties of fuzzy closure filters.

DEFINITION 3.1. A fuzzy subset ν of $M_0(L)$ is called a fuzzy ideal of $M_0(L)$ if $\nu((1)^+) = 1$ and $\nu((a)^+ \sqcup (b)^+) \geq \nu((a)^+) \wedge \nu((b)^+))$ and $\nu((a)^+ \cap (b)^+) \geq \nu((a)^+) \vee \nu((b)^+))$, for all $(a)^+, (b)^+ \in M_0(L)$.

EXAMPLE 3.2. Let L be a non-empty set and $\vee, \wedge, '$ be binary operations and unary operations respectively which are defined by

$'$	0	1	2	3
	1	0	0	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	3
3	3	1	3	3

Then $(L, \vee, \wedge, ', 0, 1)$ is a decomposable MS-algebra. Define $\nu((1)^+) = 1,\ \nu((0)^+) = 0.5,\ \nu((2)^+) = \nu((3)^+) = 0.8$. Clearly, ν is a fuzzy ideal of $M_0(L)$.

In the following definition, we define two operators α and $\overleftarrow{\alpha}$ in L

DEFINITION 3.3. Let L be an $MS-$algebra.
(1) For any fuzzy filter θ of L and for any a in L, define an operator α as $\alpha(\theta)((a)^+) = \sup\{\theta(b) \mid (a)^+ = (b)^+, b \in L\}$.
(2) For any fuzzy ideal ν of $M_0(L)$ and for any a in L, define an operator $\overleftarrow{\alpha}$ as $\overleftarrow{\alpha}(\nu)(a) = \nu((a)^+)$.

LEMMA 3.4. *In any $MS-$algebra L, The following three statements hold:*

(1) For any fuzzy filter θ of L, $\alpha(\theta)$ is a fuzzy ideal of $M_0(L)$
(2) For any fuzzy ideal ν of $M_0(L)$, $\overleftarrow{\alpha}(\nu)$ is a fuzzy filter of L
(3) The maps α and $\overleftarrow{\alpha}$ are isotone.

Proof.
(1). For any fuzzy filter θ, we have $\alpha(\theta)((1)^+) = 1$. Let $(x)^+, (y)^+ \in M_0(L)$. Then, we have $\alpha(\theta)((x)^+) \wedge \alpha(\theta)((y)^+) = \sup\{\theta(a) \mid (a)^+ = (x)^+\} \wedge \sup\{\theta(b) \mid (b)^+ = (y)^+\} = \sup\{\theta(a) \wedge \theta(b) \mid (a)^+ = (x)^+, (b)^+ = (y)^+\} \leq \sup\{\theta(a \wedge b) \mid (a \wedge b)^+ = (x \wedge y)^+\} = \alpha(\theta)((x \wedge y)^+) = \alpha(\theta)((x)^+ \sqcup (y)^+)$ and $\alpha(\theta)((x)^+) \vee \alpha(\theta)((y)^+) = \sup\{\theta(a) \mid (a)^+ = (x)^+\} \vee \sup\{\theta(b) \mid (b)^+ = (y)^+\} = \sup\{\theta(a) \vee \theta(b) \mid (a)^+ = (x)^+, (b)^+ = (y)^+\} \leq \sup\{\theta(a \vee b) \mid (a \vee b)^+ = (x \vee y)^+\} = \alpha(\theta)((x \vee y)^+) = \alpha(\theta)((x)^+ \cap (y)^+)$. Therefore, $\alpha(\theta)$ is a fuzzy ideal of $M_0(L)$.
(2). Let ν be any fuzzy ideal of $M_0(L)$. Then $\overleftarrow{\alpha}(\nu)(1) = \nu((1)^+) = 1$. For any $x, y \in L$, $\overleftarrow{\alpha}(\nu)(x \wedge y) = \nu((x \wedge y)^+) = \nu((x)^+ \sqcup (y)^+) \geq \nu((x)^+) \wedge \nu((y)^+) = \overleftarrow{\alpha}(\nu)(x) \wedge \overleftarrow{\alpha}(\nu)(y)$ and $\overleftarrow{\alpha}(\nu)(x \vee y) = \nu((x \vee y)^+) = \nu((x)^+ \cap (y)^+) \geq \nu((x)^+) \vee \nu((y)^+) = \overleftarrow{\alpha}(\nu)(x) \vee \overleftarrow{\alpha}(\nu)(y)$.

(3). Assume that ν and θ are fuzzy filters of L with $\nu \subseteq \theta$. Now $\alpha(\nu)((a)^+) = \sup\{\nu(b) \mid (b)^+ = (a)^+\} \leq \sup\{\theta(b) \mid (b)^+ = (a)^+\} = \alpha(\theta)((a)^+)$. Therefore, α is an isotone mapping. Similarly, we deduce that $\overleftarrow{\alpha}$ is also an isotone mapping. □

THEOREM 3.5. *The mapping $\nu \to \overleftarrow{\alpha}\alpha(\nu)$ is a closure operator on the lattice of fuzzy filters of L. i.e., for any fuzzy filters ν and θ of L,*
(1) $\nu \subseteq \overleftarrow{\alpha}\alpha(\nu)$
(2) $\nu \subseteq \theta \Rightarrow \overleftarrow{\alpha}\alpha(\nu) \subseteq \overleftarrow{\alpha}\alpha(\theta)$
(3) $\overleftarrow{\alpha}\alpha\{\overleftarrow{\alpha}\alpha(\nu)\} = \overleftarrow{\alpha}\alpha(\nu)$.

Proof.
(1). Clearly, we have the following equality:

$$\overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (a)^+ = (b)^+\} \geq \nu(a),$$

for all $a \in L$
(2). This part is clear.
(3). Let $a \in L$. Now, we have $\overleftarrow{\alpha}\alpha\{\overleftarrow{\alpha}\alpha(\nu)\}(a) = \alpha\{\overleftarrow{\alpha}\alpha(\nu)\}((a)^+) = \sup\{\overleftarrow{\alpha}\alpha(\nu)(b) \mid (b)^+ = (a)^+,\ b \in L\} = \sup\{\alpha(\nu)((b)^+) \mid (b)^+ = (a)^+,\ b \in L\} = \alpha(\nu)((a)^+) = \overleftarrow{\alpha}\alpha(\nu)(a)$.
□

THEOREM 3.6. *Let L be an $MS-$algebra. Then α is a homomorphism of the lattice of fuzzy filters of L into the lattice of fuzzy ideals of $M_0(L)$.*

Proof. Let $\mathcal{FF}(L)$ be the set of all fuzzy filters of L and $\mathcal{FI}M_0(L)$ be the set of all fuzzy ideals in $M_0(L)$. Then, for any $\nu, \theta \in \mathcal{FF}(L)$, we have $\nu \cap \theta \subseteq \nu$ and $\nu \cap \theta \subseteq \theta$. These results imply that $\alpha(\nu \cap \theta) \subseteq \alpha(\nu)$ and $\alpha(\nu \cap \theta) \subseteq \alpha(\theta)$. The above results further imply that $\alpha(\nu \cap \theta) \subseteq \alpha(\nu) \cap \alpha(\theta)$. Now, we have $(\alpha(\nu) \cap \alpha(\theta))((a)^+) = \alpha(\nu)((a)^+) \wedge \alpha(\theta)((a)^+) = \sup\{\nu(x) \mid (x)^+ = (a)^+\} \wedge \sup\{\theta(y) \mid (y)^+ = (a)^+\} \leq \sup\{\nu(x \vee y) \mid (x \vee y)^+ = (a)^+\} \wedge \sup\{\theta(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \sup\{\nu(x \vee y) \wedge \theta(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \sup\{(\nu \cap \theta)(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \alpha(\nu \cap \theta)((a)^+)$. Therefore, we deduce that $\alpha(\nu) \cap \alpha(\theta) = \alpha(\nu \cap \theta)$. Since $\nu \subseteq \nu \vee \theta$, we have $\theta \subseteq \nu \vee \theta$, we obtain that $\alpha(\nu) \subseteq \alpha(\nu \vee \theta)$ and $\alpha(\theta) \subseteq \alpha(\nu \vee \theta)$. The above results imply that $\alpha(\nu) \sqcup \alpha(\theta) \subseteq \alpha(\nu \vee \theta)$. Now, we have $(\alpha(\nu \vee \theta))((a)^+) = \sup\{(\nu \vee \theta)(x) \mid (x)^+ = (a)^+\} = \sup\{\sup\{\nu(x_1) \wedge \theta(x_2) \mid x = x_1 \wedge x_2\} \mid (x)^+ = (a)^+\} \leq \sup\{\sup\{\nu(y_1) \wedge \theta(y_2) \mid (y_1)^+ = (x_1)^+, (y_2)^+ = (x_2)^+\} \mid (x_1 \wedge x_2)^+ = (a)^+\} = \sup\{\sup\{\nu(y_1) \mid (y_1)^+ = (x_1)^+\} \wedge \sup\{\theta(y_2) \mid (y_2)^+ = (x_2)^+\} \mid (x_1)^+ \sqcup (x_2)^+ = (a)^+\} = \sup\{\alpha(\nu)((x_1)^+) \wedge \alpha(\theta)((x_2)^+) \mid (x_1)^+ \sqcup (x_2)^+ = (a)^+\} = (\alpha(\nu) \sqcup \alpha(\theta))((a)^+)$. The above equalities imply that $\alpha(\nu \vee \theta) \subseteq \alpha(\nu) \sqcup \alpha(\theta)$. Hence, $\alpha(\nu \vee \theta) = \alpha(\nu) \sqcup \alpha(\theta)$. Clearly, we have shown that $\chi_{\{1\}},\ \chi_L$ are the smallest and the largest fuzzy filters of L, respectively and also we have that $\alpha(\chi_{\{1\}}),\ \alpha(\chi_L)$ are smallest and greatest fuzzy ideals of $M_0(L)$, respectively. Hence ,α is indeed a homomorphism from $\mathcal{FF}(L)$ into $\mathcal{FI}M_0(L)$. □

Corollary 3.7. *Let ν and θ be any two fuzzy filters of an $MS-$algebra L. Then, we have $\overleftarrow{\alpha}\alpha(\nu \cap \theta) = \overleftarrow{\alpha}\alpha(\nu) \cap \overleftarrow{\alpha}\alpha(\theta)$.*

Proof. By using the above result, we obtain that $\alpha(\nu) \cap \alpha(\theta) = \alpha(\nu \cap \theta)$. Now, $\overleftarrow{\alpha}\alpha(\nu \cap \theta)(b) = \alpha(\nu \cap \theta)((b)^+) = \alpha(\nu)((b)^+) \wedge \alpha(\theta)((b)^+) = \overleftarrow{\alpha}\alpha(\nu)(b) \wedge \overleftarrow{\alpha}\alpha(\theta)(b)$. Therefore, we have $\overleftarrow{\alpha}\alpha(\nu \cap \theta) = \overleftarrow{\alpha}\alpha(\nu) \cap \overleftarrow{\alpha}\alpha(\theta)$. □

Now, we introduce the concept of fuzzy closure filters in $MS-$algebras.

Definition 3.8. A fuzzy filter ν of an $MS-$algebra L is called a fuzzy closure filter if $\overleftarrow{\alpha}\alpha(\nu) = \nu$.

Example 3.9. Consider Example 3.2 and define $\nu(1) = 1,\ \nu(0) = 0.5,\ \nu(2) = \nu(3) = 0.8$. Clearly, ν is a fuzzy filter of L. Clearly, we have $\overleftarrow{\alpha}\alpha\nu(x) = \nu(x)$, for all $x \in L$. Hence, ν is a fuzzy closure filter of L. Define $\theta(1) = 1,\ \theta(0) = 0,\ \theta(2) = 0.3,\ \theta(3) = 0.6$. Clearly, θ is a fuzzy filter of L. But θ is not a fuzzy closure filter of L, because $\overleftarrow{\alpha}\alpha\theta(2) \neq \theta(2)$.

Now we characterize the fuzzy closure filters in terms of its level subsets and characteristic functions.

Theorem 3.10. *Let ν be any proper fuzzy subset of L. Then ν is a fuzzy closure filter if and only if ν_t, for all $t \in [0,1]$, is a closure filter of L.*

Proof. Let ν is a fuzzy closure filter of L. Then $(\overleftarrow{\alpha}\alpha(\nu))_t = (\nu)_t$ Now we prove every level subset of ν is a closure filter of L. It is enough to show $\overleftarrow{\alpha}\alpha(\nu_t) = \nu_t$. Clearly, we have that $\nu_t \subseteq \overleftarrow{\alpha}\alpha(\nu_t)$. Let $a \in \overleftarrow{\alpha}\alpha(\nu_t)$. That implies $(a)^+ \in \alpha(\nu_t)$. Then there exists $b \in \nu_t$ such that $(a)^+ = (b)^+$ and so, we have $\nu(b) \geq \alpha$ with $(a)^+ = (b)^+$. That implies $\alpha(\nu)((a)^+) = \sup\{\nu(b) \mid (a)^+ = (b)^+\} \geq \alpha$ and so $\overleftarrow{\alpha}\alpha(\nu)(a) \geq t$. That implies $a \in (\overleftarrow{\alpha}\alpha(\nu))_t$. Therefore, we have $\overleftarrow{\alpha}\alpha(\nu_t) \subseteq \nu_t$ and hence $\overleftarrow{\alpha}\alpha(\nu_t) = \nu_t$. Clearly, we arrive that $\nu \subseteq \overleftarrow{\alpha}\alpha(\nu)$. Let $\alpha = \overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (b)^+ = (a)^+\}$. Then for each $\epsilon > 0$, there is $x \in L,\ (x)^+ = (a)^+$ such that $\nu(a) > \alpha - \epsilon$. Since ϵ is arbitrary chosen, we have $\nu(a) \geq \alpha$ such that $(x)^+ = (a)^+$. This result implies $x \in \nu_t$. Therefore, we have

$a \in \overleftarrow{\alpha}\alpha(\nu_t) = \nu_\alpha$ and hence $\nu(a) \geq \alpha = \overleftarrow{\alpha}\alpha(\nu_t)$. Thus, we conclude that $\nu \supseteq \overleftarrow{\alpha}\alpha(\nu)$. □

Corollary 3.11. *Let F be any non-empty subset F of an $MS-$algebra L. Then F is a closure filter if and only if χ_F is a fuzzy closure filter of L.*

Now we characterize the fuzzy closure filters in terms of boosters in the following result.

Theorem 3.12. *Let ν be a fuzzy filter of L. Then ν is a fuzzy closure filter if and only if for any $a, b \in L$, $(a)^+ = (b)^+$ implies $\nu(a) = \nu(b)$.*

Proof. Assume that ν is a fuzzy closure filter of L. Then we have the following equality $\nu(a) = \overleftarrow{\alpha}\alpha(\nu)(a)$, for all $a \in L$. Let $a, b \in L$ such that $(a)^+ = (b)^+$. Then, we have $\nu(a) = \overleftarrow{\alpha}\alpha(\nu)(a) = \alpha(\nu)((a)^+) = \alpha(\nu)((b)^+) = \overleftarrow{\alpha}\alpha(\nu)(b) = \nu(b)$. Conversely, assume that for any $a, b \in L$, $(a)^+ = (b)^+$ implies $\nu(a) = \nu(b)$. Now $\overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (b)^+ = (a)^+\} = \nu(a)$. Therefore. we have $\overleftarrow{\alpha}\alpha(\nu) = \nu$. □

We now establish the following main theorem of fuzzy closure filters.

Theorem 3.13. *Let $\{\nu_i \mid i \in \Omega\}$ be any family of fuzzy closure filters of an $MS-$algebra L. Then $\bigcap_{i\in\Omega} \nu_i$ is a fuzzy closure filter of L.*

Corollary 3.14. *Let L be an $MS-$algebra. Then the set $\mathcal{FF}_{\mathcal{C}}(L)$ of all fuzzy closure filters of L is a complete distributive lattice with relation $\subseteq$. The sup and inf of any subfamily $\{\nu_i \mid i \in \Omega\}$ of fuzzy closure filters are $\overleftarrow{\alpha}\alpha(\bigvee \nu_i)$ and i) and $\bigcap_{i\in\Omega} \nu_i$ respectively, where $\bigvee \nu_i$ is their supremum in the lattice of fuzzy filters of L.*

Lemma 3.15. *Let ν be any fuzzy ideal of $M_0(L)$. Then $\nu = \alpha\overleftarrow{\alpha}(\nu)$.*

Proof. Let $(a)^+ \in M_0(L)$. Now $\alpha\overleftarrow{\alpha}(\nu)((a)^+) = \sup\{\overleftarrow{\alpha}(\nu)(b) \mid (b)^+ = (a)^+\} = \sup\{\nu((b)^+) \mid (b)^+ = (a)^+\} = \nu((a)^+)$. Therefore $\alpha\overleftarrow{\alpha}(\nu) = \nu$. □

Using the above Corollary 3.14 and Lemma 3.15, we are able to prove that the lattice of fuzzy closure filters of L is isomorphic to the lattice of fuzzy ideals of $M_0(L)$.

Theorem 3.16. *Let L be an $MS-$algebra. Then there is an isomorphism of the lattice of fuzzy closure filters of L onto the lattice of fuzzy ideals of $M_0(L)$.*

Proof. Let $\mathcal{FF}_{\mathcal{C}}(L)$ be the set of all fuzzy filters of L, $\mathcal{FI}M_0(L)$ be the set of all fuzzy ideals of $M_0(L)$. Define $f : \mathcal{FF}_{\mathcal{C}}(L) \to \mathcal{FI}M_0(L)$ by $f(\nu) = \alpha(\nu)$, for any $\nu \in \mathcal{FF}_{\mathcal{C}}(L)$. It is easy to see that f is one one. Let ν be an fuzzy ideal of $M_0(L)$. Then $\overleftarrow{\alpha}(\nu)$ is a fuzzy filter of L. Now By applying the above Lemma, we deduce that $\overleftarrow{\alpha}\alpha(\overleftarrow{\alpha}(\nu)) = \overleftarrow{\alpha}(\alpha\overleftarrow{\alpha}(\nu)) = \overleftarrow{\alpha}(\nu)$. Thus $\overleftarrow{\alpha}(\nu)$ is a fuzzy closure filter of L. Hence, we derive that $f(\overleftarrow{\alpha}(\nu)) = \alpha(\overleftarrow{\alpha}(\nu)) = \nu$. This result gives that f is onto. Let ν, θ be any two fuzzy closure filters of L. Clearly, we have $f(\nu \cap \theta) = \alpha(\nu \cap \theta) = \alpha(\nu) \cap \alpha(\theta)$. Now, we further obtain $f(\overleftarrow{\alpha}\alpha(\nu \vee \theta)) = \alpha((\overleftarrow{\alpha}\alpha(\nu \vee \theta))) = \alpha(\nu \vee \theta) = \alpha(\nu) \sqcup \alpha(\theta)$. Therefore, we have shown that f is an isomorphism. □

4. Prime Fuzzy closure Filters and Maximal Fuzzy closure Filters of $MS-$algebras

In this section, we continue to study some important properties of prime fuzzy closure filters and maximal fuzzy closure filters in $MS-$algebras.

Definition 4.1. A proper fuzzy closure filter ν of an $MS-$algebra L is said to be prime if for any fuzzy filters θ and μ such that $\theta \cap \mu \subseteq \nu$, we have $\theta \subseteq \nu$ or $\mu \subseteq \nu$.

Lemma 4.2. *Let P be a proper filter of L. Then P is a prime closure filter of L, $t \in [0, 1)$ if and only if*

$$P_t^1(a) = \begin{cases} 1 \text{ if } a \in P \\ t \text{ otherwise} \end{cases}$$

is a prime closure filter of L.

Proof. Assume that P is a proper closure filter of L and $t \in [0, 1)$. It can be easily verified that P_t^1 is a proper fuzzy filter of L. Now, we prove that P_t^1 is a prime fuzzy filter of L. Let θ and λ be fuzzy filters of L such that $\theta \not\subseteq P_t^1$ and $\lambda \not\subseteq P_t^1$. Then there exist $a, b \in L$ such that $\theta(a) > P_t^1(a)$ and $\lambda(b) > P_t^1(b)$. This implies $a \notin P$ and $b \notin P$, and so we have $a \vee b \notin P$ and $P_t^1(a \vee b) = \alpha$. It follows that $\theta(x) \wedge \lambda(b) > t$. Since θ and λ are isotone mappings, we have $(\theta \cap \lambda)(a \vee b) = \theta(a \vee b) \wedge \lambda(a \vee b) \geq$

$\theta(a) \wedge \lambda(a) > t = P_t^1(a \vee b)$. This implies $\theta \cap \lambda \nsubseteq P_t^1$. Thus, we have shown that P_t^1 is a prime fuzzy filter of L. Next, we prove that P_t^1 is a prime fuzzy closure filter of L. Since P is a prime closure filter of L and $t \in [0, 1)$, for any $a, b \in L$ such that $(a)^+ = (b)^+$. If $P_t^1(a) = 1$, then $a \in P$. This implies that $b \in P$ and $P_t^1(b) = 1$. If $P_t^1(a) = t$; then $a \notin P$. This implies that $b \notin P$ and $P_t^1(b) = t$. Hence, P_t^1 is a prime fuzzy closure filter of L. Conversely, assume that P_t^1 is a prime fuzzy filter of L. If F and G are any filters of L such that $F \cap G \subseteq P$, then $(F \cap G)_t^1 = F_t^1 \cap G_t^1 \subseteq P_t^1$. This implies $F_t^1 \subseteq P_t^1$ or $G_t^1 \subseteq P_t^1$, so that $F \subseteq P$ or $G \subseteq P$. Therefore, we have shown that P is a prime filter of L. Now, suppose that P_t^1 is a prime fuzzy closure filter of L and for any $a, b \in L$ such that $(a)^+ = (b)^+$. Let $a \in P$. Then, we deduce that $1 = P_t^1(a) = P_t^1(b)$. This implies $b \in P$. Hence, P is indeed a prime closure filter of L. □

COROLLARY 4.3. *A proper filter P is a prime closure filter of L if and only if χ_P is a prime fuzzy closure filter of L.*

Proof. Assume that P is a prime closure filter of L. Now we prove that χ_P is a prime fuzzy filter of L. Let ν and λ be any fuzzy filters of L such that $\theta \cap \lambda \subseteq \chi_P$. Suppose $\theta \nsubseteq \chi_P$ and $\lambda \nsubseteq \chi_P$. Then there exist $a, b \in L$ such that $\lambda(a) > \chi_P(a)$ and $\theta(b) > \chi_P(b)$. This implies $a \notin P$ and $b \notin P$. Since P is a prime filter, $a \vee b \notin P$. Thus $\chi_P(a \vee b) = 0$. Now, $(\lambda \cap \theta)(a \vee b) = \lambda(a \vee b) \wedge \theta(a \vee b) \geq \lambda(a) \wedge \theta(b) > \chi_P(a) \wedge \chi_P(b) = 0 = \chi_P(a \vee b)$. This implies $\theta \cap \lambda \nsubseteq \chi_P$, which is a contradiction. Thus χ_P is a prime filter of L. Next we prove that χ_P is a prime fuzzy closure filter. Let $a, b \in L$ such that $(a)^+ = (b)^+$. If $\chi_P(a) = 1$, then $a \in P$. This implies $b \in P$. Thus $\chi_P(b) = 1$. If $\chi_P(a) = 0$, then $a \notin P$. This implies $b \notin P$. Thus $\chi_P(b) = 0$. Hence χ_P is a prime fuzzy closure filter of L. Conversely, assume that χ_P is a prime closure filter of L. Now we show that P is a prime filter of L. Let F and G be any filters of L such that $F \cap G \subseteq P$. Then $\chi_{F \cap G} \subseteq \chi_P$. That implies $\chi_F \subseteq \chi_P$ or $\chi_G \subseteq \chi_P$ and hence $F \subseteq P$ or $G \subseteq P$. Therefore P is a prime filter. We prove that P is a prime closure filter of L. Let $a, b \in L$ such that $(a)^+ = (b)^+$. Let $a \in P$. Then $\chi_P(a) = 1 = \chi_P(b)$. Thus $b \in P$. Hence P is a prime fuzzy closure filter of L. □

THEOREM 4.4. *proper fuzzy filter ν of L is a prime fuzzy closure filter if and only if $Img(\nu) = \{1, t\}$, where $t \in [0, 1)$ and the set $\nu_* = \{x \in L \mid \nu(x) = 1\}$ is a prime closure filter of L.*

Proof. From the above lemma, we have the converse part. Assume that ν is a prime fuzzy closure filter. Clearly, we have $1 \in Im(\nu)$. Since ν is proper, there is $a \in L$ such that $\nu(a) < 1$. We show that $\nu(a) = \nu(b)$, for all $a, b \in L \setminus \nu_*$. Suppose $\nu(a) \neq \nu(b)$, for some $a, b \in L \setminus \nu_*$. Without loss of generality we can assume that $\nu(b) < \nu(a) < 1$. Define fuzzy subsets θ and λ as follows:

$$\theta(x) = \begin{cases} 1 \text{ if } x \in [a) \\ 0 \text{ otherwise} \end{cases}$$

and

$$\lambda(x) = \begin{cases} 1 \text{ if } x \in \nu_* \\ \nu(a) \text{ otherwise} \end{cases}$$

for all $x \in L$. Clearly, we see immediately that both θ and λ are fuzzy filters of L. Let $x \in L$. If $x \in \nu_*$, then $(\theta \cap \lambda)(x) \leq 1 = \nu(x)$. If $x \in [a) \setminus \nu_*$, then $x = a \vee x$, and we have $(\theta \cap \lambda)(x) = \theta(x) \wedge \lambda(x) = 1 \wedge \nu(a) = \nu(a) \leq \nu(x)$. Also if $x \notin [a)$, then $\theta(x) = 0$ and hence $(\theta \cap \lambda)(x) = 0 \leq \nu(x)$. Therefore, we get $\theta \cap \lambda \subseteq \nu$. Since $\theta(x) = 1 > \nu(x)$ and $\lambda(y) = \nu(x) > \nu(y)$, we arrive that $\lambda \nsubseteq \nu$ and $\theta \nsubseteq \lambda$, which is a contradiction. Thus $\nu(a) = \nu(b)$ for all $a, b \in L \setminus \nu_*$ and hence $Im(\nu) = \{1, t\}$ for some $t \in [0, 1)$. Let $P = \{a \in L \mid \nu(a) = 1\}$. Since ν is proper, we get that P is a proper filter of L. Let $t \neq 1$. Then

$$\nu(x) = \begin{cases} 1 \text{ if } x \in P \\ t \text{ if } x \notin P. \end{cases}$$

By the above lemma, we have shown that $P = \nu_*$. □

Definition 4.5. A proper fuzzy filter ν of an $MS-$algebra L is said to be maximal if $Im\nu = \{1, t\}$, where $t \in [0, 1)$ and the level filter $\nu_* = \{a \in L \mid \nu(a) = 1\}$ is a maximal filter.

A proper fuzzy filter ν of an $MS-$algebra L is said to be a maximal fuzzy closure filter of L if $Im\nu = \{1, t\}$, where $t \in [0, 1)$ and the level filter ν_* is a maximal closure filter.

Theorem 4.6. *Every maximal fuzzy filter of an $MS-$algebra is a fuzzy closure filter.*

Proof. Let ν be a maximal fuzzy filter of L. Then ν_* is a maximal filter and $Im\nu = \{1, t\}$. That implies ν_* is maximal and $\nu_t = L$. Since every maximal filter is a closure filter of L, we get that the level subsets of L is closure filters of L. Hence ν is a fuzzy closure filter of L. □

The following corollaries follow immediately.

Corollary 4.7. *Every maximal fuzzy closure filter of L is a maximal fuzzy filter.*

Corollary 4.8. *Every maximal fuzzy closure filter of L is a prime fuzzy closure filter.*

Theorem 4.9. *Let L be an $MS-$algebra. If ν is minimal in the class of all prime fuzzy filters containing a given fuzzy closure filter, then ν is a fuzzy closure filter.*

Proof. Let ν be a minimal in the class of all prime fuzzy filters containing a fuzzy closure filter θ of L. Since ν is a prime fuzzy filter of L, there exists a prime filter P of L such

$$\nu(z) = \begin{cases} 1 \text{ if } x \in P \\ t \text{ otherwise,} \end{cases}$$

for some $t \in [0, 1)$. Suppose that ν is not a fuzzy closure filter of L. Then there exist $a, b \in L,\ (a)^+ = (b)^+$ such that $\nu(a) \neq \nu(b)$. Without loss of generality, we may assume that $\nu(a) = 1$ and $\nu(b) = t$. Consider a fuzzy ideal ϕ of L defined by

$$\phi(x) = \begin{cases} 1 \text{ if } x \in (L \setminus P) \vee (a \vee b) \\ t \text{ otherwise.} \end{cases}$$

Then we have $\theta \cap \phi \leq t$. For if otherwise, then there exists $y \in L$ such that $\phi(y) = 1$. This implies $y \in (L \setminus P) \vee (a \vee b]$. This result again implies $y = r \vee s$ for some $r \in (L \setminus P)$ and $s \in (a \vee b]$ and hence, $y = r \vee s = r \vee (s \wedge (a \vee b)) = (r \vee s) \wedge (s \vee a \vee b) \leq s \vee a \vee b$. Since θ is a fuzzy closure filter of $L,\ t < \theta(r \vee s) \leq \theta(r \vee a \vee b)\nu(r \vee a \vee b)$. Also, $(a)^+ = (b)^+$ implies $(r \vee a \vee b)^+ = (r \vee b)^+$. These results imply that $\theta(r \vee a \vee b) = \theta(r \vee b) \leq \nu(r \vee b) = 1$. Since ν is a prime filter, we have $\nu(r) = 1$ or $\nu(y) = 1$, which is a contradiction. Thus, we arrive that $\theta \cap \phi \leq t$. This result implies that there exists a prime fuzzy filter η such that $\eta \cap \phi \leq t$ and $\theta \subseteq \eta$. Clearly, we have $a \vee b \in (L \setminus P) \vee (a \vee b]$. This result implies $\phi(a \vee b) = 1$ and $\phi \cap \eta \leq t$. Hence, we have $\eta(a \vee b) \leq t < \nu(a \vee b) = 1$. This implies $\nu \nsubseteq \eta$. Therefore, ν is not minimal in the class of all prime fuzzy filters containing a given fuzzy closure filter, which is a contradiction. Finally, we have shown that ν is indeed a fuzzy closure filter. □

Corollary 4.10. *Let L be an $MS-$algebra. Then prime fuzzy closure filters of L are one to one correspondence with the prime fuzzy ideals of $M_0(L)$.*

Proof. Clearly, we see that fuzzy closure filters of L are one to one correspondence with the fuzzy ideals of $M_0(L)$. Now we prove that if ν is a prime fuzzy closure filter, then $\alpha(\nu)$ is also a prime fuzzy ideal of $M_0(L)$ and vice versa. Let ν be a prime fuzzy closure filter of L. Then $\alpha(\nu)$ is a fuzzy ideal of $M_0(L)$. Let θ and ν be any ideals of $M_0(L)$. Then there exist a fuzzy closure filter of L, ϕ and ψ such that $\theta = \alpha(\phi)$ and $\nu = \alpha(\psi)$. Assume that $\alpha(\phi) \cap \alpha(\psi) \subseteq \alpha(\nu)$. Then $\alpha(\phi \cap \psi) \subseteq \alpha(\nu)$ and so $\phi \cap \psi \subseteq \nu$. Since ν is a prime closure filter of L, then $\phi \subseteq \nu$ or $\psi \subseteq \nu$. This gives $\alpha(\phi) \subseteq \alpha(\nu)$ or $\alpha(\psi) \subseteq \alpha(\nu)$. Let ν be a prime ideal of $M_0(L)$. Then there exists a fuzzy closure filter of η of L such that $\nu = \alpha(\eta)$. Let ϕ, ψ be any fuzzy filters of L such that $\phi \cap \psi \subseteq \eta$. Then $\alpha(\phi \cap \psi) = \alpha(\phi) \cap \alpha(\psi) \subseteq \alpha(\eta)$. Since $\alpha(\eta)$ is a prime ideal of L, then we have $\alpha(\phi) \subseteq \alpha(\eta)$ or $\alpha(\psi) \subseteq \alpha(\eta)$ and so $\phi \subseteq \eta$ or $\psi \subseteq \eta$. This result implies η is a prime fuzzy closure filter of L. Thus, we have shown that prime fuzzy closure filters of L are one to one correspondence with the prime fuzzy ideals of $M_0(L)$.

□

Now we turn to prove the existence of prime fuzzy closure filters in $MS-$algebra in the following theorem.

Theorem 4.11. *Let $\alpha \in [0, 1)$, ν be a fuzzy closure filter and σ be a fuzzy ideal of an $MS-$algebra L such that$\nu \cap \sigma \leq \alpha$. Then there exists a prime fuzzy closure filter η such that $\nu \subseteq \eta$ and $\eta \cap \sigma \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF}_\mathcal{C}(L) \mid \nu \subseteq \theta,\ \theta \cap \sigma \leq \alpha\}$. Clearly, $\nu \in \xi$, $\xi \neq \emptyset$ and $(\xi, \subseteq)$ is a poset. Let $Q = \{\nu_i \mid i \in \Omega\}$ be a chain in ξ. We prove that $\bigcup\limits_{i\in\Omega} \nu_i \in \xi$. Clearly $(\bigcup\limits_{i\in\Omega} \nu_i)(1) = 1$. For any $a, b \in L$, $(\bigcup\limits_{i\in\Omega} \nu_i)(a) \wedge (\bigcup\limits_{i\in\Omega} \nu_i)(b) = \sup\{\nu_i(a) \mid i \in \Omega\} \wedge \sup\{\nu_j(b) \mid j \in \Omega\} = \sup\{\nu_i(a) \wedge \nu_j(b) \mid i, j \in \Omega\} \leq \sup\{(\nu_i \cup \nu_j)(a) \wedge (\nu_i \cup \nu_j)(b) \mid i; j \in \Omega\}$. Since Q is a chain, $\nu_i \subseteq \nu_j$ or $\nu_j \subseteq \nu_i$. Without loss of generality, we can assume that $\nu_j \subseteq \nu_i$ This implies $\nu_i \cup \nu_j = \nu_i$. That implies $(\bigcup\limits_{i\in\Omega} \nu_i)(a) \wedge (\bigcup\limits_{i\in\Omega} \nu_i)(b) \leq \sup\{\nu_i(a) \wedge \nu_j)(b) \mid i \in \Omega\} = \sup\{\nu_i(a \wedge b) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega} \nu_i)(a \wedge b)$.

Again $(\bigcup\limits_{i\in\Omega}\nu_i)(a) = \sup\{\nu_i(a) \mid i \in \Omega\} \leq \sup\{\nu_i(a \vee b) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega}\nu_i)(a \vee b)$. Similarly, we get that $(\bigcup\limits_{i\in\Omega}\nu_i)(b) \leq (\bigcup\limits_{i\in\Omega}\nu_i)(a \vee b)$. This implies $(\bigcup\limits_{i\in\Omega}\nu_i)(a) \vee (\bigcup\limits_{i\in\Omega}\nu_i)(b) \leq (\bigcup\limits_{i\in\Omega}\nu_i)(a \vee b)$. Hence $(\bigcup\limits_{i\in\Omega}\nu_i)$ is a fuzzy filter of L. Now prove that $(\bigcup\limits_{i\in\Omega}\nu_i)$ is a fuzzy closure filter. $\overleftarrow{\alpha}\alpha(\bigcup\limits_{i\in\Omega}\nu_i)(a) = \sup\{(\bigcup\limits_{i\in\Omega}\nu_i)(x) \mid (a)^+ = (x)^+,\ x \in L\} = \sup\{\sup\{\nu_i)(x) \mid i \in \Omega\} \mid (a)^+ = (x)^+,\ x \in L\} = \sup\{\sup\{\nu_i)(x) \mid (a)^+ = (x)^+,\ x \in L\} \mid i \in \Omega\} = \sup\{\overleftarrow{\alpha}\alpha(\nu_i) \mid i \in \Omega\} = \sup\{\nu_i(a) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega}\nu_i)(a)$. Thus $\bigcup\limits_{i\in\Omega}\nu_i$ is a fuzzy closure filter of L. Since $\nu_i \cap \sigma \leq \alpha$, for each $((\bigcup\limits_{i\in\Omega}\nu_i) \cap \sigma)(a) = (\bigcup\limits_{i\in\Omega}\nu_i)(a) \wedge \sigma)(a) = \sup\{\nu_i(a) \mid i \in \Omega\} \wedge \sigma(a) = \sup\{\nu_i(a) \wedge \sigma(a) \mid i \in \Omega\} = \sup\{(\nu_i \wedge \sigma)(a) \mid i \in \Omega\} \leq \alpha$. Thus $(\bigcup\limits_{i\in\Omega}\nu_i) \cap \sigma \leq \alpha$. Hence $\bigcup\limits_{i\in\Omega}\nu_i \in \xi$. By Zorn's Lemma, ξ has a maximal element, say δ, i.e, δ is a fuzzy closure filter of L such that $\nu \subseteq \delta$ and $\delta \cap \theta \leq \alpha$. Now we show that δ is a prime fuzzy closure filter of L. Assume that δ is not a prime fuzzy closure filter. Let $\lambda_1, \lambda_2 \in \mathcal{FF}_{\mathcal{C}}(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. Suppose $\delta_1 = \overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)$. Then both δ_1, δ_2 are fuzzy closure filters of L properly containing δ. Since δ is a maximal in ξ, we get that $\delta_1, \delta_2 \notin \xi$. That implies $\delta_1 \cap \theta \lneq \alpha$ and $\delta_1 \cap \theta \lneq \alpha$. That implies there exist $a, b \in L$ such that $(\delta_1 \cap \sigma)(a) > \alpha$ and $(\delta_2 \cap \sigma)(a) > \alpha$. We have $(\delta_1 \cap \sigma)(a \vee b) \wedge (\delta_2 \cap \sigma)(a \vee b) \geq (\delta_1 \cap \sigma)(a) \wedge (\delta_2 \cap \sigma)(b) \geq \alpha$, which implies $(\delta_1 \cap \sigma)(a \vee b) \wedge (\delta_2 \cap \sigma)(a \vee b) = ((\delta_1 \cap \theta) \cap (\delta_2 \cap \sigma))(a \vee b) = ((\delta_1\ \ \delta_2) \cap \sigma)(a \vee b) = ((\overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta) \cap \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)) \cap \sigma)(a \vee b) = (\overleftarrow{\alpha}\alpha(\lambda_1 \cap \lambda_2) \vee \delta) \cap \sigma)(a \vee b) = (\overleftarrow{\alpha}\alpha(\delta) \cap \sigma)(a \vee b) = (\delta \cap \theta)(a \vee b) > \alpha$. That implies $(\delta \cap \sigma)(a \vee b) > \alpha$, which is a contradiction to $\delta \cap \sigma \leq \alpha$. Therefore δ is a prime fuzzy closure filter of L. □

Corollary 4.12. *Let ν be a fuzzy closure filter and σ be a fuzzy ideal of an $MS-$algebra L such that $\nu \cap \sigma = 0$. Then there exists a prime fuzzy closure filter η such that $\nu \subseteq \eta$ and $\eta \cap \sigma = 0$.*

Corollary 4.13. *Let $t \in [0, 1)$, ν be a fuzzy closure filter of an $MS-$algebra L and $\nu(x) \leq \alpha$. Then there exists a prime fuzzy closure filter θ of L such that $\nu \subseteq \theta$ and $\theta(x) \leq t$.*

Proof. Consider $\xi = \{\theta \in \mathcal{FF}_{\mathcal{C}}(L) \mid \nu \subseteq \theta$ and $\theta(x) \leq t\}$. Clearly, we have that $\nu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\nu_i \mid i \in \Omega\}$

be a chain in ξ. By above theorem, $\bigcup_{i\in\Omega} \nu_i$ is a fuzzy closure filter of L. Since $\nu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(a) \leq t$. $(\bigcup_{i\in\Omega} \nu_i)(a) = \sup\{\nu_i(x) \mid i \in \Omega\} \leq \theta(a) \leq t$. Hence $\bigcup_{i\in\Omega} \nu_i \in \xi$. By Zorn's Lemma, ξ has a maximal element say δ, i.e, δ is a fuzzy closure filter of L such that $\nu \subseteq \delta$ and $\nu(a) \leq t$. Next we show that δ is a prime fuzzy closure filter of L. Assume that δ is not a prime fuzzy closure filter. Let $\lambda_1, \lambda_2 \in \mathcal{FF}(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are fuzzy closure filters of L properly containing δ. Since δ is maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. This we show that $\delta_1(a) \lneq t$ and $\delta_2(a) \lneq t$. Thus implies $\delta_1(a) > t$ and $\delta_2(a) > t$. We get $(\delta_1(a) \wedge (\delta_2)(a) = (\delta_1 \cap \delta_2)(a) > t$, which implies $\delta_1(a) \wedge \delta_2(a) = (\overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta) \cap \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta))(a) = (\overleftarrow{\alpha}\alpha((\lambda_1 \cap \lambda_2) \vee \delta))(a) = \overleftarrow{\alpha}\alpha(\delta)(a) = \delta(a) > t$. That implies $\delta(a) > t$, which is a contradiction $\delta(a) \leq t$. Thus δ is a prime fuzzy closure filter of L.

□

Corollary 4.14. *Let L be an $MS-$algebra. Then every proper fuzzy closure filters of L is the intersection of all prime fuzzy closure filters containing it.*

Proof. Let ν be a proper fuzzy closure filter of L.
Put $\eta = \bigcap\{\theta \mid \theta$ is a prime fuzzy closure filter such that $\nu \subseteq \theta\}$. Now, we proceed to prove that $\nu = \eta$. Clearly, $\nu \subseteq \eta$. Put $t = \nu(x)$, for some $x \in L$. This implies $\nu \subseteq \nu$ and $\nu(a) \leq t$. By the above Corollary, there exists a prime fuzzy closure filter δ such that $\nu \subseteq \delta$ and $\delta(x) \leq t$. Thus,we have $\eta \subseteq \nu$. Hence, $\nu = \eta$. This result implies that every proper fuzzy closure filters of L is the intersection of all prime fuzzy closure filters containing it. □

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